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# EXISTENCE AND ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Manabu Naito

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**Abstract.** We consider the half-linear differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \quad t \geq t_0,$$

under the condition

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}.$$

It is shown that if certain additional conditions are satisfied, then the above equation has a pair of nonoscillatory solutions with specific asymptotic behavior as $t \to \infty$.

**Keywords:** asymptotic behavior, nonoscillatory solution, half-linear differential equation.

**Mathematics Subject Classification:** 34C11, 34C10.

## 1. INTRODUCTION

In this paper we consider the half-linear ordinary differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \quad t \geq t_0, \tag{1.1}$$

where $\alpha$ is a positive constant and $q(t)$ is a real-valued continuous function on an interval $[t_0, \infty)$, $t_0 > 1$.

If $\alpha = 1$, then (1.1) becomes the linear equation

$$x'' + q(t)x = 0, \quad t \geq t_0. \tag{1.2}$$

It is known that basic results and qualitative results for the linear equation (1.2) can be generalized to the half-linear equation (1.1). The important works for (1.1) are summarized in the book of Došlý and Řehák [2].

 

In this paper the half-linear equation (1.1) is considered under the condition that

$$\int_{t_0}^{\infty} q(s)ds = \lim_{t\to\infty} \int_{t_0}^{t} q(s)ds \quad \text{exists and is finite.} \tag{1.3}$$

Then we define the function $Q(t)$ by

$$Q(t) = \int_{t}^{\infty} q(s)ds, \quad t \ge t_0. \tag{1.4}$$

Now, for the equation (1.1), we put

$$E(\alpha) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{1.5}$$

Let $c$ be an arbitrary and fixed real number, and consider the following equation with respect to $\rho$

$$|\rho|^{(\alpha+1)/\alpha} - \rho + c = 0. \tag{1.6}$$

It is easily checked that if $c < E(\alpha)$, then (1.6) has two real roots $\rho_1$, $\rho_2$ ($\rho_1 < \rho_2$), and they satisfy $\rho_1 < (\alpha+1)E(\alpha) < \rho_2$. It is also seen that if $c = E(\alpha)$, then (1.6) has the double root $\rho = (\alpha+1)E(\alpha)$. If $c > E(\alpha)$, then the left-hand side of (1.6) is positive for all $\rho \in \mathbb{R}$, and so (1.6) has no real root. For the case $c \le E(\alpha)$, the condition

$$\lim_{t\to\infty} t^{\alpha} Q(t) = c \tag{1.7}$$

plays an important role in the asymptotic analysis of nonoscillatory solutions of (1.1). It is known that if

$$-(2\alpha+1)E(\alpha) < \liminf_{t\to\infty} t^{\alpha}Q(t) \le \limsup_{t\to\infty} t^{\alpha}Q(t) < E(\alpha),$$

then (1.1) is nonoscillatory (see [2, Theorem 2.2.9]); and if

$$\liminf_{t\to\infty} t^{\alpha}Q(t) > E(\alpha),$$

then (1.1) is oscillatory (see [2, Theorem 2.3.2]). If $c = E(\alpha)$ in (1.7), then it is a critical case in the sense that $c = E(\alpha)$ is the borderline between nonoscillation and oscillation of (1.1). As an important work in the critical case, we refer the reader to [4], in which oscillation criteria and nonoscillation criteria are obtained for a perturbed Euler type half-linear differential equation in the critical case. See also the recent paper [1].

For the case $c < E(\alpha)$, the following theorem has been proved by Jaroš, Kusano and Tanigawa [5, Theorem 3.1]. As usual, the asterisk notation

$$\xi^{\gamma*} = |\xi|^{\gamma} \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \quad \gamma > 0,$$

is used.

**Theorem 1.1** ([5]). *Consider the equation* (1.1) *under the condition* (1.3). *Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* $\rho_1$, $\rho_2$ ($\rho_1 < \rho_2$) *be the real roots of* (1.6). *If* (1.7) *is satisfied, then* (1.1) *has a pair of solutions* $x_i(t)$ ($i = 1$ *and* 2) *such that*

$$\lim_{t\to\infty} t\frac{x_i'(t)}{x_i(t)} = \rho_i^{(1/\alpha)*} \quad (i = 1, 2). \tag{1.8}$$

In the paper [5], the results are stated in terms of regularly varying functions.

Now, put

$$\varepsilon(t) = t^\alpha Q(t) - c, \quad t \geq t_0, \tag{1.9}$$

where $Q(t)$ is given by (1.4). Then the condition

$$\int_{t_0}^{\infty} \frac{|\varepsilon(t)|}{t} dt < \infty \tag{1.10}$$

plays an important part. In fact, the author has recently proved the following theorem ([11, Theorem 1.3]).

**Theorem 1.2** ([11]). *Consider the equation* (1.1) *under the condition* (1.3). *Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* $\rho_1$, $\rho_2$ *be the real roots of* (1.6) *such that* $\rho_1 < \rho_2$ *and* $\rho_1 \neq 0$. *Suppose that* (1.7) *holds and define* $\varepsilon(t)$ *by* (1.9). *If* (1.10) *is satisfied, then* (1.1) *has a pair of solutions* $x_i(t)$ ($i = 1$ *and* 2) *such that*

$$\begin{cases} x_i(t) \sim t^{\lambda_i} & \text{as} \quad t \to \infty, \\ x_i'(t) \sim \lambda_i t^{\lambda_i - 1} & \text{as} \quad t \to \infty, \end{cases} \tag{1.11}$$

*where* $\lambda_i = \rho_i^{(1/\alpha)*}$ ($i = 1, 2$).

Here, the notation $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty}[f(t)/g(t)] = 1$.

Note that (1.11) implies (1.8) ($i = 1, 2$).

For the critical case $c = E(\alpha)$, the following theorem has been proved by Jaroš, Kusano and Tanigawa [5, Theorem 3.2]. The theorem requires the additional condition

$$\int_{t_0}^{\infty} \frac{1}{t} \left( \int_t^{\infty} \frac{|\varepsilon(s)|}{s} ds \right) dt < \infty. \tag{1.12}$$

**Theorem 1.3** ([5]). *Consider the equation* (1.1) *under the condition* (1.3). *Suppose that* (1.7) *with* $c = E(\alpha)$ *holds and define* $\varepsilon(t)$ *by* (1.9) ($c = E(\alpha)$). *If* (1.10) *and* (1.12) *are satisfied, then* (1.1) *has a solution* $x(t)$ *such that*

$$x(t) \sim t^{\alpha/(\alpha+1)} \quad \text{and} \quad x'(t) \sim \frac{\alpha}{\alpha+1} t^{-1/(\alpha+1)} \quad \text{as} \quad t \to \infty.$$

It is also known ([2, Section 1.4.2], [3]) that, for the case $c = E(\alpha)$ and $\varepsilon(t) \equiv 0$, i.e., $q(t) = \alpha E(\alpha)/t^{\alpha+1}$, the equation (1.1) has a pair of nonoscillatory solutions $x_i(t)$ ($i = 1, 2$) such that

$$x_1(t) = t^{\alpha/(\alpha+1)} \quad \text{and} \quad x_2(t) \sim t^{\alpha/(\alpha+1)} (\log t)^{2/(\alpha+1)} \quad (t \to \infty).$$

In the present paper, we restrict our attention to the critical case $c = E(\alpha)$ and make deeper discussions on the existence and asymptotic behavior of nonoscillatory solutions of (1.1). In what follows, we assume (1.7) with $c = E(\alpha)$, i.e.,

$$\lim_{t\to\infty} t^{\alpha} Q(t) = E(\alpha). \tag{1.13}$$

As in the above, we put

$$\varepsilon(t) = t^{\alpha} Q(t) - E(\alpha). \tag{1.14}$$

Then, for the case where (1.10) holds, the conditions

$$\int_{t_0}^{\infty} \frac{1}{s} \left( \int_{s}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 ds < \infty \tag{1.15}$$

and

$$\int_{t}^{\infty} \frac{1}{s} \left( \int_{s}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 ds = o\left( \int_{t}^{\infty} \frac{|\varepsilon(s)|}{s} ds \right) \quad \text{as} \quad t \to \infty \tag{1.16}$$

are also important. We have the following theorem.

**Theorem 1.4.** *Consider the equation* (1.1) *under the condition* (1.3). *Define* $Q(t)$ *by* (1.4). *Suppose that* (1.13) *holds and define* $\varepsilon(t)$ *by* (1.14). *If the conditions* (1.10), (1.15) *and* (1.16) *are satisfied, then* (1.1) *has a nonoscillatory solution* $x(t)$ *such that*

$$\lim_{t\to\infty} t \frac{x'(t)}{x(t)} = \frac{\alpha}{\alpha+1}. \tag{1.17}$$

For the case $c < E(\alpha)$ (i.e., the case where (1.6) has two distinct real roots $\rho_i$), if (1.7) holds, then (1.1) has nonoscillatory solutions $x_i(t)$ satisfying (1.8) (see Theorem 1.1). For the critical case $c = E(\alpha)$ (i.e., the case where (1.6) has the double root $\rho = (\alpha+1)E(\alpha)$), the condition (1.13) alone is not sufficient to guarantee the existence of a nonoscillatory solution of (1.1). For instance, the half-linear equation

$$(|x'|^{\alpha} \operatorname{sgn} x')' + \left( \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\nu}{t^{\alpha+1}(\log t)^2} \right) |x|^{\alpha} \operatorname{sgn} x = 0, \quad t \geq 1,$$

is oscillatory if $\nu > (\alpha+1)E(\alpha)/2$ and nonoscillatory if $\nu \leq (\alpha+1)E(\alpha)/2$ (see [4]). Theorem 1.4 shows that, under the additional conditions (1.10), (1.15) and (1.16), the equation (1.1) has a nonoscillatory solution $x(t)$ satisfying (1.17).

We can show that, for the case where (1.10) holds, the condition

$$(\log t) \int_{t}^{\infty} \frac{|\varepsilon(s)|}{s} ds \to 0 \quad \text{as} \quad t \to \infty \tag{1.18}$$

implies (1.15) and (1.16) (see (II) of Lemma 2.5 in the next section). Therefore, Theorem 1.4 produces the following corollary.

**Corollary 1.5.** *Consider the equation* (1.1) *under the condition* (1.3)*. Define* $Q(t)$ *by* (1.4)*. Suppose that* (1.13) *holds and define* $\varepsilon(t)$ *by* (1.14)*. If* (1.10) *and* (1.18) *hold, then* (1.1) *has a nonoscillatory solution* $x(t)$ *which satisfies* (1.17)*.*

The next theorem shows that a pair of nonoscillatory solutions can be obtained under the additional condition

$$\lim_{t\to\infty} (\log t)\varepsilon(t) = 0. \tag{1.19}$$

**Theorem 1.6.** *Consider the equation* (1.1) *under the condition* (1.3)*. Define* $Q(t)$ *by* (1.4)*. Suppose that* (1.13) *holds and define* $\varepsilon(t)$ *by* (1.14)*. If* (1.10) *and* (1.18) *and* (1.19) *are satisfied, then* (1.1) *has a pair of nonoscillatory solutions* $x_i(t)$ $(i = 1, 2)$ *such that*

$$\lim_{t\to\infty} (\log t) \left( t\frac{x_1'(t)}{x_1(t)} - \frac{\alpha}{\alpha+1} \right) = 0, \tag{1.20}$$

*and*

$$\lim_{t\to\infty} (\log t) \left( t\frac{x_2'(t)}{x_2(t)} - \frac{\alpha}{\alpha+1} \right) = \frac{2}{\alpha+1}. \tag{1.21}$$

In the next section we will prove that, under the condition (1.10), the condition (1.12) implies (1.18), that is, (1.12) is stronger than (1.18). For the case where (1.12) holds, at least one nonoscillatory solution is obtained in Theorem 1.3. The next theorem shows that another nonoscillatory solution is obtained under the same conditions.

**Theorem 1.7.** *Consider the equation* (1.1) *under the condition* (1.3)*. Define* $Q(t)$ *by* (1.4)*. Suppose that* (1.13) *holds and define* $\varepsilon(t)$ *by* (1.14)*. If* (1.10) *and* (1.12) *are satisfied, then* (1.1) *has a pair of nonoscillatory solutions* $x_i(t)$ $(i = 1, 2)$ *such that*

$$\begin{cases} x_1(t) \sim t^{\alpha/(\alpha+1)} & (t\to\infty), \\ x_1'(t) \sim \dfrac{\alpha}{\alpha+1} t^{-1/(\alpha+1)} & (t\to\infty), \end{cases} \tag{1.22}$$

*and*

$$\begin{cases} x_2(t) \sim t^{\alpha/(\alpha+1)} (\log t)^{2/(\alpha+1)} & (t\to\infty), \\ x_2'(t) \sim \dfrac{\alpha}{\alpha+1} t^{-1/(\alpha+1)} (\log t)^{2/(\alpha+1)} & (t\to\infty). \end{cases} \tag{1.23}$$

Throughout the paper the following fact plays an essential part. Let $x(t)$ be a nonoscillatory solution of (1.1). We suppose that $x(t) > 0$ for $t \geq T$ $(\geq t_0)$. Put

$$y(t) = \left(\frac{x'(t)}{x(t)}\right)^{\alpha *}, \quad t \geq T. \tag{1.24}$$

Then, $y(t)$ satisfies the generalized Riccati equation

$$y'(t) = -q(t) - \alpha |y(t)|^{(\alpha+1)/\alpha}, \quad t \geq T. \tag{1.25}$$

Conversely, if $y(t)$ is a solution of (1.25) on $[T,\infty)$, then

$$x(t) = \exp\left(\int_T^t y(s)^{(1/\alpha)*} ds\right), \quad t \geq T, \tag{1.26}$$

is a positive solution of (1.1) on $[T,\infty)$. The proof is immediate.

Now, suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that

$$\lim_{t\to\infty} t\frac{x'(t)}{x(t)} = \lambda \quad \text{for some} \quad \lambda \in \mathbb{R}. \tag{1.27}$$

The solution $x = x(t)$ in Theorem 1.4 and the solutions $x = x_i(t)$ $(i = 1, 2)$ in Theorems 1.6 and 1.7 satisfy (1.27) $(\lambda = \alpha/(\alpha+1))$. We suppose that $x(t) > 0$ for $t \geq T$ $(\geq t_0)$, and define the function $y(t)$ by (1.24). Then it is seen (for example, see [11, Proposition 1.2]) that (1.3) holds and $y(t)$ satisfies

$$y(t) = Q(t) + \alpha \int_t^\infty |y(s)|^{(\alpha+1)/\alpha} ds, \quad t \geq T, \tag{1.28}$$

where $Q(t)$ is defined by (1.4). This fact also plays a crucial part in the present paper.

In the next section we give a few number of preparatory lemmas. The solution $x(t)$ in Theorem 1.4 and the solutions $x_i(t)$ $(i = 1, 2)$ in Theorems 1.6 and 1.7 are obtained by solving certain integral equations similar to those used in [11]. The proofs of Theorem 1.4 and Theorems 1.6 and 1.7 in the case $i = 1$ are presented in Section 3. The proofs of Theorems 1.6 and 1.7 in the case $i = 2$ are given in Section 4. Examples illustrating the main results are provided in Section 5.

The present paper is related to regularly varying solutions of (1.1). A function $x(t)$ which satisfies (1.27) is a normalized regularly varying function of index $\lambda$. Recent results on regularly varying solutions of the half-linear equation (1.1) are found, e.g., in [5–9, 13, 14]. However, in the present paper, the theory of regularly varying functions is not used.

The papers [6, 8–14] deal with a more general equation

$$(p(t)|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \quad t \geq t_0, \tag{1.29}$$

where $p(t)$ is a positive continuous function on $[t_0,\infty)$. For the case

$$\int_{t_0}^\infty p(s)^{-1/\alpha} ds = \infty,$$

the change of variables $(t, x) \to (\tau, y)$ given by

$$\tau = \int_{t_0}^t p(s)^{-1/\alpha} ds, \quad y(\tau) = x(t)$$

transforms (1.29) into

$$(|\dot{y}|^{\alpha}\mathrm{sgn}\,\dot{y})^{\cdot} + p(t)^{1/\alpha}q(t)|y|^{\alpha}\mathrm{sgn}\,y = 0, \quad \tau \geq 0 \quad \left(\cdot = \frac{d}{d\tau}\right), \tag{1.30}$$

which is the same form as (1.1). Then we can apply the results for (1.1) to (1.30), and, after transforming back, we get the results for (1.29).

For the complementary case

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha} ds < \infty, \tag{1.31}$$

it is impossible to apply the results in the present paper to the equation (1.29). Nevertheless, it may be conjectured that analogous results for (1.29) with (1.31) can be established. For example, observe an analogy between [11] and [12]. The precise statements, however, are unclear at this stage.

## 2. PREPARATORY RESULTS

To prove our results, we prepare a few number of lemmas.

**Lemma 2.1.** *Let $\lambda \neq 0$ be fixed. Let $w$ and $\varepsilon$ be real numbers with $|\varepsilon| \leq |\lambda|^{\alpha}/4$. The function*

$$\begin{aligned} F(w,\varepsilon) = |w + \lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} \\ - \frac{\alpha+1}{\alpha}(\lambda^{\alpha *} + \varepsilon)^{(1/\alpha)*} w \end{aligned} \tag{2.1}$$

*satisfies*

$$0 \leq F(w,\varepsilon) \leq K(\alpha)|\lambda|^{-\alpha+1}w^2 \quad \left(|w| \leq \frac{|\lambda|^{\alpha}}{4},\ |\varepsilon| \leq \frac{|\lambda|^{\alpha}}{4}\right),$$

*where*

$$K(\alpha) = \begin{cases} \dfrac{\alpha+1}{2\alpha^2}\left(\dfrac{3}{2}\right)^{(-\alpha+1)/\alpha} & (0 < \alpha \leq 1), \\ \dfrac{\alpha+1}{2\alpha^2}\left(\dfrac{1}{2}\right)^{(-\alpha+1)/\alpha} & (\alpha > 1). \end{cases}$$

Note that the function $F(w,\varepsilon)$ defined by (2.1) arises naturally in [5]. For a brief proof of Lemma 2.1, see Naito [10, Lemma 2.4].

**Lemma 2.2.** *Let $\lambda > 0$, $v_0 > 0$ and $T > 1$ be fixed. Let $t$, $v$ and $\varepsilon$ be real numbers with $t \geq T$ and $|\varepsilon| \leq \lambda^\alpha/4$. If*

$$v_0 + 2\lambda^\alpha \leq \frac{1}{4}\lambda^\alpha \log T, \tag{2.2}$$

*then the function*

$$\begin{aligned} G(t,v,\varepsilon) = &\left|\frac{v+2\lambda^\alpha}{\log t} + \lambda^\alpha + \varepsilon\right|^{(\alpha+1)/\alpha} - (\lambda^\alpha+\varepsilon)^{(\alpha+1)/\alpha} \\ &- \frac{\alpha+1}{\alpha}(\lambda^\alpha+\varepsilon)^{1/\alpha}\frac{v+2\lambda^\alpha}{\log t} \\ &- \frac{\alpha+1}{2\alpha^2}(\lambda^\alpha+\varepsilon)^{(1/\alpha)-1}\left(\frac{v+2\lambda^\alpha}{\log t}\right)^2 \end{aligned} \tag{2.3}$$

*satisfies*

$$|G(t,v,\varepsilon)| \leq L(\alpha)\lambda^{-2\alpha+1}\left|\frac{v+2\lambda^\alpha}{\log t}\right|^3 \quad \left(t \geq T,\ |v| \leq v_0,\ |\varepsilon| \leq \frac{\lambda^\alpha}{4}\right),$$

*where*

$$L(\alpha) = \begin{cases} \dfrac{(\alpha+1)|\alpha-1|}{6\alpha^3}\left(\dfrac{3}{2}\right)^{(1/\alpha)-2} & (0 < \alpha \leq 1/2), \\ \dfrac{(\alpha+1)|\alpha-1|}{6\alpha^3}\left(\dfrac{1}{2}\right)^{(1/\alpha)-2} & (\alpha > 1/2). \end{cases}$$

*Proof.* By Taylor's theorem, there exists $\theta \in (0,1)$ such that

$$(x+a)^p = a^p + pa^{p-1}x + \frac{p(p-1)}{2}a^{p-2}x^2 + \frac{p(p-1)(p-2)}{6}(\theta x + a)^{p-3}x^3,$$

where $a > 0$, $x > -a$ and $p \in \mathbb{R}$. Let $t \geq T$, $|v| \leq v_0$ and $|\varepsilon| \leq \lambda^\alpha/4$. Then we apply the above equality to the case $a = \lambda^\alpha + \varepsilon$, $x = (v + 2\lambda^\alpha)/\log t$ and $p = (\alpha+1)/\alpha$. To this aim, note first that $a \geq 3\lambda^\alpha/4 > 0$. By (2.2), we have

$$v_0 \leq \frac{1}{4}\lambda^\alpha \log T, \tag{2.4}$$

and so $x + a \geq (v/\log t) + (3\lambda^\alpha/4) \geq (-v_0/\log T) + (3\lambda^\alpha/4) \geq \lambda^\alpha/2 > 0$. Then it is seen that

$$|G(t,v,\varepsilon)| = \frac{(\alpha+1)|\alpha-1|}{6\alpha^3}\left|\theta\frac{v+2\lambda^\alpha}{\log t} + \lambda^\alpha + \varepsilon\right|^{(1/\alpha)-2}\left|\frac{v+2\lambda^\alpha}{\log t}\right|^3.$$

Since $-v_0 \leq v + 2\lambda^\alpha \leq v_0 + 2\lambda^\alpha$ and $0 < \theta < 1$, it follows from (2.2) and (2.4) that

$$\frac{1}{2}\lambda^\alpha \leq \theta\frac{v+2\lambda^\alpha}{\log t} + \lambda^\alpha + \varepsilon \leq \frac{3}{2}\lambda^\alpha.$$

Then the assertion of Lemma 2.2 is immediate. □

**Lemma 2.3.** *Let* $\lambda \neq 0$. *Then*

$$\left| |\lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda|^{\alpha+1} \right| \leq 2\frac{\alpha+1}{\alpha}|\lambda||\varepsilon|, \tag{2.5}$$

$$\left| (\lambda^{\alpha *} + \varepsilon)^{(1/\alpha)*} - \lambda \right| \leq \frac{2}{\alpha}|\lambda|^{-\alpha+1}|\varepsilon|, \quad \textit{and} \tag{2.6}$$

$$\left| |\lambda^{\alpha *} + \varepsilon|^{(1/\alpha)-1} - |\lambda|^{1-\alpha} \right| \leq 2\frac{|\alpha-1|}{\alpha}|\lambda|^{-2\alpha+1}|\varepsilon| \tag{2.7}$$

*for all sufficiently small* $|\varepsilon|$.

*Proof.* Since

$$\lim_{\varepsilon\to 0} \frac{|\lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda|^{\alpha+1}}{\varepsilon} = \frac{\alpha+1}{\alpha}\lambda,$$

$$\lim_{\varepsilon\to 0} \frac{(\lambda^{\alpha *} + \varepsilon)^{(1/\alpha)*} - \lambda}{\varepsilon} = \frac{1}{\alpha}|\lambda|^{-\alpha+1} \quad (\lambda \neq 0), \quad \text{and}$$

$$\lim_{\varepsilon\to 0} \frac{|\lambda^{\alpha *} + \varepsilon|^{(1/\alpha)-1} - |\lambda|^{1-\alpha}}{\varepsilon} = \frac{-\alpha+1}{\alpha}|\lambda|^{-2\alpha+1}\operatorname{sgn}\lambda \quad (\lambda \neq 0),$$

the assertions (2.5)–(2.7) are clear. □

**Lemma 2.4.** *Let* $\lambda > 0$. *Then*

$$\left| (\lambda^{\alpha} + \delta)^{(1/\alpha)*} - \lambda - \frac{\lambda^{-\alpha+1}}{\alpha}\delta \right| \leq \frac{|\alpha-1|}{\alpha^2}\lambda^{-2\alpha+1}\delta^2 \tag{2.8}$$

*for all sufficiently small* $|\delta|$.

*Proof.* Since

$$\lim_{\delta\to 0} \frac{(\lambda^{\alpha} + \delta)^{(1/\alpha)*} - \lambda - (\lambda^{-\alpha+1}/\alpha)\delta}{\delta^2} = \frac{1}{2}\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\lambda^{-2\alpha+1},$$

the assertion (2.8) is obvious. □

**Lemma 2.5.** *Let* $\varepsilon(t)$ *be a continuous function on* $[t_0, \infty)$, $t_0 > 1$, *and suppose that* (1.10) *holds. Then:*

(I) *the condition* (1.12) *implies* (1.18),
(II) *the condition* (1.18) *implies* (1.15) *and* (1.16).

*Proof.* (I) Let $\tau > t_0$. An integration by parts gives

$$\begin{aligned}
&\int_{t_0}^{\tau} \frac{1}{t}\left(\int_t^{\infty} \frac{|\varepsilon(s)|}{s} ds\right) dt \\
&= (\log\tau)\int_{\tau}^{\infty} \frac{|\varepsilon(s)|}{s} ds - (\log t_0)\int_{t_0}^{\infty} \frac{|\varepsilon(s)|}{s} ds + \int_{t_0}^{\tau} (\log s)\frac{|\varepsilon(s)|}{s} ds \\
&\geq -(\log t_0)\int_{t_0}^{\infty} \frac{|\varepsilon(s)|}{s} ds + \int_{t_0}^{\tau} (\log s)\frac{|\varepsilon(s)|}{s} ds.
\end{aligned}$$

Therefore, by (1.12), we find that

$$\int_{t_0}^{\infty} (\log s) \frac{|\varepsilon(s)|}{s} ds < \infty, \tag{2.9}$$

and hence

$$\lim_{\tau \to \infty} (\log \tau) \int_{\tau}^{\infty} \frac{|\varepsilon(s)|}{s} ds \quad \text{exists and is a nonnegative finite value.}$$

Put

$$\lim_{\tau \to \infty} (\log \tau) \int_{\tau}^{\infty} \frac{|\varepsilon(s)|}{s} ds = \ell,$$

where $0 \le \ell < \infty$. Assume that $0 < \ell < \infty$. Then there is $T > t_0$ such that

$$\log \tau \ge \frac{\ell}{2} \left( \int_{\tau}^{\infty} \frac{|\varepsilon(s)|}{s} ds \right)^{-1} \quad \text{for} \quad \tau \ge T.$$

Consequently we have

$$\begin{aligned}
\int_{T}^{\infty} (\log s) \frac{|\varepsilon(s)|}{s} ds &= \lim_{\tau \to \infty} \int_{T}^{\tau} (\log s) \frac{|\varepsilon(s)|}{s} ds \\
&\ge \frac{\ell}{2} \lim_{\tau \to \infty} \int_{T}^{\tau} \left( \int_{s}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^{-1} \frac{|\varepsilon(s)|}{s} ds \\
&= -\frac{\ell}{2} \lim_{\tau \to \infty} \left[ \log \left( \int_{\tau}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right) - \log \left( \int_{T}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right) \right] \\
&= \infty,
\end{aligned}$$

which is a contradiction to (2.9). Therefore we have $\ell = 0$. This implies (1.18).

(II) Let $t_0 \le t < \tau$. An integration by parts gives

$$\begin{aligned}
&\int_{t}^{\tau} \frac{1}{s} \left( \int_{s}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 ds \\
&= (\log \tau) \left( \int_{\tau}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 - (\log t) \left( \int_{t}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 \\
&\quad + 2 \int_{t}^{\tau} (\log s) \left( \int_{s}^{\infty} \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right) \frac{|\varepsilon(s)|}{s} ds.
\end{aligned}$$

By (1.10) and (1.18), the first term of the right-hand side of the above equality tends to 0 as $\tau \to \infty$, and the last term is convergent as $\tau \to \infty$. Therefore, (1.15) holds and

$$\begin{aligned}
&\int_t^\infty \frac{1}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)^2 ds \\
&= -(\log t)\left(\int_t^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)^2 + 2\int_t^\infty (\log s)\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)\frac{|\varepsilon(s)|}{s}ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_t^\infty \frac{1}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)^2 ds &\le 2\int_t^\infty (\log s)\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)\frac{|\varepsilon(s)|}{s}ds \\
&\le 2M(t)\int_t^\infty \frac{|\varepsilon(s)|}{s}ds,
\end{aligned}$$

where

$$M(t) = \sup_{s\ge t}\left\{(\log s)\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right\}.$$

Since $M(t) \to 0$ as $t \to \infty$, this implies (1.16). □

**Lemma 2.6.** *Let $p$ be a constant such that $p > 1$. Suppose that $f(t)$ is a continuous function on $[a, b]$, $a > 1$. Then we have*

$$\int_a^b \frac{1}{s(\log s)^p}\left[\int_a^s \frac{1}{\sigma}|f(\sigma)|d\sigma\right]^p ds \le \left(\frac{p}{p-1}\right)^p \int_a^b \frac{1}{s}|f(s)|^p ds. \tag{2.10}$$

*Proof.* For the left-hand side of (2.10) we make the change of variable twice, first $\log s = u$, and next $\sigma = e^v$. Then we have

$$\int_a^b \frac{1}{s(\log s)^p}\left[\int_a^s \frac{1}{\sigma}|f(\sigma)|d\sigma\right]^p ds = \int_{\log a}^{\log b} \frac{1}{u^p}\left[\int_{\log a}^{u} |f(e^v)|dv\right]^p du.$$

An application of the Hardy-type inequality which is proved in [11, Theorem 2.2] yields

$$\int_{\log a}^{\log b} \frac{1}{u^p}\left[\int_{\log a}^{u} |f(e^v)|dv\right]^p du \le \left(\frac{p}{p-1}\right)^p \int_{\log a}^{\log b} |f(e^u)|^p du.$$

By the change of variable $e^u = s$, the right-hand of the above inequality is equal to the right-hand of (2.10). This proves the inequality (2.10). □

## 3. PROOFS OF THE RESULTS

Now, let us return to the half-linear equation (1.1). The number

$$\lambda = \frac{\alpha}{\alpha+1} \tag{3.1}$$

is a unique real root of the equation

$$|\lambda|^{\alpha+1} - \lambda^{\alpha *} + E(\alpha) = 0,$$

where $E(\alpha)$ is defined by (1.5). In what follows, the letter $\lambda$ is the number which is given by (3.1). Further, in what follows, we suppose that (1.13) holds and define $\varepsilon(t)$ by (1.14). Since $\varepsilon(t) \to 0$ $(t \to \infty)$, there is $T > t_0$ such that $|\varepsilon(t)| \le \lambda^\alpha/4$ for $t \ge T$.

Let $x(t)$ be a nonoscillatory solution of (1.1) which satisfies the condition (1.17). We may suppose that $x(t) > 0$ for $t \ge T$, and define the function $y(t)$ by (1.24). As mentioned in Section 1, the function $y(t)$ satisfies (1.28).

Put

$$w(t) = t^\alpha y(t) - \lambda^\alpha - \varepsilon(t), \quad t \ge T.$$

Noting that $\lambda^\alpha = \lambda^{\alpha+1} + E(\alpha)$ and using the formula (1.28), we have

$$w(t) = -\lambda^{\alpha+1} + \alpha t^\alpha \int_t^\infty \frac{|w(s) + \lambda^\alpha + \varepsilon(s)|^{(\alpha+1)/\alpha}}{s^{\alpha+1}} ds, \quad t \ge T. \tag{3.2}$$

Then it is easy to see that

$$w'(t) = \frac{\alpha}{t} w(t) + \frac{\alpha\lambda^{\alpha+1}}{t} - \frac{\alpha}{t}|w(t) + \lambda^\alpha + \varepsilon(t)|^{(\alpha+1)/\alpha}, \quad t \ge T. \tag{3.3}$$

We have $\lambda^\alpha + \varepsilon(t) \ge 3\lambda^\alpha/4 > 0$ for $t \ge T$. The above equality can be written as

$$\begin{aligned} w'(t) = &-\frac{\alpha}{t}\left\{(\lambda^\alpha + \varepsilon(t))^{(\alpha+1)/\alpha} - \lambda^{\alpha+1}\right\} \\ &- \frac{\alpha+1}{t}\left\{(\lambda^\alpha + \varepsilon(t))^{1/\alpha} - \lambda\right\} w(t) \\ &- \frac{\alpha}{t} F(w(t), \varepsilon(t)), \quad t \ge T, \end{aligned}$$

where $F(w, \varepsilon)$ is defined by (2.1) with $\lambda = \alpha/(\alpha+1)$.

For simplicity of notation, we put

$$f_1(t) = (\lambda^\alpha + \varepsilon(t))^{(\alpha+1)/\alpha} - \lambda^{\alpha+1}, \quad f_2(t) = (\lambda^\alpha + \varepsilon(t))^{1/\alpha} - \lambda, \tag{3.4}$$

and so

$$w'(t) = -\frac{\alpha}{t} f_1(t) - \frac{\alpha+1}{t} f_2(t) w(t) - \frac{\alpha}{t} F(w(t), \varepsilon(t)), \quad t \ge T. \tag{3.5}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, it follows from (2.5) and (2.6) in Lemma 2.3 that

$$|f_1(t)| \le 2\frac{\alpha+1}{\alpha}\lambda|\varepsilon(t)|, \quad |f_2(t)| \le \frac{2}{\alpha}\lambda^{-\alpha+1}|\varepsilon(t)| \tag{3.6}$$

for all large $t$. Without loss of generality we assume that (3.6) holds for $t \ge T$.

Under the above preparation, we prove Theorem 1.4.

*Proof of Theorem 1.4.* Let $\lambda > 0$ be the number which is defined by (3.1). We take $T > t_0$ sufficiently large so that $|\varepsilon(t)| \leq \lambda^\alpha/4$ for $t \geq T$. Define the functions $f_1(t)$ and $f_2(t)$ by (3.4). We may suppose that (3.6) holds for $t \geq T$. Put

$$\eta(t) = \int_t^\infty \frac{|\varepsilon(s)|}{s} ds, \quad t \geq t_0. \tag{3.7}$$

By (1.10), the function $\eta(t)$ is well-defined. Since $\eta(t) \to 0$ ($t \to \infty$), we can suppose that

$$4(\alpha+1)\lambda\eta(t) \leq \lambda^\alpha/4, \quad t \geq T,$$

and

$$\frac{4(\alpha+1)}{\alpha}\lambda^{-\alpha+2}\eta(t) \leq \lambda, \quad t \geq T. \tag{3.8}$$

The condition (1.16) is rewritten as

$$\int_t^\infty \frac{\eta(s)^2}{s} ds = o\left(\eta(t)\right) \quad \text{as } \ t \to \infty.$$

Therefore we can suppose that

$$\int_t^\infty \frac{\eta(s)^2}{s} ds \leq \frac{(\alpha+1)\lambda}{16\alpha(\alpha+1)^2 K(\alpha)\lambda^{-\alpha+3}}\eta(t) \quad \text{for } \ t \geq T, \tag{3.9}$$

where $K(\alpha)$ is the positive constant appearing in Lemma 2.1.

Denote by $W$ the set of all functions $w \in C[T,\infty)$ such that

$$|w(t)| \leq 4(\alpha+1)\lambda\eta(t), \quad t \geq T. \tag{3.10}$$

Moreover, keeping (3.5) in mind, we define the operator $\mathcal{F} : W \to C[T,\infty)$ by

$$(\mathcal{F}w)(t) = \alpha \int_t^\infty \frac{f_1(s)}{s} ds + (\alpha+1) \int_t^\infty \frac{f_2(s)}{s} w(s) ds$$
$$+ \alpha \int_t^\infty \frac{1}{s} F(w(s), \varepsilon(s)) ds, \quad t \geq T.$$

Here, $F(w,\varepsilon)$ is given by (2.1) with $\lambda = \alpha/(\alpha+1)$. As is easily verified, the set $W$ is a nonempty closed convex subset of the Fréchet space $C[T,\infty)$ of all continuous functions on $[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Note that if $w \in W$, then $|w(t)| \leq \lambda^\alpha/4$ for $t \geq T$, and so, by Lemma 2.1,

$$0 \leq F(w(t), \varepsilon(t)) \leq K(\alpha)\lambda^{-\alpha+1} w(t)^2, \quad t \geq T. \tag{3.11}$$

Then it can be easily checked that $\mathcal{F}w$ is well-defined and continuous on $[T,\infty)$ for $w \in W$.

Let $w \in W$. Then, by (3.6), (3.10) and (3.11), we have

$$|(\mathcal{F}w)(t)| \le 2(\alpha+1)\lambda \int_t^\infty \frac{|\varepsilon(s)|}{s}ds + \frac{8(\alpha+1)^2}{\alpha}\lambda^{-\alpha+2}\int_t^\infty \frac{|\varepsilon(s)|}{s}\eta(s)ds$$
$$+ 16\alpha(\alpha+1)^2K(\alpha)\lambda^{-\alpha+3}\int_t^\infty \frac{\eta(s)^2}{s}ds, \quad t \ge T.$$

Note that the first and the second integrals in the right-hand side of the above inequality are equal to $\eta(t)$ and $(1/2)\eta(t)^2$, respectively. Therefore, using (3.8) and (3.9), we obtain

$$\begin{aligned}|(\mathcal{F}w)(t)| &\le 2(\alpha+1)\lambda\eta(t) + (\alpha+1)\lambda\eta(t) + (\alpha+1)\lambda\eta(t)\\ &= 4(\alpha+1)\lambda\eta(t), \quad t \ge T.\end{aligned}$$

This shows that

(i) $\mathcal{F}$ maps $W$ into $W$.

Moreover it can be checked that:

(ii) $\mathcal{F}$ is continuous on $W$,
(iii) $\mathcal{F}W$ is uniformly bounded and equicontinuous at every point of $[T,\infty)$.

The Schauder–Tychonoff fixed point theorem implies that $\mathcal{F}$ has a fixed element $w \in W$: $w(t) = (\mathcal{F}w)(t)$, $t \ge T$.

It is clear the above fixed element $w(t)$ satisfies (3.3) and (3.10). Since $\eta(t) \to 0$ $(t \to \infty)$, it follows from (3.10) that $w(t) \to 0$ as $t \to \infty$. In addition, it can be shown without difficulty that $w(t)$ satisfies (3.2) for $t \ge T$. Put

$$y(t) = \frac{w(t) + \lambda^\alpha + \varepsilon(t)}{t^\alpha}, \quad t \ge T.$$

We find that $y(t)$ satisfies (1.28), and hence it satisfies (1.25). Therefore, the function $x(t)$ which is defined by (1.26) is a positive solution of (1.1) on $[T,\infty)$. Furthermore, we have

$$t\frac{x'(t)}{x(t)} = (w(t) + \lambda^\alpha + \varepsilon(t))^{(1/\alpha)*}, \quad t \ge T. \tag{3.12}$$

Noting that $\varepsilon(t) \to 0$ and $w(t) \to 0$ $(t \to \infty)$ and using (2.6) in Lemma 2.3, we get

$$\left|(w(t) + \lambda^\alpha + \varepsilon(t))^{(1/\alpha)*} - \lambda\right| \le \frac{2}{\alpha}\lambda^{-\alpha+1}\{|w(t)| + |\varepsilon(t)|\} \tag{3.13}$$

for all large $t$. The inequality (3.13) is not used at the present stage, but it is used for the proofs of Theorem 1.6 ($i = 1$) and Theorem 1.7 ($i = 1$). It follows from (3.12) that

$$\lim_{t\to\infty} t\frac{x'(t)}{x(t)} = \lambda. \tag{3.14}$$

Since $\lambda = \alpha/(\alpha+1)$, this implies (1.17). The proof of Theorem 1.4 is complete. □

In what follows, we will show that if the conditions (1.10), (1.18) and (1.19) hold, then the solution $x(t)$ which is obtained in the proof of Theorem 1.4 satisfies the asymptotic condition of the form (1.20). Similarly, if the conditions (1.10) and (1.12) hold, then the solution $x(t)$ which is obtained in the proof of Theorem 1.4 satisfies the asymptotic condition of the form (1.22). In other words, we prove Theorem 1.6 in the case $i = 1$ and Theorem 1.7 in the case $i = 1$ as a continuation of the proof of Theorem 1.4.

*Proof of Theorem 1.6 in the case $i = 1$.* The proof is done as a continuation of the proof of Theorem 1.4. Let $x(t)$ be the solution of (1.1) which is obtained in the proof of Theorem 1.4. We have (3.12) and (3.13). The function $w(t)$ in (3.12) and (3.13) is estimated as in (3.10). Here, $\lambda$ and $\eta(t)$ are given by (3.1) and (3.7), respectively. Then, since

$$t\frac{x'(t)}{x(t)} - \lambda = (w(t) + \lambda^\alpha + \varepsilon(t))^{(1/\alpha)*} - \lambda, \quad t \geq T,$$

we have

$$(\log t)\left|t\frac{x'(t)}{x(t)} - \lambda\right| \leq \frac{2}{\alpha}\lambda^{-\alpha+1}\{(\log t)|w(t)| + (\log t)|\varepsilon(t)|\} \tag{3.15}$$

for all large $t$. By (3.7) and (3.10) we have

$$(\log t)|w(t)| \leq 4(\alpha+1)\lambda(\log t)\int_t^\infty \frac{|\varepsilon(s)|}{s}ds, \quad t \geq T,$$

and so it follows from (1.18) that $(\log t)|w(t)| \to 0$ $(t \to \infty)$. Then, by this fact and (1.19) and (3.15), we obtain (1.20) with $x_1(t) = x(t)$. The proof of Theorem 1.6 in the case $i = 1$ is complete. □

*Proof of Theorem 1.7 in the case $i = 1$.* Let $x(t)$ be the solution of (1.1) which is obtained in the proof of Theorem 1.4. We have (3.10) and (3.12)–(3.14). Since

$$\frac{x'(t)}{x(t)} = \frac{\lambda}{t} + \frac{(w(t) + \lambda^\alpha + \varepsilon(t))^{(1/\alpha)*} - \lambda}{t}, \quad t \geq T,$$

it is easily seen that

$$x(t) = \frac{x(T)}{T^\lambda}\exp\left(\int_T^t \frac{(w(s) + \lambda^\alpha + \varepsilon(s))^{(1/\alpha)*} - \lambda}{s}ds\right)t^\lambda, \quad t \geq T. \tag{3.16}$$

The condition (1.12) implies

$$\int_T^\infty \frac{\eta(t)}{t}dt < \infty,$$

and so (3.10) gives

$$\int_T^\infty \frac{|w(t)|}{t}dt < \infty. \tag{3.17}$$

By (1.10), (3.13) and (3.17), the solution $x(t)$ expressed as (3.16) can be written in the form

$$x(t) = c_0(t)t^{\lambda} \quad \text{with} \quad c_0(t) \to c_0 \in (0,\infty) \ \text{ as } \ t \to \infty, \tag{3.18}$$

and hence, by (3.14), the derivative $x'(t)$ is presented as

$$x'(t) = c_0(t)t\frac{x'(t)}{x(t)}t^{\lambda-1} = c_1(t)t^{\lambda-1} \quad \text{with} \quad c_1(t) \to c_0\lambda \ \text{ as } \ t \to \infty. \tag{3.19}$$

In general, if $x(t)$ is a solution of the half-linear equation (1.1) and if $c$ is a constant, then $cx(t)$ is also a solution of (1.1). Therefore, without loss of generality, we may suppose that $c_0 = 1$ in (3.18) and (3.19). Since $\lambda = \alpha/(\alpha+1)$, this shows (1.22) with $x_1(t) = x(t)$. The proof of Theorem 1.7 in the case $i = 1$ is complete. □

## 4. PROOFS OF THE RESULTS (CONTINUED)

Next, let us prove Theorem 1.6 in the case $i = 2$ and Theorem 1.7 in the case $i = 2$. As before, the letter $\lambda$ is the number defined by (3.1). We suppose that (1.13) holds and define $\varepsilon(t)$ by (1.14). Since $\varepsilon(t) \to 0$ $(t \to \infty)$, there is $T > t_0$ such that $|\varepsilon(t)| \le \lambda^{\alpha}/4$ for $t \ge T$. Further, we suppose that (1.10) and (1.18) hold. If (1.10) and (1.12) are satisfied, then (1.18) holds (see (I) of Lemma 2.5).

Let $x(t)$ be a nonoscillatory solution of (1.1) which satisfies the condition (1.17). We suppose that $x(t) > 0$ for $t \ge T$, and define the function $y(t)$ by (1.24). The function $y(t)$ satisfies (1.28).

Put

$$v(t) = (\log t)\big[t^{\alpha}y(t) - \lambda^{\alpha} - \varepsilon(t)\big] - 2\lambda^{\alpha}, \quad t \ge T.$$

Noting that $\lambda^{\alpha} = \lambda^{\alpha+1} + E(\alpha)$ and using the formula (1.28), we obtain

$$\begin{aligned} v(t) &= -2\lambda^{\alpha} - \lambda^{\alpha+1}\log t \\ &\quad + \alpha t^{\alpha}\log t \int_t^{\infty} \frac{1}{s^{\alpha+1}} \left| \frac{v(s)+2\lambda^{\alpha}}{\log s} + \lambda^{\alpha} + \varepsilon(s) \right|^{(\alpha+1)/\alpha} ds \end{aligned} \tag{4.1}$$

for $t \ge T$. By differentiation of (4.1) we get

$$\begin{aligned} v'(t) &= \left(\frac{\alpha}{t} + \frac{1}{t\log t}\right) v(t) + 2\alpha\lambda^{\alpha}\frac{1}{t} + 2\lambda^{\alpha}\frac{1}{t\log t} \\ &\quad + \alpha\lambda^{\alpha+1}\frac{\log t}{t} - \alpha\frac{\log t}{t}\left| \frac{v(t)+2\lambda^{\alpha}}{\log t} + \lambda^{\alpha} + \varepsilon(t) \right|^{(\alpha+1)/\alpha} \end{aligned} \tag{4.2}$$

for $t \ge T$. Therefore it can be shown that

$$\begin{aligned} ((\log t)v(t))' &= \left(\alpha\frac{\log t}{t} + \frac{2}{t}\right) v(t) + 2\alpha\lambda^{\alpha}\frac{\log t}{t} + 2\lambda^{\alpha}\frac{1}{t} \\ &\quad + \alpha\lambda^{\alpha+1}\frac{(\log t)^2}{t} - \alpha\frac{(\log t)^2}{t}\left| \frac{v(t)+2\lambda^{\alpha}}{\log t} + \lambda^{\alpha} + \varepsilon(t) \right|^{(\alpha+1)/\alpha} \end{aligned}$$

for $t \ge T$.

Recall that $\lambda = \alpha/(\alpha+1)$. Then the above equality can be written as

$$\begin{aligned}((\log t)v(t))' &= \frac{2}{t}v(t) - \frac{\alpha+1}{2\alpha}\lambda^{1-\alpha}\frac{1}{t}(v(t)+2\lambda^{\alpha})^2 + 2\lambda^{\alpha}\frac{1}{t}\\ &- \alpha\frac{(\log t)^2}{t}\{(\lambda^{\alpha}+\varepsilon(t))^{(\alpha+1)/\alpha} - \lambda^{\alpha+1}\}\\ &- (\alpha+1)\frac{\log t}{t}\{(\lambda^{\alpha}+\varepsilon(t))^{1/\alpha} - \lambda\}(v(t)+2\lambda^{\alpha})\\ &- \frac{\alpha+1}{2\alpha}\frac{1}{t}\{(\lambda^{\alpha}+\varepsilon(t))^{(1/\alpha)-1} - \lambda^{1-\alpha}\}(v(t)+2\lambda^{\alpha})^2\\ &- \alpha\frac{(\log t)^2}{t}G(t, v(t), \varepsilon(t)),\end{aligned}$$

where $G(t, v, \varepsilon)$ is given by (2.3) with $\lambda = \alpha/(\alpha+1)$. For simplicity of notation, we put

$$\begin{cases} f_1(t) = (\lambda^{\alpha}+\varepsilon(t))^{(\alpha+1)/\alpha} - \lambda^{\alpha+1}, \quad f_2(t) = (\lambda^{\alpha}+\varepsilon(t))^{1/\alpha} - \lambda,\\ f_3(t) = (\lambda^{\alpha}+\varepsilon(t))^{(1/\alpha)-1} - \lambda^{1-\alpha}.\end{cases} \tag{4.3}$$

Then it is easily seen that

$$\begin{aligned}((\log t)v(t))' = &-\frac{\lambda^{-\alpha}}{2}\frac{1}{t}v(t)^2 - \alpha\frac{(\log t)^2}{t}f_1(t)\\ &- (\alpha+1)\frac{\log t}{t}f_2(t)(v(t)+2\lambda^{\alpha})\\ &- \frac{\alpha+1}{2\alpha}\frac{1}{t}f_3(t)(v(t)+2\lambda^{\alpha})^2 - \alpha\frac{(\log t)^2}{t}G(t, v(t), \varepsilon(t)).\end{aligned} \tag{4.4}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, it follows from Lemma 2.3 that

$$\begin{cases} |f_1(t)| \le 2\dfrac{\alpha+1}{\alpha}\lambda|\varepsilon(t)|, \quad |f_2(t)| \le \dfrac{2}{\alpha}\lambda^{-\alpha+1}|\varepsilon(t)|,\\ |f_3(t)| \le 2\dfrac{|\alpha-1|}{\alpha}\lambda^{-2\alpha+1}|\varepsilon(t)|\end{cases} \tag{4.5}$$

for all large $t$.

We are now ready to prove Theorem 1.6 $(i = 2)$ and Theorem 1.7 $(i = 2)$.

*Proof of Theorem 1.6 in the case* $i = 2$. Let $\lambda > 0$ be the number which is defined by (3.1). We take $T > \max\{t_0, e\}$ sufficiently large so that $|\varepsilon(t)| \le \lambda^{\alpha}/4$ for $t \ge T$ ($e$ is Napier's constant). Define the functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ by (4.3). We may suppose that (4.5) holds for $t \ge T$. Now, put

$$\varphi(t) = \frac{\lambda^{\alpha}}{4} + \frac{1}{(\log t)^{1/2}} + \frac{9\alpha}{\log t}\int_{t_0}^{t}\frac{\log s}{s}\left(\int_{s}^{\infty}\frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)ds, \quad t \ge t_0.$$

From the condition (1.18), we see that $\lim \varphi(t) = \lambda^\alpha/4$ as $t \to \infty$. Put

$$v_0 = \sup_{t \ge t_0} \varphi(t),$$

which is a positive finite number. Taking the number $T$ sufficiently large, we may suppose that (2.2) in Lemma 2.2 holds. Define the function $\psi(t;T)$ by

$$\psi(t;T) = \frac{\lambda^\alpha}{4}\frac{\log T}{\log t} + \frac{1}{(\log t)^{1/2}} + \frac{9\alpha}{\log t}\int_T^t \frac{\log s}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)ds, \quad t \ge T.$$

It is clear that

$$0 < \psi(t;T) \le \varphi(t) \le v_0 \quad \text{for} \quad t \ge T.$$

Denote by $V$ the set of all functions $v \in C[T,\infty)$ such that

$$|v(t)| \le \psi(t;T), \quad t \ge T. \tag{4.6}$$

Keeping in mind the above preparatory calculation, we define the operator $\mathcal{F} : V \to C[T,\infty)$ by

$$\begin{aligned}(\mathcal{F}v)(t) = &-\frac{\lambda^{-\alpha}}{2}\frac{1}{\log t}\int_T^t \frac{1}{s}v(s)^2 ds - \frac{\alpha}{\log t}\int_T^t \frac{(\log s)^2}{s}f_1(s)ds \\ &- \frac{\alpha+1}{\log t}\int_T^t \frac{\log s}{s}f_2(s)(v(s)+2\lambda^\alpha)ds \\ &- \frac{\alpha+1}{2\alpha}\frac{1}{\log t}\int_T^t \frac{1}{s}f_3(s)(v(s)+2\lambda^\alpha)^2 ds \\ &- \frac{\alpha}{\log t}\int_T^t \frac{(\log s)^2}{s}G(s,v(s),\varepsilon(s))ds, \quad t \ge T.\end{aligned} \tag{4.7}$$

Here, $G(t,v,\varepsilon)$ is given by (2.3) with $\lambda = \alpha/(\alpha+1)$. The set $V$ is a nonempty closed convex subset of the Fréchet space $C[T,\infty)$ of all continuous functions on $[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$.

Denote the $i$-th term of the right-hand side of (4.7) by $R_i(t)$, $i = 1,2,\dots,5$. Let $v \in V$. Since $(A+B+C)^2 \le 4A^2+4B^2+4C^2$ for all $A,B,C \in \mathbb{R}$, we have

$$\begin{aligned}\int_T^t \frac{1}{s}v(s)^2 ds \le \frac{\lambda^{2\alpha}}{4}(\log T)^2\int_T^t \frac{1}{s(\log s)^2}ds + 4\int_T^t \frac{1}{s\log s}ds \\ + 4(9\alpha)^2\int_T^t \frac{1}{s(\log s)^2}\left[\int_T^s \frac{\log\sigma}{\sigma}\left(\int_\sigma^\infty \frac{|\varepsilon(r)|}{r}dr\right)d\sigma\right]^2 ds\end{aligned}$$

for $t \geq T$. Denote by $R_{1,3}(t)$ the last term of the right-hand side of the above inequality. To estimate $R_{1,3}(t)$, we use Lemma 2.6 of the case $p = 2$, $a = T$, $b = t$ and

$$f(t) = (\log t) \int_t^\infty \frac{|\varepsilon(r)|}{r} dr.$$

Then we find that

$$\begin{aligned} |R_{1,3}(t)| &\leq 4^2 (9\alpha)^2 \int_T^t \frac{(\log s)^2}{s} \left( \int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right)^2 ds \\ &\leq 4^2 (9\alpha)^2 M(T) \int_T^t \frac{\log s}{s} \left( \int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right) ds, \end{aligned}$$

where

$$M(T) = \sup_{s \geq T} \left\{ (\log s) \int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right\}. \tag{4.8}$$

Therefore we have

$$\begin{aligned} |R_1(t)| \leq{}& \frac{\lambda^\alpha}{8} \frac{\log T}{\log t} + 2\lambda^{-\alpha} \frac{\log(\log t)}{\log t} \\ &+ 8(9\alpha)^2 \lambda^{-\alpha} M(T) \frac{1}{\log t} \int_T^t \frac{\log s}{s} \left( \int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma} d\sigma \right) ds \end{aligned} \tag{4.9}$$

for $t \geq T$. To estimate $|R_2(t)|$, $|R_3(t)|$ and $|R_4(t)|$, we use (4.5). We have

$$|R_2(t)| \leq \frac{2\alpha}{\log t} \int_T^t \frac{(\log s)^2}{s} |\varepsilon(s)| ds \tag{4.10}$$

for $t \geq T$. It is clear that $|v(t)| \leq v_0$ for $t \geq T$, and so $|v(t) + 2\lambda^\alpha| \leq v_0 + 2\lambda^\alpha$ for $t \geq T$. Therefore we get

$$|R_3(t)| \leq \frac{2(\alpha+1)}{\alpha} \lambda^{-\alpha+1} (v_0 + 2\lambda^\alpha) \frac{1}{\log t} \int_T^t \frac{\log s}{s} |\varepsilon(s)| ds \tag{4.11}$$

and

$$|R_4(t)| \leq \frac{(\alpha+1)|\alpha-1|}{\alpha^2} \lambda^{-2\alpha+1} (v_0 + 2\lambda^\alpha)^2 \frac{1}{\log t} \int_T^t \frac{1}{s} |\varepsilon(s)| ds \tag{4.12}$$

for $t \geq T$. We may suppose that $T$ is sufficiently large so that

$$\frac{2(\alpha+1)}{\alpha} \lambda^{-\alpha+1} (v_0 + 2\lambda^\alpha) \leq \alpha \log T$$

and

$$\frac{(\alpha+1)|\alpha-1|}{\alpha^2}\lambda^{-2\alpha+1}(v_0+2\lambda^\alpha)^2 \le \alpha(\log T)^2.$$

Then it follows from (4.11) and (4.12) that

$$|R_3(t)| \le \frac{\alpha}{\log t}\int_T^t \frac{(\log s)^2}{s}|\varepsilon(s)|ds \tag{4.13}$$

and

$$|R_4(t)| \le \frac{\alpha}{\log t}\int_T^t \frac{(\log s)^2}{s}|\varepsilon(s)|ds \tag{4.14}$$

for $t \ge T$. Therefore, by (4.10), (4.13) and (4.14), we get

$$\begin{aligned}
&|R_2(t)| + |R_3(t)| + |R_4(t)| \\
&\le \frac{4\alpha}{\log t}\int_T^t \frac{(\log s)^2}{s}|\varepsilon(s)|ds \\
&\le 4\alpha\frac{(\log T)^2}{\log t}\int_T^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma + \frac{8\alpha}{\log t}\int_T^t \frac{\log s}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)ds \\
&\le 4\alpha M(T)\frac{\log T}{\log t} + \frac{8\alpha}{\log t}\int_T^t \frac{\log s}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)ds
\end{aligned} \tag{4.15}$$

for $t \ge T$. Here, $M(T)$ is given by (4.8). Since $|\varepsilon(t)| \le \lambda^\alpha/4$ and $|v(t)| \le v_0$ for $t \ge T$, it follows from Lemma 2.2 that

$$|G(t, v(t), \varepsilon(t))| \le L(\alpha)\lambda^{-2\alpha+1}\left|\frac{v(t)+2\lambda^\alpha}{\log t}\right|^3, \quad t \ge T.$$

Therefore

$$\begin{aligned}
|R_5(t)| &\le \alpha L(\alpha)\lambda^{-2\alpha+1}(v_0+2\lambda^\alpha)^3\frac{1}{\log t}\int_T^t \frac{1}{s\log s}ds \\
&\le \alpha L(\alpha)\lambda^{-2\alpha+1}(v_0+2\lambda^\alpha)^3\frac{\log(\log t)}{\log t}, \quad t \ge T.
\end{aligned} \tag{4.16}$$

Consequently, by (4.9), (4.15) and (4.16), it is found that

$$\begin{aligned}|(\mathcal{F}v)(t)| &\le \sum_{i=1}^{5} |R_i(t)| \\ &\le \left(\frac{\lambda^\alpha}{8} + 4\alpha M(T)\right)\frac{\log T}{\log t} \\ &\quad + \left(2\lambda^{-\alpha} + \alpha L(\alpha)\lambda^{-2\alpha+1}(v_0 + 2\lambda^\alpha)^3\right)\frac{\log(\log t)}{\log t} \\ &\quad + \left(8(9\alpha)^2\lambda^{-\alpha}M(T) + 8\alpha\right)\frac{1}{\log t}\int_T^t \frac{\log s}{s}\left(\int_s^\infty \frac{|\varepsilon(\sigma)|}{\sigma}d\sigma\right)ds\end{aligned}$$

for $t \ge T$. Since $M(T) \to 0$ as $T \to \infty$, we may suppose that

$$4\alpha M(T) \le \frac{\lambda^\alpha}{8} \quad \text{and} \quad 8(9\alpha)^2\lambda^{-\alpha}M(T) \le \alpha.$$

Further, we may suppose that

$$\left(2\lambda^{-\alpha} + \alpha L(\alpha)\lambda^{-2\alpha+1}(v_0 + 2\lambda^\alpha)^3\right)\frac{\log(\log t)}{\log t} \le \frac{1}{(\log t)^{1/2}} \quad \text{for} \quad t \ge T.$$

Then we get

$$|(\mathcal{F}v)(t)| \le \psi(t;T), \quad t \ge T.$$

This means that

(i) $\mathcal{F}$ maps $V$ into $V$.

Moreover it can be checked that:

(ii) $\mathcal{F}$ is continuous on $V$,
(iii) $\mathcal{F}V$ is uniformly bounded and equicontinuous at every point of $[T, \infty)$.

Therefore, by the Schauder–Tychonoff fixed point theorem, we conclude that $\mathcal{F}$ has a fixed element $v \in V$: $v(t) = (\mathcal{F}v)(t)$, $t \ge T$.

It is clear that the above fixed element $v(t)$ satisfies (4.4) and (4.6). Since $\lim \psi(t;T) = 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} v(t) = 0. \tag{4.17}$$

Note also that $|v(t)| \le v_0$ for $t \ge T$. Moreover we can verify without difficulty that $v(t)$ satisfies (4.2) for $t \ge T$, and, in consequence, $v(t)$ satisfies (4.1) for $t \ge T$. Put

$$y(t) = \frac{1}{t^\alpha}\left(\frac{v(t) + 2\lambda^\alpha}{\log t} + \lambda^\alpha + \varepsilon(t)\right), \quad t \ge T.$$

It is shown that $y(t)$ satisfies (1.28), and hence it satisfies (1.25). Therefore, the function $x(t)$ which is defined by (1.26) is a positive solution of (1.1) on $[T,\infty)$. Furthermore we have

$$t^{\alpha}\left(\frac{x'(t)}{x(t)}\right)^{\alpha*} = t^{\alpha}y(t) = \frac{v(t)+2\lambda^{\alpha}}{\log t} + \lambda^{\alpha} + \varepsilon(t), \quad t \geq T,$$

and so

$$t\frac{x'(t)}{x(t)} = \left(\lambda^{\alpha} + \frac{v(t)+2\lambda^{\alpha}}{\log t} + \varepsilon(t)\right)^{(1/\alpha)*}, \quad t \geq T.$$

Consequently, by (2.8) in Lemma 2.4, the function $tx'(t)/x(t)$ has the form

$$t\frac{x'(t)}{x(t)} = \lambda + \frac{\lambda^{-\alpha+1}}{\alpha}\left(\frac{v(t)+2\lambda^{\alpha}}{\log t} + \varepsilon(t)\right) + R(t) \tag{4.18}$$

with

$$|R(t)| \leq \frac{|\alpha-1|}{\alpha^2}\lambda^{-2\alpha+1}\left|\frac{v(t)+2\lambda^{\alpha}}{\log t} + \varepsilon(t)\right|^2 \tag{4.19}$$

for all large $t$.

It should be noticed that the condition (1.19) is not used in the arguments up to now.

From (4.18) it follows that

$$(\log t)\left(t\frac{x'(t)}{x(t)} - \lambda\right) = \frac{2\lambda}{\alpha} + \frac{\lambda^{-\alpha+1}}{\alpha}\left(v(t) + (\log t)\varepsilon(t)\right) + (\log t)R(t)$$

for all large $t$. Then, by (1.19), (4.17) and (4.19), we can conclude that

$$\lim_{t\to\infty}(\log t)\left(t\frac{x'(t)}{x(t)} - \lambda\right) = \frac{2\lambda}{\alpha}.$$

Since $\lambda = \alpha/(\alpha+1)$, this gives (1.21) with $x_2(t) = x(t)$. The proof of Theorem 1.6 in the case $i = 2$ is complete. □

*Proof of Theorem 1.7 in the case $i = 2$.* Since (1.10) and (1.12) imply (1.18) (see (I) of Lemma 2.5), almost all of the proof of Theorem 1.6 in the case $i = 2$ remain valid. Let $x(t)$ be the solution of (1.1) which is obtained in the proof of Theorem 1.6 in the case $i = 2$. We have (4.18), (4.19) and (4.6). Use of (4.18) yields

$$\frac{x'(t)}{x(t)} = \frac{\lambda}{t} + \frac{2\lambda}{\alpha}\frac{1}{t\log t} + \frac{\lambda^{-\alpha+1}}{\alpha}\left(\frac{v(t)}{t\log t} + \frac{\varepsilon(t)}{t}\right) + \frac{R(t)}{t}$$

for all large $t$. Then it is easily seen that

$$\begin{aligned} x(t) = {} & \frac{x(T_1)}{T_1^{\lambda}(\log T_1)^{2\lambda/\alpha}}\exp\left(\frac{\lambda^{-\alpha+1}}{\alpha}\int_{T_1}^{t}\left(\frac{v(s)}{s\log s} + \frac{\varepsilon(s)}{s}\right)ds\right) \\ & \times \exp\left(\int_{T_1}^{t}\frac{R(s)}{s}ds\right)t^{\lambda}(\log t)^{2\lambda/\alpha} \end{aligned} \tag{4.20}$$

for $t \geq T_1$, where $T_1$ is a constant and is taken sufficiently large. Recall that the condition (1.10) is assumed. Note furthermore that the condition (1.12) implies

$$\int_{T_1}^{\infty} \frac{\psi(s;T)}{s \log s} ds < \infty.$$

Then it follows from (4.6) that

$$\int_{T_1}^{\infty} \frac{|v(s)|}{s \log s} ds < \infty.$$

We find without difficulty that

$$\int_{T_1}^{\infty} \frac{|R(s)|}{s} ds < \infty.$$

Then, by (4.20), it is shown that $x(t)$ is written in the form

$$x(t) = c_0(t) t^{\lambda} (\log t)^{2\lambda/\alpha} \quad \text{with} \quad c_0(t) \to c_0 \in (0, \infty) \quad \text{as} \quad t \to \infty. \tag{4.21}$$

It is clear that (4.18) and (4.19) give

$$\lim_{t\to\infty} t \frac{x'(t)}{x(t)} = \lambda,$$

and so (4.21) implies

$$\begin{cases} x'(t) = c_0(t) t \dfrac{x'(t)}{x(t)} t^{\lambda-1} (\log t)^{2\lambda/\alpha} = c_1(t) t^{\lambda-1} (\log t)^{2\lambda/\alpha} \\ \text{with} \quad c_1(t) \to c_0 \lambda \quad \text{as} \quad t \to \infty. \end{cases} \tag{4.22}$$

We may suppose that $c_0 = 1$ in (4.21) and (4.22). Since $\lambda = \alpha/(\alpha+1)$, this shows (1.23) with $x_2(t) = x(t)$. The proof of Theorem 1.7 in the case $i = 2$ is complete. □

## 5. EXAMPLES

We now present some examples illustrating our main results.

**Example 5.1.** Consider the equation (1.1) with

$$q(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{d}{dt} \left\{ \frac{1}{t^{\alpha} (\log t)^2 (\log \log t)} + \frac{1}{t^{\alpha} (\log t)^2 (\log \log t)^2} \right\}$$

for $t \geq e + 1$. We easily see that (1.3) and (1.13) are satisfied, and the function $\varepsilon(t)$ which is defined by (1.14) is calculated as follows:

$$\varepsilon(t) = -\frac{1}{(\log t)^2 (\log \log t)} - \frac{1}{(\log t)^2 (\log \log t)^2}, \quad t \geq e + 1.$$

Hence, the condition (1.10) is also satisfied and

$$\int_t^\infty \frac{|\varepsilon(s)|}{s} ds = \frac{1}{(\log t)(\log\log t)}, \quad t \geq e+1.$$

Then it is clear that (1.18) and (1.19) are satisfied. Therefore it follows from Theorem 1.6 that (1.1) has a nonoscillatory solution $x_1(t)$ satisfying (1.20) and a nonoscillatory solution $x_2(t)$ satisfying (1.21).

It should be noticed that this $q(t)$ does not satisfy the condition (1.12), and so Theorem 1.3 does not work.

**Example 5.2.** Let $\omega(t)$ be a continuously differentiable function on $[t_0, \infty)$, $t_0 > 1$, such that $\omega(t) \geq 0$ for $t \geq t_0$ and

$$\int_{t_0}^\infty \omega(t) dt < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{t\omega(t)}{\log t} = 0. \tag{5.1}$$

Then, consider the equation (1.1) with

$$q(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} - \frac{d}{dt}\left\{ \frac{t\omega(t)}{t^\alpha \log t} + \frac{1}{t^\alpha (\log t)^2} \int_t^\infty \omega(s) ds \right\}, \quad t \geq t_0.$$

It is easy to see that (1.3) and (1.13) are satisfied and the function $\varepsilon(t)$ defined by (1.14) is

$$\varepsilon(t) = \frac{t\omega(t)}{\log t} + \frac{1}{(\log t)^2} \int_t^\infty \omega(s) ds.$$

Therefore it is shown without difficulty that (1.10) is satisfied and

$$\int_t^\infty \frac{|\varepsilon(s)|}{s} ds = \frac{1}{\log t} \int_t^\infty \omega(s) ds.$$

Hence, (1.18) holds. By Corollary 1.5 we deduce that (1.1) has a nonoscillatory solution $x(t)$ satisfying (1.17).

If $\omega(t)$ satisfies the condition

$$\lim_{t\to\infty} t\omega(t) = 0, \tag{5.2}$$

which is stronger than the latter half of (5.1), then (1.19) also holds. Therefore, by Theorem 1.6, we conclude that (1.1) has a nonoscillatory solution $x_1(t)$ satisfying (1.20) and a nonoscillatory solution $x_2(t)$ satisfying (1.21).

If $\omega(t)$ satisfies the condition

$$\int_{t_0}^{\infty} \frac{1}{t \log t} \left( \int_t^{\infty} \omega(s) ds \right) dt < \infty, \tag{5.3}$$

then (1.12) holds, and, by Theorem 1.7, the equation (1.1) has a nonoscillatory solution $x_1(t)$ satisfying (1.22) and a nonoscillatory solution $x_2(t)$ satisfying (1.23).

For example, the function $\omega(t)$ behaving like

$$\omega(t) \sim \frac{1}{t(\log t)(\log \log t)^2} \quad (t \to \infty)$$

satisfies both (5.1) and (5.2), while it does not satisfy (5.3). The function $w(t)$ behaving like

$$\omega(t) \sim \frac{1}{t(\log t)(\log \log t)^3} \quad (t \to \infty)$$

satisfies all of the conditions (5.1), (5.2) and (5.3).

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Manabu Naito
jpywm078@yahoo.co.jp

Ehime University
Faculty of Science
Department of Mathematics
Matsuyama 790-8577, Japan

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