ANALIZA DYNAMICZNEJ ODPOWIEDZI TORU KOLEJOWEGO W OPARCIU O PODEJŚCIE PÓŁ-ANALITYCZNE[1]

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Streszczenie. *W artykule opisano dwa nieliniowe modele drogi szynowej wraz z ich rozwiązaniem za pomocą metody pół-analitycznej wykorzystującej aproksymacje falkowe. Pokazano, że metoda ta pozwala uzyskać prędkości i przyspieszenia drgań szyny w przypadku nieliniowej sztywności podłoża, dla obciążenia generowanego przez pojazd poruszający się ze stałą prędkością. Jednocześnie podano przykłady pokazujące, że zastosowana metoda oraz model nie dają prawidłowych wyników w odniesieniu do naprężeń pionowych w osi szyny.*

Słowa kluczowe: *dynamika dróg szynowych, metody pół-analityczne*

1. Introduction

Problems of rail track dynamics are usually being solved by using numerical methods, especially when one deals with parameters other than vertical displacements [13,1]. For this purpose, the rail track should be appropriately modelled. One can find in the literature two main approaches to this modelling problem [4]. The first one, called the one-layer model, assumes that the rail track can be represented by either the Euler-Bernoulli beam or the Timoshenko beam resting on viscoelastic foundation representing rail foundation. The second one, the two-layer model, consists of two layers associated with rails and sleepers. These layers are represented by two beams, usually the Euler-Bernoulli ones due to simplicity of analytical formulas, also resting on viscoelastic foundation, under condition that bending in the layer representing sleepers is neglected. The layers are described by dynamic beam equations and the force or set of forces associated with train load is considered in various configurations [8]. Examples of numerical solutions to these models can be easily found in the literature, whereas analytical approaches are relatively rare. Although they are highly valued for their universality and usefulness in parametrical analyses, one can observe a lack of mathematical tools allowing solution of more complex systems, being closer to real structures. Therefore ap-

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propriate approximation methods are sought. Recent progress in mathematical sciences gives nowadays opportunity to develop modern efficient semi-analytical techniques that can be effectively applied to engineering analysis of such systems [11]. This paper presents fundamentals of two approximate approaches to analysis of one-layer or two-layer models [4, 8]. The first one based on the classical Fourier series application and the second one using wavelet based estimation, which allows to solve, in conjunction with Adomian expansion, also nonlinear and stochastic systems. Some examples of vibration velocity and acceleration are shown in the case of two-layer model with nonlinear stiffness of cubic type included in foundation description.

2. Models

The one-layer model can be described by the following equation [4, 8]:

$$EI\frac{\partial^4 y_r}{\partial x^4}+N\frac{\partial^2 y_r}{\partial x^2}+m_r\frac{\partial^2 y_r}{\partial t^2}+c_r\frac{\partial y_r}{\partial t}+k_r y_r+k_N {y_r}^3=q(x,t) \tag{1}$$

in which the used parameters are: EI [Nm^2] – bending stiffness of rail steel, N [N] – axial force, m_r [kg/m] – rail unit mass, k_r [N/m^2] – stiffness of rail foundation and c_r [Ns/m^2] – viscous damping of rail foundation. The load q [N/m] is described as a set of distributed forces produced by wheels of train. Geometrical imperfections of rail head surface are represented by regular cosine wave with frequency Ω [rad/m] and amplitude s_o [m]. These imperfections produce forces changing in time with circular frequency $\omega=\Omega v$ and the amplitude $F_s=s_o k_c$ in the case of constant speed v [km/h] [4]. The parameter k_N [N/m^4] represents nonlinear part of rail track foundation stiffness and k_c is the contact spring between wheel and rail [N/m] [2,14].

The load $q(x,t)$ generated by moving train is represented by a set of forces (concentrated or distributed) associated with wheels of train:

$$q(x,t)=q_S(x,t)+q_V(x,t) \tag{2}$$

where two terms are distinguished: the first one produced by the train weight

$$q_S(x,t)=\sum_{l=0}^{l_T}\frac{Q}{a}\cos^2\left(\frac{\pi(x-vt-s_l)}{2a}\right)H(a^2-(x-vt-s_l)^2) \tag{3}$$

and the second one related to rail head surface imperfections producing additional force varying in time

$$P_V(x,t)=\sum_{l=0}^{l_T}\frac{\Delta Q}{a}e^{i\omega t}\cos^2\left(\frac{\pi(x-vt-s_l)}{2a}\right)H(a^2-(x-vt-s_l)^2) \tag{4}$$

where $H(.)$, $2a$, s_l, l_T, ω and v are the Heaviside function, the span of single load, the distance of consecutive forces produced by wheels, the number of axles, the frequency of load and the speed of train, respectively. The axles configuration can be also introduced by inclusion of the phase shift associated with the wheel position on the surface of irregularity. It is assumed that the rail head imperfections are placed directly side by side, that is with no space between them. Their amplitude is constant and the shape of rail surface is regular, having a cosine form.

The two-layer model is represented by the following system of joint differential equations [4,6,8]:

$$EI\frac{\partial^4 y_r}{\partial x^4}+N\frac{\partial^2 y_r}{\partial x^2}+m_r\frac{\partial^2 y_r}{\partial t^2}+c_r(\frac{\partial y_r}{\partial t}-\frac{\partial y_s}{\partial t})+k_r(y_r-y_s)-k_N{y_s}^3=q(x,t)$$
$$m_s\frac{\partial^2 y_s}{\partial t^2}+c_s\frac{\partial y_s}{\partial t}+k_s y_s+k_N{y_s}^3=c_r(\frac{\partial y_r}{\partial t}-\frac{\partial y_s}{\partial t})+k_r(y_r-y_s) \tag{5}$$

in which the first equation represents rail and the second one describes sleepers layer. The additional used parameters are: m_s [kg/m] – unit mass of sleepers uniformly distributed along the track, k_s [N/m²] – stiffness of sleepers foundation and c_s [Ns/m²] – viscous damping of sleepers foundation. The effect of sleepers bending is neglected in this model.

Introducing the moving coordinate system (η=*x*, ζ=*x-vt*) leads to re-formulation of the two described above models in which equations (1-5) can be rewritten as follows:

$$EI\frac{\partial^4 y_r}{\partial \xi^4}+N\frac{\partial^2 y_r}{\partial \xi^2}+m_r(\frac{\partial^2 y_r}{\partial t^2}-2v\frac{\partial^2 y_r}{\partial t\partial \xi}+v^2\frac{\partial^2 y_r}{\partial \xi^2})$$
$$+c_r(\frac{\partial y_r}{\partial t}-v\frac{\partial y_r}{\partial \xi})+k_r y_r+k_N{y_r}^3=q(\xi,t) \tag{6}$$

in the case of one-layer model (eq. (1)), and

$$EI\frac{\partial^4 y_r}{\partial \xi^4}+N\frac{\partial^2 y_r}{\partial \xi^2}+m_r(\frac{\partial^2 y_r}{\partial t^2}-2v\frac{\partial^2 y_r}{\partial t\partial \xi}+v^2\frac{\partial^2 y_r}{\partial \xi^2})$$
$$+c_r(\frac{\partial y_r}{\partial t}-v\frac{\partial y_r}{\partial \xi}-\frac{\partial y_s}{\partial t}+v\frac{\partial y_s}{\partial \xi})+k_r(y_r-y_s)-k_N{y_s}^3=q(\xi,t) \tag{7a}$$

$$m_s(\frac{\partial^2 y_s}{\partial t^2}-2v\frac{\partial^2 y_s}{\partial t\partial \xi}+v^2\frac{\partial^2 y_s}{\partial \xi^2})+c_s(\frac{\partial y_r}{\partial t}-\frac{\partial y_s}{\partial \xi})+k_s y_s$$
$$-c_r(\frac{\partial y_r}{\partial t}-v\frac{\partial y_r}{\partial \xi}-\frac{\partial y_s}{\partial t}+v\frac{\partial y_s}{\partial \xi})-k_r(y_r-y_s)+k_N{y_s}^3=0 \tag{7b}$$

in the case of two-layer model (eqs. (5)). The load representation takes the form:

$$q(\xi,t)=Q_S(\xi,t)+Q_V(\xi,t)=\sum_{l=0}^{l_T}(\frac{Q}{a}+\frac{\Delta Q}{a}e^{i\omega t})\cos^2(\frac{\pi(\xi-s_l)}{2a})H(a^2-(\xi-s_l)^2) \tag{8}$$

where

$$Q_S(\xi,t)=\sum_{l=0}^{l_T}\frac{Q}{a}\cos^2\left(\frac{\pi(\xi-s_l)}{2a}\right)H(a^2-(\xi-s_l)^2) \tag{9}$$

is the stationary part of load, i.e. constant in time, and

$$Q_V(\xi,t)=\sum_{l=0}^{l_T}\frac{\Delta Q}{a}e^{i\omega t}\cos^2\left(\frac{\pi(\xi-s_l)}{2a}\right)H(a^2-(\xi-s_l)^2) \tag{10}$$

is the varying in time force generated by rail head surface irregularities.

3. Solutions

The two models described in section 3 can be solved by using the Fourier series [4] or the Fourier transform [8], depending on assumptions regarding the length of layers (rails and sleepers layer) and nonlinear properties of systems.

In the case of linear foundation stiffness and finite length of layers, one can apply the Fourier series. If the load is expanded in the following way:

$$q(\xi,t)=q_c(\xi)\cdot\cos\omega t+q_s(\xi)\cdot\sin\omega t\,, \tag{11}$$

then the stationary solution of equation (6) has the form of the system of the ordinary differential equations:

$$\begin{aligned}
&EI\frac{d^4Y_c}{d\xi^4}-m_r\omega^2Y_c-2m_rv\omega Y_s+(N+m_rv^2)\frac{d^2Y_c}{d\xi^2}+c_r\omega Y_s-c_rv\frac{dY_c}{d\xi}-k_rY_c=q_c(\xi)\\
&EI\frac{d^4Y_s}{d\xi^4}-m_r\omega^2Y_s+2m_rv\omega Y_c+(N+m_rv^2)\frac{d^2Y_s}{d\xi^2}-c_r\omega Y_c-c_rv\frac{dY_s}{d\xi}+k_rY_s=q_s(\xi)
\end{aligned} \tag{12}$$

which can be solved by applying the Fourier series to both the distributed loads

$$\begin{aligned}
&q_c(\xi)=\frac{q_{c0}}{2}+\sum_{i=1}^{\infty}(a_i\cdot\cos\Omega_i\xi+b_i\cdot\sin\Omega_i\xi);\\
&q_s(\xi)=\frac{q_{s0}}{2}+\sum_{i=1}^{\infty}(c_i\cdot\cos\Omega_i\xi+d_i\cdot\sin\Omega_i\xi);\\
&\xi\in[0,\lambda];\Omega_i=\frac{2\pi\cdot i}{\lambda}
\end{aligned} \tag{13}$$

and the sought functions

$$\begin{aligned}
&Y_c(\xi)=\frac{Y_{c01}}{2}+\sum_{i=1}^{\infty}(A_i\cdot\cos\Omega_i\xi+B_i\cdot\sin\Omega_i\xi);\\
&Y_s(\xi)=\frac{Y_{s01}}{2}+\sum_{i=1}^{\infty}(C_i\cdot\cos\Omega_i\xi+D_i\cdot\sin\Omega_i\xi).
\end{aligned} \tag{14}$$

If the distribution of load is given in analytical form, e.g. $\cos^2$ function (see eq. (8)) [6,8,12], the parameters in equations (13) can be determined in explicit form and the solution of equation (1), i.e. the unknown parameters Y_{c01}, Y_{s01}, A_i, B_i, C_i, D_i can be obtained from the system of six algebraic equations. Similar procedure is appropriate for solution of the system (5), in the case of the two-layer model (with the number of algebraic equations increased to 12).

The procedure becomes more complex when one deals with nonlinear or stochastic properties of the system. Such features of physical structures associated with rail tracks are widely discussed in the literature showing their importance for accurate enough representation of detailed dynamic properties of rail tracks [8,9,11,12].

In the case of the systems analysed in the present paper, it is assumed that track foundation is described by classical representation of nonlinearity, i.e. its cubic form, which is experimentally verified and widely used in the literature [11,12].

The following semi-analytical approach is used for solving the nonlinear systems defined by equations (1) and (5), along with the load description given by formulas (2)-(4) [8,11,12]. Under condition that the considered layers are infinitely long, one can apply the Fourier transform to both models. However, the nonlinear term must be approximated in some way in order to eliminate its nonlinear representation. For this purpose, the Adomian's decomposition method is proposed. In this paper, classical form of the Adomian polynomials is used, leading to the following expansion of the nonlinear factor [8,9,11,12]

$$y_i^3(x) = \sum_{j=0}^{\infty} P_j(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\frac{d^j}{ds^j}\left(\sum_{k=0}^{\infty} \lambda^k y_{ik}(x)\right)^3\right]_{s=0} \tag{15}$$

for $i = r, s$, where $y_{i0}(x)$ is a solution of linear equation (i.e. for $k_N = 0$).

Then the Fourier transform can be applied to both the one-layer and the two-layer models:

$$\tilde{y}_j(\omega) = F\left[y_j(\xi)\right] = \int_{-\infty}^{\infty} y_j(\xi) e^{-i\omega\xi} d\xi \tag{16}$$

for $j = r, s$. After obtaining the solution in the transform domain, the response of the system can be found by applying wavelet based approximation using coiflet coefficients:

$$y_j(\xi) = \lim_{n\to\infty} f_n(\xi) = \lim_{n\to\infty} \frac{1}{2^{n+1}\pi} \prod_{k=1}^{k_N} \left(\sum_{j=0}^{3N-1} p_j e^{ij\xi/2^{n+k}}\right) \sum_{k=-\infty}^{+\infty} \tilde{y}_j((k+M)2^{-n}) e^{i\xi k 2^{-n}} . \tag{17}$$

Here, $M = \sum_{k=0}^{3N-1} k p_k$, N is the degree of the used coiflet filter (p_k) and k_N is a number evaluated on the basis of stabilization of the obtained solution. The analyses carried out so far, show that the two approaches presented in this section give similar results. These results are also validated experimentally in the case of vertical displacements [5,8,10]. One should underline that sufficiently exact solutions can be found in this case and successfully used in parametrical analysis when one takes into account engineering applications. Also other characteristics obtained by differentiation of the solution with respect to the time variable are close to values obtained experimentally.

However, in the case of other characteristics, involving derivatives of the obtained solution in relation to the space variable, such as e.g. bending stress, they give wrong results and the problem of analytical modelling becomes more difficult [3,7].

4. Examples

In this section, some examples using the system of parameters taken from real structures are presented: $EI = 6.4*10^6$ Nm^2, $m_r = 60$ kg/m, $k_r = 8.8*10^7$ N/m^2, $c_r = 3950$ Ns/m^2, $m_s = 267$ kg/m, $k_s = 8.5*10^7$ N/m^2 and $c_s = 8.2*10^3$ Ns/m^2. The nonlinear factor and axial force are neglected for simplicity of numerical simulations and left for further analysis. It is assumed that there are no factors producing the additional force changing in time (eq. (4)), i.e. imperfections on rolling surface of rail head, other than the deflection between sleepers. The axles configuration of fast train Pendolino EMU250 is considered. The two-layer model is used for obtaining the presented results.

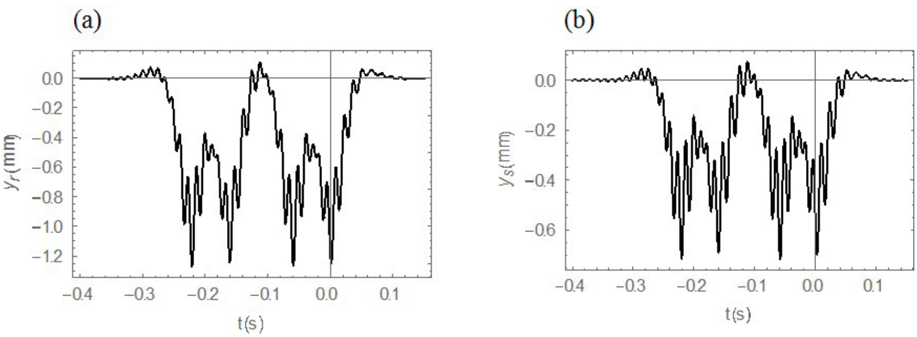

Fig. 1. Vertical vibrations of: (a) rails, (b) sleepers (v=160 km/h)

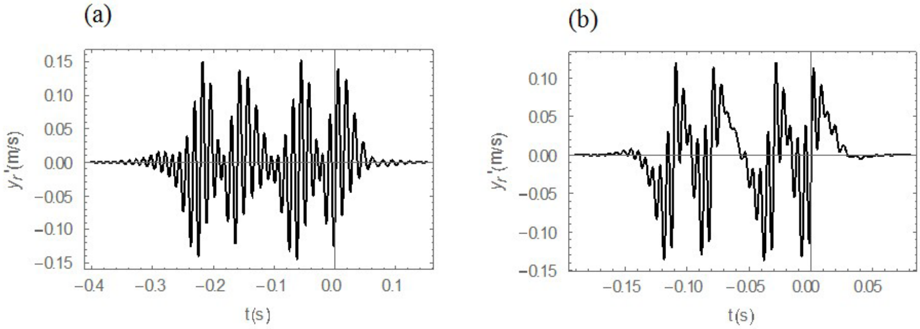

Fig. 2. Velocity of vertical vibrations of rails (a) v=160 km/h, (b) v=320 km/h

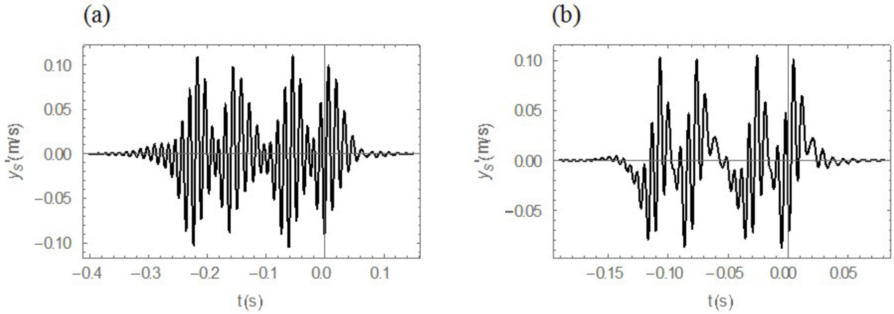

Fig. 3. Velocity of vertical vibrations of sleepers (a) v=160 km/h, (b) v=320 km/h

Figure 1 shows vertical vibrations of rails and sleepers. One can observe smaller amplitude of sleepers vibrations but their more intensive dynamic character should be underlined. These results were confirmed experimentally before [5,10,11].

Figures 2 and 3 present velocity of vertical vibrations for rails and sleepers. It is worth to note that the velocity of rail vibrations becomes smaller along with the train speed increase, whereas this tendency cannot be observed for sleepers. Figures 4 and 5 show examples of vibrations acceleration for the same speeds of train. Maximal values are close to real ones, at least in the case of the track in relatively good condition [5]. The obtained results for vibrations velocity or acceleration, need to be verified by experimental measurements or other methods of solution for extended range of parameters.

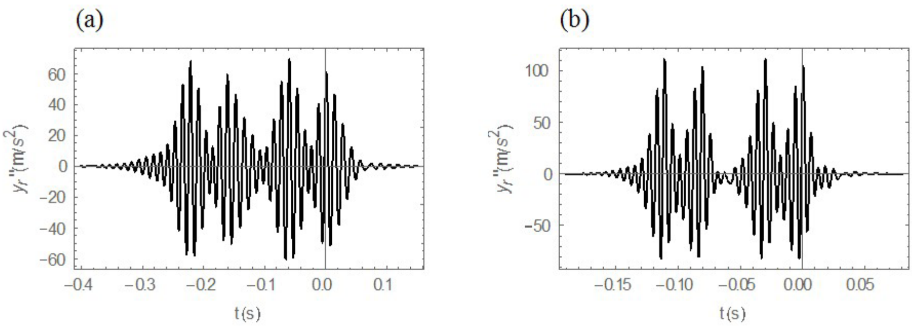

Fig. 4. Acceleration of rails vertical vibrations (a) v=160 km/h, (b) v=320 km/h

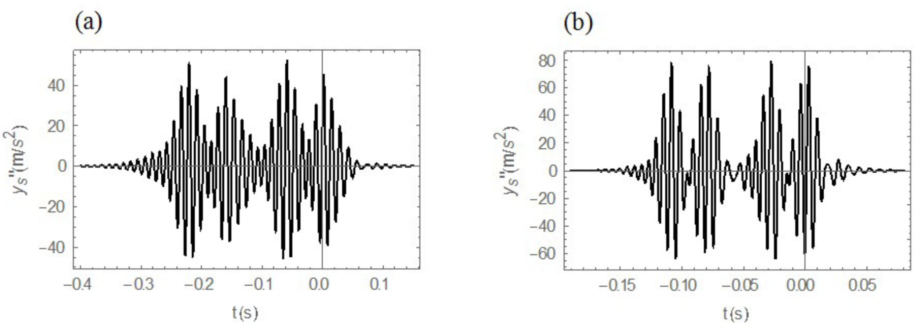

Fig. 5. Acceleration of sleepers vertical vibrations (a) v=160 km/h, (b) v=320 km/h

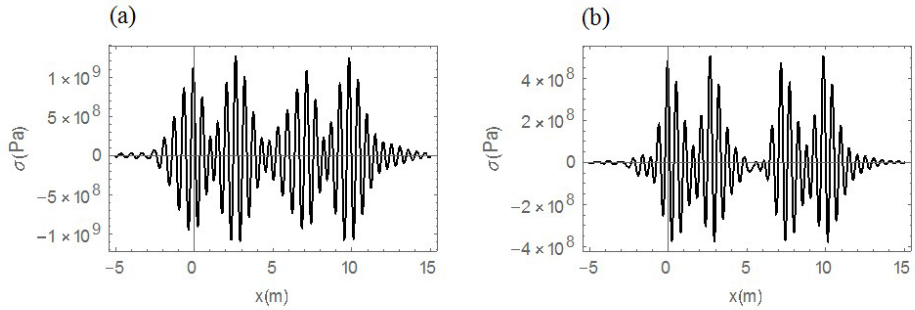

Fig. 6. Normal bending stress in rail axis (a) $v=160$ km/h, (b) $v=320$ km/h

Serious doubts arise from the analysis of characteristics involving derivatives of higher order with respect to the space variable. It can be observed that analytically obtained values for the normal bending stresses significantly exceed limit values of the strength of rail steel R260 (around 800 MPa) [3,7].

Figure 6 is prepared for the system of parameters used in previous calculations. It shows the normal stress distribution in rail with irregularity related to the sleeper spacing. The purpose of this analysis is to show the method ability to analytically calculate stresses.

Although in the case presented in figure 6, the limit values are reached or only slightly exceeded, more extensive analyses showed that for some lengths of waves, the maximal value of the normal stress can extremely exceed the limit value (a few hundred percent and more). For example, when one consider the length on waves associated with the length of irregularity appearing on rail head rolling surface equal to 30 cm (cosine type of imperfection), which is twice less than the distance between sleepers, and the train speed $v=200$ km/h, the maximal value of the normal stress in rail axis can be estimated at around 1500 MPa [3].

The main aim of the project [3] was to show examples of analytically derived stresses using models (methods) validated and known from the literature in simplified models of rail tracks. During the project realization, several examples of results exceeding limit values have been found. The analysis was carried out for a relatively wide range of physical parameters and train speeds. Conclusions arising from this project show that a lack of appropriate analytical approaches for the rail normal stress calculation is noticeable and should be underlined.

5. Conclusions

The analysis carried out during paper preparation show that the obtained results for normal bending stress in rail axis exceed, for some systems of parameters, limit values. Such examples can be found for various models and methods of analytical solution. One can conclude that the applied approaches are inappropriate for stresses analysis. The natural question about improvements leading to better representation of rail stresses remains open. However, there are some possibilities of further work continuation. One might suggest consideration of additional mechanical features of the investigated rail track models, leading to their reformulation if future. Nevertheless, these desired features remain unrecognised so far. The interesting possibility might be sought in modification of rail model, which should avoid its consideration as a homogeneous beam. Some preliminary results, strongly dependent on the load frequency, are obtained by the authors in the case of "head on web" problem or rail internal damping. This analysis is not entirely satisfactory, so far, giving results valid for specific sets of physical parameters only. One should underline that the analytical models, the wavelet based method used in this paper and the mentioned approach using the Fourier transform are verified and experimentally validated for rails and sleepers vertical displacement analysis.

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