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Totally positive polynomials with small length*

by

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Abstract: In this work we provide a list of irreducible monic, totally positive polynomials with integer coefficients and lengths below 2.3679^d for a degree d polynomial. Our approach combines techniques based on auxiliary functions and mixed-integer linear programming algorithms. The list obtained shows that there are polynomials of this type up to degree 16, with the exception of degree 11 where we are very close to the threshold.

Keywords: auxiliary functions, length of polynomials

1. Introduction

Let $P(x) = b_0x^d + \cdots + b_d = b_0(x-\alpha_1)\cdots(x-\alpha_d)$, $b_0b_d \neq 0$, $P \neq x$, be a polynomial with complex coefficients. M. Langevin (1980) defined three families of measures of polynomials which are, for $p > 0$:

$$\mathrm{M}_p(P) = \Big(\int_0^1 |P(e^{2i\pi t})|^p dt\Big)^{\frac{1}{p}},$$

$$\mathrm{L}_p(P) = \Big(\sum_{i=1}^{d} |a_i|^p\Big)^{\frac{1}{p}},$$

$$\mathrm{R}_p(P) = |a_0|\prod_{i=1}^{d}(1+|\alpha_i|^p)^{\frac{1}{p}}.$$

Note first that $\mathrm{M}(P) = \lim_{p\to 0} \mathrm{M}_p(P) = \exp\Big(\int_0^1 \log|P(e^{2i\pi t})|dt\Big)$ is the well known *Mahler measure* of P. If α is an algebraic integer and $\mathrm{M}(\alpha) = 1$, then the

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classical theorem of Kronecker (1857) tells us that α is a root of unity. This raises the question of whether

$$\inf_{\alpha \text{ not a root of unity}} \mathrm{M}(\alpha) > 1,$$

which is known as the *Lehmer's problem* and is still open. The smallest known value is due to Lehmer himself and is $\mathrm{M}(P) = 1.176280\ldots$ where

$$P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

For a complete survey on the Mahler measure, see Smyth (2008).

Now, note that $\mathrm{L}(P) = \mathrm{L}_1(P)$ is the well known *length* of P. In 1994, inspired by the works related to the Mahler measure, the first author (Flammang, 1994) studied the structure of the set $\mathcal{R}$ of the quantities $\mathrm{L}(P)^{1/\deg(P)}$, P irreducible, (namely the *absolute length* of P) in case of P totally positive (i.e., all its roots are positive real numbers) with integer coefficients and leading coefficient equal to 1. Using the principle of auxiliary functions, she found the five smallest points of $\mathcal{R}$ in the interval $(2, 2.361101]$. At last, she showed that $\mathcal{R}$ is dense in $[2.376841\ldots, \infty)$. In 2013, Q. Mu and Q. Wu (2013) extended the description of the spectrum $\mathcal{R}$ to 2.364950, finding no new point in that spectrum. In 2014, the first author (Flammang, 2014) continued this extension to 2.365827 but found again no new point in the spectrum. However, she conjectured that the next point is given by

$$x^{10}-19x^9+143x^8-557x^7+1231x^6-1599x^5+1231x^4-557x^3+143x^2-19x+1$$

with measure 2.366160. Both results (Mu and Wu, 2013, and Flammang, 2014) are based on the use of auxiliary functions, which require finding relevant polynomials in order to construct them. Since the work mentioned above did not lead to the sixth point of the spectrum, the idea came to seek for higher degree totally positive integer polynomials with small length, with a view to using them in the construction of new more efficient auxiliary functions.

Note that exhibiting such polynomials serves a threefold purpose. On the one hand, such polynomials can provide improved auxiliary functions that could lead to the sixth point of the spectrum. On the other hand, drawing up lists of polynomials with small length has its own interest, as does the search for small Mahler measures, for example. Finally, the values of these small lengths may indicate the presence of possible unknown limit points for the absolute length of polynomials defined above. Previously, the first author, Flammang (1994), was able to get exhaustive results from degree 3 to degree 7, but the higher degrees remained out of reach using the techniques available at that time. New methods, based on optimization techniques, have since been developed by the second author so that it is now possible to deal with the problem up to degree 16.

The optimization method used is a linear optimization method, in that the constraints imposed on unknown variables are linear, and the function to be minimized, i.e. the length of the polynomials, can also be expressed as a linear form for the particular polynomials (the sign of their coefficients changes alternately) that we consider. So, we are in a mixed integer programming-based optimization framework, because our problem has the important peculiarity of imposing an integer condition on the variables (the coefficients of unknown polynomials) to be found. We write our problem as a mixed-integer programming problem rather than a pure-integer programming problem, because the coefficients of linear constraints are real numbers. Of course, we could multiply the left and right members of linear constraints to have only integer coefficients, but this is not efficient, because it leads to dealing with very large integers, hence our choice to keep real coefficients in linear constraints.

The main lines of solving a mixed-integer linear problem are as follows (see Rustem, 2000):

1. Solve the continuous linear problem (i.e. ignore integrality).
2. If the optimal variables are all integer then the optimum solution is reached. Otherwise:
3. Generate a *cut* (a constraint) which is satisfied by all integer solutions to the problem but not by the current solution of the linear (continuous) problem.
4. Add this new constraint and go to 1.

In this work, we have extended the list of polynomials with small length up to degree 16, using both the principle of auxiliary functions and our algorithmic method based on mixed-integer linear programming. Again, this did not find the sixth point of the spectrum. Nevertheless, the list obtained is very interesting in itself, in particular because it highlights four previously unknown polynomials with length very close to 2.366160. As in Flammang (1994), we focus on the search for polynomials P of degree d with length less than or equal to 2.3769^d (remember that the known limit point is 2.376841). The first step is to find a lower bound and an upper bound for coefficients a_1 and a_{d-1}. This will allow us to find an interval containing all the roots of P, depending on the values of a_1. The next step is to give precise estimates of lower and upper bounds relative to each coefficient of the sought polynomials. The full details of this preparatory work are given in Section 2.

2. Preparation of the computations

Let P be an irreducible totally positive polynomial with integer coefficients. Since all its roots are positive real numbers, we deduce from the relations between coefficients and roots that P can be written as

$$P = x^d - a_1 x^{d-1} + \cdots + (-1)^{d-1} a_{d-1} x + (-1)^d = (x - \alpha_1) \cdots (x - \alpha_d)$$

where the a_i's are positive integers. An important consequence of the alternative signs in P is that $\mathrm{L}(P) = |P(-1)| = \prod_{i=1}^{d}(\alpha_i + 1)$. We will use this equality in a later demonstration.

2.1. Bounds for a_1 and a_{d-1}

Recall that the *trace* of P is defined as $\mathrm{tr}(P) = \sum_{i=1}^{d} \alpha_i = a_1$. If P^* is the reciprocal polynomial of P (i.e., $P^*(x) = x^d P(1/x)$), then $\mathrm{tr}(P^*) = a_{d-1}$. In Flammang (2016) the first author gave a lower bound for all but finitely totally positive polynomials P with integer coefficients: $\mathrm{tr}(P) \geq 1.792812d$. Hence, we have a lower bound for the coefficients a_1 and a_{d-1}. We then prove the following result:

Proposition 1 *If P is a totally positive polynomial with integer coefficients of degree d, different from $x-1$ then*

$$\mathrm{tr}(P) \leq 5.336355 \log \mathrm{L}(P) - 2.466204d.$$

Proof: The auxiliary function involved here is of the type:

$$\forall x > 0, \;\; f(x) = -x + c_0 \log(x+1) - c_1 \log|x-1|.$$

Thanks to the semi infinite linear programming (introduced into Number Theory by C. J. Smyth, 1984, for more details, see Flammang, 2016), we are able to find c_0 and c_1 such that the minimum m of the function f is as large as possible. We obtain $c_0 = 5.336355$, $c_1 = 1.170114$ and $m = 2.466204$. If $x-1$ does not divide P, then we have

$$\sum_{i=1}^{d} f(\alpha_i) \geq md$$

i.e.,

$$-\mathrm{tr}(P) + c_0 \log \prod_{i=1}^{d}(\alpha_i + 1) \geq c_1 \log \prod_{i=1}^{d} |\alpha_i - 1| + md.$$

Since $x-1$ does not divide P, then $\prod_{i=1}^{d} |\alpha_i - 1|$ is a nonzero integer, because it is the resultant of P and $x-1$. Hence, we have

$$\mathrm{tr}(P) \leq c_0 \log L(P) - md.$$

Now, remember that $\mathrm{L}(P) \leq 2.3769^d$. Thus, we obtain an upper bound for $\mathrm{tr}(P) = a_1$ and $\mathrm{tr}(P^*) = a_{d-1}$. □

2.2. Localization of the roots of the polynomials

Let

$$P = x^d - a_1x^{d-1} + \cdots + (-1)^{d-1}a_{d-1}x + (-1)^d = (x - \alpha_1)\cdots(x - \alpha_d)$$

where the a_i's are positive integers and $\alpha_1 < \alpha_2 < \cdots < \alpha_d = \alpha$ are the roots of P. Suppose that $\mathrm{tr}(P)$ is a fixed integer. The auxiliary function f, introduced in Flammang (2016), is such that $f(x) = x - \sum_{i=1}^{79} c_j \log |Q_j(x)| \geq 1.792812$ for $x > 0$, where the coefficients c_j are positive real numbers and the Q_j's are polynomials with integer coefficients. The c_j's and the Q_j's are explicitly given in Flammang (2016). It follows that $\mathrm{tr}(P) \geq f(\alpha) + 1.792812(d-1)$. Now let A be the point, from which the function f is increasing. We seek $B > A$ such that, if $\alpha \geq B$ then $f(\alpha) + 1.792812(d-1) > \mathrm{tr}(P)$, which is fixed, as stated above. Then we have $\alpha < B$ and the roots of P lie in $(0, B)$. This interval depends on the degree and trace of the polynomials.

2.3. Global bounds for the coefficients of the polynomials

In this subsection, we explain how to give precise estimates for the upper and lower bounds relative to each coefficient of a polynomial for a given degree and trace.

For a given degree d and a given trace t, the estimation method consists of sampling at random (from a uniform distribution) several million sets of d real numbers contained in the interval $(0, B)$ of the previous subsection, and reconstructing for each of these sets the polynomial with real coefficients, whose roots are the d real numbers it contains. We continue sampling until we obtain 100,000 polynomials with real coefficients whose traces are in a neighborhood very close to t. For each coefficient, we thus have 100,000 real values coming from the 100,000 selected polynomials, and we give an interval containing 99.9% of these values. This provides very good estimates of lower and upper bounds for each coefficient of the sought polynomials. More details are given in Otmani et al. (2014).

3. Formulation as an optimization problem

The method, which allows to express the search for polynomial with small length as a mixed-integer programming problem is in the spirit of the method, devoted to a problem of a similar nature, which is explained in detail in Otmani et al. (2014). We will therefore only give the broad lines in this section. Suppose we are looking for polynomials of degree d. The basic idea is that these polynomials, which have only real roots, change sign d times, and are strictly positive or strictly negative between their roots. We translate this property in terms of linear constraints in the following way: we sample at random (uniform distribution) d real numbers in the interval $(0, B)$, corresponding to the degree and the trace which interest us. These real numbers are sorted in ascending order, and we impose on the sought polynomial to change alternately sign for each of these real numbers. This provides inequalities involving integer coefficients (i.e. unknown variables) to be found. We add new constraints using the intervals found in Section 2.3, which restrict the possible values for these unknown integer coefficients. Finally, with

$$P = x^d - a_1 x^{d-1} + \cdots + (-1)^{d-1} a_{d-1} x + (-1)^d,$$

we minimize the linear form

$$(a_{d-1}, \cdots, a_0) \to -a_{d-1} + a_{d-2} - \cdots + (-1)^d$$

with the constraints previously mentioned. It is not sure that for the given constraints, the mixed-integer programming problem has a solution. If there is one, it remains to check the irreducibility of the found polynomial, which is done using the Pari library (see Pari/GP, 2016).

In summary, the search algorithm is as follows:

1. Randomly sample d real numbers in $(0, B)$.
2. Solve the mixed-integer programming problem by using the randomly sampled real numbers to construct linear constraints.
3. If a solution is found, keep it if the corresponding polynomial is irreducible in $\mathbb{Z}[X]$.
4. Back to 1.

4. Implementation and results

For the implementation of our mixed-integer programming algorithm, we used the Matlab Optimization toolbox, which, in particular, allowed us to deal with the fact that unknown coefficients are integers. The most difficult polynomials to find, in the case where d is an even integer, are non-reciprocal polynomials, since reciprocal polynomials can be obtained from half-degree polynomials via the change of variables $x \to x + \dfrac{1}{x} - 2$. When d is an odd integer, this question does not arise, and the polynomials found are all non-reciprocal.

In the following lines, we list by degree the reciprocal and non-reciprocal polynomials of degree d, whose length is less than 2.3769^d.

Rather than giving a long list of polynomials, we provide for each degree, the reciprocal and the non-reciprocal polynomials of smallest lengths (in the case of even degrees), and the non-reciprocal polynomial of smallest length (in the case of odd degrees) found by our algorithm. The full list is available at http://www.iecl.univ-lorraine.fr/ Jean-Marc.Sac-Epee/SmallLength.html

- **Degree 8**: $2.3769^8 = 1018.79\ldots$

  Reciprocal polynomial

  Length = 941: $x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$

  Non-reciprocal polynomial

  Length = 999: $x^8-16x^7+91x^6-240x^5+323x^4-228x^3+84x^2-15x+1$

- **Degree 9**: $2.3769^9 = 2421.57\ldots$

  Length = 2389: $x^9-17x^8+111x^7-366x^6+668x^5-690x^4+396x^3-121x^2+18x-1$

- **Degree 10**: $2.3769^{10} = 5755.84\ldots$

  Reciprocal polynomial

  Length = 5501: $x^{10}-19x^9+143x^8-557x^7+1231x^6-1599x^5+1231x^4-557x^3+143x^2-19x+1$

  Non-reciprocal polynomial

  Length = 5741: $x^{10}-19x^9+144x^8-567x^7+1269x^6-1670x^5+1302x^4-595x^3+153x^2-20x+1$

- **Degree 11**: $2.3769^{11} = 13681.06\ldots$

  Length = 13683: $x^{11}-21x^{10}+178x^9-805x^8+2150x^7-3535x^6+3618x^5-2287x^4+873x^3-192x^2+22x-1$

- **Degree 12**: $2.3769^{12} = 32518.53\ldots$

  Reciprocal polynomial

  Length = 31169: $x^{12}-23x^{11}+218x^{10}-1118x^9+3438x^8-6651x^7+8271x^6-6651x^5+3438x^4-1118x^3+218x^2-23x+1$

  Non-reciprocal polynomial

  Length = 31703: $x^{12}-23x^{11}+219x^{10}-1132x^9+3506x^8-6802x^7+8441x^6-6750x^5+3466x^4-1121x^3+218x^2-23x+1$

- **Degree 13**: $2.3769^{13} = 77293.29\ldots$

  Length = 74839: $x^{13}-24x^{12}+245x^{11}-1401x^{10}+4980x^9-11549x^8+17848x^7-18464x^6+12694x^5-5702x^4+1625x^3-279x^2+26x-1$

- **Degree 14**: $2.3769^{14} = 183718.43\ldots$

  Reciprocal polynomial

  Length = 175449: $x^{14}-27x^{13}+308x^{12}-1963x^{11}+7790x^{10}-20307x^9+35763x^8-43131x^7+35763x^6-20307x^5+7790x^4-1963x^3+308x^2-27x+1$

Non-reciprocal polynomial

Length = 179271: $x^{14} - 27x^{13} + 310x^{12} - 1993x^{11} + 7969x^{10} - 20864x^9 + 36760x^8 - 44197x^7 + 36446x^6 - 20561x^5 + 7840x^4 - 1967x^3 + 308x^2 - 27x + 1$

- **Degree 15**: $2.3769^{15} = 436680.36\ldots$

  Length = 414157: $x^{15} - 28x^{14} + 339x^{13} - 2349x^{12} + 10389x^{11} - 30960x^{10} + 63969x^9 - 92910x^8 + 95232x^7 - 68646x^6 + 34443x^5 - 11805x^4 + 2676x^3 - 379x^2 + 30x - 1$

- **Degree 16**: $2.3769^{16} = 1037945.54\ldots$

  Reciprocal polynomial

  Length = 969581: $x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10} - 189036x^9 + 222621x^8 - 189036x^7 + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3 + 413x^2 - 31x + 1$

  Non-reciprocal polynomial

  Length = 1024159: $x^{16} - 31x^{15} + 414x^{14} - 3164x^{13} + 15484x^{12} - 51391x^{11} + 119441x^{10} - 197861x^9 + 235603x^8 - 202018x^7 + 124235x^6 - 54218x^5 + 16465x^4 - 3364x^3 + 436x^2 - 32x + 1$

5. Remarks

In conclusion of this work, we note that the reciprocal polynomials have smaller lengths than the non-reciprocal polynomials, the fact that we do not know how to explain at the moment.

Another very interesting and rather promising observation is relative to the absolute length of the polynomials found, which we can calculate from the lengths given in our list. Indeed, it appears that four absolute lengths are very close to 2.366160, namely 2.366160..., 2.366799..., 2.368519... and 2.368523..., which gives some consistency to conjecture of the existence of a limit point smaller than the known limit point 2.376841.

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