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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

ON SOME CONVERGENCE RESULTS FOR FRACTIONAL PERIODIC SOBOLEV SPACES

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Abstract. In this note we extend the well-known limiting formulas due to Bourgain–Brezis–Mironescu and Maz'ya–Shaposhnikova, to the setting of fractional Sobolev spaces on the torus. We also give a Γ-convergence result in the spirit of Ponce. The main theorems are obtained by using the nice structure of Fourier series.

Keywords: fractional periodic Sobolev spaces, Fourier series, Γ-convergence.

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1. INTRODUCTION

Let $s>0$ and $N\geq 1$, and denote by $\mathbb{T}^N$ the flat torus $\mathbb{R}^N/\mathbb{Z}^N$. We define the periodic Sobolev spaces on $\mathbb{T}^N$ (see [18, 19]) as follows

$$H^s(\mathbb{T}^N)=\left\{u=\sum_{k\in\mathbb{Z}^N} c_k\frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}}\in L^2(\mathbb{T}^N):[u]_s^2=\sum_{k\in\mathbb{Z}^N}|k|^{2s}|c_k|^2<+\infty\right\},$$

endowed with the norm

$$\|u\|_s^2=\sum_{k\in\mathbb{Z}^N}(1+|k|^{2s})|c_k|^2=\|u\|^2_{L^2(\mathbb{T}^N)}+[u]_s^2,$$

where

$$c_k=\frac{1}{\sqrt{(2\pi)^N}}\int\limits_{\mathbb{T}^N} u(x)e^{-\imath k\cdot x}\,dx \quad (k\in\mathbb{Z}^N)$$

are the Fourier coefficients of u. Clearly, $H^s(\mathbb{T}^N)$ is a Hilbert space with inner product

$$(u,v)_s=\sum_{k\in\mathbb{Z}^N}(1+|k|^{2s})c_k\bar{d}_k=(u,v)_{L^2(\mathbb{T}^N)}+\sum_{k\in\mathbb{Z}^N}|k|^{2s}c_k\bar{d}_k$$

for all $u,v\in H^s(\mathbb{T}^N)$, where c_k and d_k are the Fourier coefficients of u and v, respectively.

We also introduce the following subspace of $H^s(\mathbb{T}^N)$

$$\dot{H}^s(\mathbb{T}^N)=\left\{u=\sum_{k\in\mathbb{Z}^N} c_k\frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}}\in H^s(\mathbb{T}^N): c_0=0\right\}.$$

For any

$$u=\sum_{k\in\mathbb{Z}^N} c_k\frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}}\in \dot{H}^s(\mathbb{T}^N)$$

we have the following Poincaré inequality

$$\|u\|^2_{L^2(\mathbb{T}^N)}=\sum_{k\in\mathbb{Z}^N}|c_k|^2\leq [u]^2_s,$$

from which we deduce that $[u]_s$ is a norm in $\dot{H}^s(\mathbb{T}^N)$ and equivalent to the standard one in $H^s(\mathbb{T}^N)$.

Our first main result can be stated as follows:

Theorem 1.1. *Let $s\in(0,1)$ and $N\geq 1$. Then, for all $u\in\dot{H}^1(\mathbb{T}^N)$ it holds*

$$\lim_{s\to 1^-}[u]^2_s=[u]^2_1$$

and

$$\lim_{s\to 0^+}[u]^2_s=\|u\|^2_{L^2(\mathbb{T}^N)}.$$

The above convergence results can be seen in some sense as an extension in the Hilbertian periodic setting of the main results for $W^{s,p}$-Sobolev spaces (see [9, 15] for their definitions and properties) due to Bourgain–Brezis–Mironescu [5] and Maz'ya–Shaposhnikova [13] who proved, respectively, that for a fixed $p\in[1,\infty)$

$$\lim_{s\to 1^-}(1-s)\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy=\begin{cases} A_{N,p}\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p dx & \text{if } u\in W^{1,p}(\mathbb{R}^N),\\ A_{1,p}\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p dx & \text{if } u\in W^{1,1}(\mathbb{R}^N),\end{cases} \tag{1.1}$$

and

$$\lim_{s\to 0^+} s\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy=B_{N,p}\int_{\mathbb{R}^N}|u|^p dx \quad \text{if } u\in\bigcup_{0<s<1}W^{s,p}(\mathbb{R}^N), \tag{1.2}$$

for some suitable positive constant $A_{N,p}$ and $B_{N,p}$ depending only on N and p, and $A_{1,p}$ depending only on p. Later, the above results have been extended to Besov spaces $B^s_{p,\theta}(\mathbb{R}^N)$ by Kolyada and Lerner [10], and Milman [14] generalized (1.1) and (1.2) to the setting of interpolation spaces, by establishing continuity of the real and complex interpolation spaces at the endpoints.

Moreover, motivated by [5], Ponce [16] established the following Γ-convergence result:

$$\Gamma - \lim_{s\to 1^-} (1-s) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy = \begin{cases} \int\limits_{\mathbb{R}^N} |\nabla u|^p dx & \text{if } u \in W^{1,p}(\mathbb{R}^N), \\ |Du|(\mathbb{R}^N) & \text{if } u \in BV(\mathbb{R}^N). \end{cases} \tag{1.3}$$

In the spirit of [16], we also have the following second main result of this work:

Theorem 1.2. *Let $s\in(0,1)$ and $N\geq 1$. Then, for all $u\in \dot{H}^1(\mathbb{T}^N)$ it holds*

$$\Gamma - \lim_{s\to 1^-} [u]_s^2 = [u]_1^2.$$

We note that in Theorem 1.1 and Theorem 1.2 the absence of the terms $(1-s)$ and s is due to the choice of the norm in $H^s(\mathbb{T}^N)$. Indeed, in the case $p=2$, it is well-known that for all $u\in\mathcal{S}(\mathbb{R}^N)$, where $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decaying functions, it holds

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy = 2C_{N,s}^{-1} \int\limits_{\mathbb{R}^N} |\zeta|^{2s} |\mathcal{F}u(\zeta)|^2 d\zeta,$$

where $\mathcal{F}u$ denotes the Fourier transform of u,

$$C_{N,s} = \left(\int\limits_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1},$$

and

$$\lim_{s\to 1^-} \frac{C_{N,s}}{s(1-s)} = \frac{4N}{\omega_{N-1}} \quad \text{and} \quad \lim_{s\to 0^+} \frac{C_{N,s}}{s(1-s)} = \frac{2}{\omega_{N-1}},$$

where ω_{N-1} stands for the $(N-1)$-dimensional measure of the unit sphere $\mathbb{S}^{N-1}$; see [9] for more details. We recall that the equivalence of the $H^s(\mathbb{R}^N)$ norms in terms of Fourier transforms and integrals of differences of functions involving a weight has been proved in [12].

Therefore, when $p=2$, (1.1), (1.2) and (1.3) can be easily deduced from

$$\lim_{s\to 1^-} \int\limits_{\mathbb{R}^N} |\zeta|^{2s} |\mathcal{F}u(\zeta)|^2 d\zeta = \int\limits_{\mathbb{R}^N} |\zeta|^{2} |\mathcal{F}u(\zeta)|^2 d\zeta, \tag{1.4}$$

$$\lim_{s\to 0^+} \int\limits_{\mathbb{R}^N} |\zeta|^{2s} |\mathcal{F}u(\zeta)|^2 d\zeta = \int\limits_{\mathbb{R}^N} |\mathcal{F}u(\zeta)|^2 d\zeta, \tag{1.5}$$

and

$$\Gamma - \lim_{s\to 1^-} \int\limits_{\mathbb{R}^N} |\zeta|^{2s} |\mathcal{F}u(\zeta)|^2 d\zeta = \int\limits_{\mathbb{R}^N} |\mathcal{F}u(\zeta)|^2 d\zeta. \tag{1.6}$$

On the other hand, in [4] the authors showed that for all $u \in \dot{H}^s(\mathbb{T}^N)$, with $s \in (0,1)$, it holds

$$[u]_s^2 \sim D_{N,s} \iint\limits_{\mathbb{T}^N \times [-\frac{1}{2}, \frac{1}{2})^N} \frac{|u(x+y) - u(y)|^2}{|x|^{N+2s}} dxdy,$$

for some not explicit positive constant $D_{N,s}$ depending only on N and s.

Motivated by the above facts, the aim of the present paper is to prove that (1.4), (1.5) and (1.6) still hold in the periodic framework. We emphasize that the proofs of Theorem 1.1 and 1.2 are completely different from the ones obtained in [5, 13] (see also [14]) and [16]. Indeed, we only take advantage of the beautiful Fourier series expressions of the norms in the periodic Sobolev space $H^s(\mathbb{T}^N)$ to obtain some useful estimates needed to achieve our main results. Moreover, we believe that the proofs given here can be easily read by a wide audience which has just basic notions about Fourier series.

2. PROOF OF THEOREM 1.1

In this section we give the proof of the first main result of this work.
Let $u \in \dot{H}^1(\mathbb{T}^N)$. Then,

$$u = \sum_{k \in \mathbb{Z}^N} c_k \frac{e^{\imath k \cdot x}}{\sqrt{(2\pi)^N}},$$

where

$$c_k = \frac{1}{\sqrt{(2\pi)^N}} \int\limits_{\mathbb{T}^N} u(x) e^{-\imath k \cdot x}\, dx \quad (k \in \mathbb{Z}^N).$$

Firstly, we aim to prove that

$$\lim_{s \to 1^-} \left(\sum_{k \in \mathbb{Z}^N} |k|^{2s} |c_k|^2 \right) = \sum_{k \in \mathbb{Z}^N} |k|^2 |c_k|^2. \tag{2.1}$$

For this reason we look for

$$\left| \sum_{k \in \mathbb{Z}^N} (|k|^{2s} - |k|^2) |c_k|^2 \right|.$$

Let us note that for all $M > 1$ we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^N} ||k|^{2s} - |k|^2||c_k|^2 &= \sum_{k \in \mathbb{Z}^N} ||k|^{2s} - |k|^2||c_k|^2 \\ &= \sum_{1 \le |k| \le M} ||k|^{2s} - |k|^2||c_k|^2 + \sum_{|k| > M} ||k|^{2s} - |k|^2||c_k|^2, \end{aligned}$$

where in the first equality we used the fact that $c_0 = 0$.

Recalling that $s \in (0,1)$, it is clear that

$$||k|^{2s} - |k|^2| = |k|^2 - |k|^{2s} \le |k|^2 \quad \forall |k| \ge 1,$$

so we get

$$\sum_{k \in \mathbb{Z}^N} ||k|^{2s} - |k|^2||c_k|^2 \le \sum_{1 \le |k| \le M} ||k|^{2s} - |k|^2||c_k|^2 + \sum_{|k|>M} |k|^2 |c_k|^2. \tag{2.2}$$

Since the first term on the right hand side in (2.2) is the sum of a finite number of terms, we easily deduce that

$$\lim_{s \to 1^-} \left(\sum_{1 \le |k| \le M} ||k|^{2s} - |k|^2||c_k|^2 \right) = 0.$$

Therefore,

$$\limsup_{s \to 1^-} \left(\sum_{k \in \mathbb{Z}^N} ||k|^{2s} - |k|^2||c_k|^2 \right) \le \sum_{|k|>M} |k|^2 |c_k|^2. \tag{2.3}$$

Taking into account that $u \in \dot{H}^1(\mathbb{T}^N)$ implies that

$$\sum_{k \in \mathbb{Z}^N} |k|^2 |c_k|^2 < +\infty,$$

we obtain that

$$\lim_{M \to +\infty} \left(\sum_{|k|>M} |k|^2 |c_k|^2 \right) = 0. \tag{2.4}$$

Putting together (2.2), (2.3) and (2.4) we see that

$$\limsup_{s \to 1^-} \left| \sum_{k \in \mathbb{Z}^N} (|k|^{2s} - |k|^2)|c_k|^2 \right| \le \limsup_{s \to 1^-} \left(\sum_{k \in \mathbb{Z}^N} ||k|^{2s} - |k|^2||c_k|^2 \right) = 0$$

and this ends the proof of (2.1).

Secondly, we prove that

$$\lim_{s \to 0^+} \left(\sum_{k \in \mathbb{Z}^N} |k|^{2s} |c_k|^2 \right) = \sum_{k \in \mathbb{Z}^N} |c_k|^2. \tag{2.5}$$

Then, for all $M > 1$ it holds

$$\sum_{k \in \mathbb{Z}^N} ||k|^{2s} - 1||c_k|^2 = \sum_{1 \le |k| \le M} ||k|^{2s} - 1||c_k|^2 + \sum_{|k|>M} ||k|^{2s} - 1||c_k|^2.$$

Noting that
$$||k|^{2s}-1|=|k|^{2s}-1\leq |k|^{2s}\leq |k|^2 \quad \forall |k|\geq 1,$$
we get
$$\sum_{k\in\mathbb{Z}^N} ||k|^{2s}-1||c_k|^2 \leq \sum_{1\leq |k|\leq M} ||k|^{2s}-1||c_k|^2+\sum_{|k|>M}|k|^2|c_k|^2. \tag{2.6}$$
As before,
$$\lim_{s\to 0^+}\left(\sum_{1\leq |k|\leq M} ||k|^{2s}-1||c_k|^2\right)=0,$$
and consequently
$$\limsup_{s\to 0^+}\left(\sum_{k\in\mathbb{Z}^N} ||k|^{2s}-1||c_k|^2\right)\leq \sum_{|k|>M}|k|^2|c_k|^2. \tag{2.7}$$
Letting $M\to+\infty$ we find
$$\limsup_{s\to 0^+}\left|\sum_{k\in\mathbb{Z}^N}(|k|^{2s}-1)|c_k|^2\right|\leq \limsup_{s\to 0^+}\left(\sum_{k\in\mathbb{Z}^N} ||k|^{2s}-1||c_k|^2\right)=0$$
and this shows that (2.5) is verified.

3. PROOF OF THEOREM 1.2

This last section is devoted to the our Γ-convergence result. For more details on this topic we refer the interested reader to [6–8].

Let $(s_n)\subset(0,1)$ be a sequence such that $s_n\to 1^-$ as $n\to+\infty$, and we introduce the following functionals $\mathcal{J}_{s_n}:L^2(\mathbb{T}^N)\to[0,\infty]$ and $\mathcal{J}_1:L^2(\mathbb{T}^N)\to[0,\infty]$ defined as
$$\mathcal{J}_{s_n}(u)=\begin{cases}[u]^2_{s_n} & \text{if } u\in \dot{H}^{s_n}(\mathbb{T}^N),\\ +\infty & \text{otherwise,}\end{cases}$$
and
$$\mathcal{J}_1(u)=\begin{cases}[u]^2_1 & \text{if } u\in \dot{H}^1(\mathbb{T}^N),\\ +\infty & \text{otherwise.}\end{cases}$$
We divide the proof of Theorem 1.2 into three steps.

Step 1. Let $(u_{s_n})\subset \dot{H}^{s_n}(\mathbb{T}^N)$ be such that $[u_{s_n}]^2_{s_n}\leq C$ for all $n\in\mathbb{N}$. Then there exists a subsequence $(u_{s_{n_j}})$ that weakly converges in $H^{1-\varepsilon}(\mathbb{T}^N)$ for all $\varepsilon>0$ and strongly in $L^2(\mathbb{T}^N)$.

Denoting by $c_k^{s_n}$ the Fourier coefficients of u_{s_n}, we have

$$\sum_{|k|\geq 1} |k|^{2s_n}|c_k^{s_n}|^2 = \sum_{k\in\mathbb{Z}^N} |k|^{2s_n}|c_k^{s_n}|^2 \leq C \quad \forall n\in\mathbb{N}.$$

Since $s_n \to 1^-$, for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $1-\varepsilon < s_n$ for all $n \geq n_\varepsilon$. Therefore, for all $n \geq n_\varepsilon$

$$\sum_{|k|\geq 1} |k|^{2(1-\varepsilon)}|c_k^{s_n}|^2 = \sum_{|k|\geq 1} |k|^{2s_n}|c_k^{s_n}|^2 \leq C$$

which implies that (u_{s_n}) is bounded in $H^{1-\varepsilon}(\mathbb{T}^N)$ for all $\varepsilon > 0$. Now, we prove that $H^\alpha(\mathbb{T}^N)$, with $\alpha \in (0,1]$, is compactly embedded into $L^2(\mathbb{T}^N)$. Assume that $u_j \rightharpoonup 0$ in $H^\alpha(\mathbb{T}^N)$ as $j \to +\infty$. Accordingly, we see that

$$\lim_{j\to\infty} |c_k^j|^2(|k|^2+1)^\alpha = 0 \quad \forall k \in \mathbb{Z}^N, \tag{3.1}$$

and

$$\sum_{k\in\mathbb{Z}^N} |c_k^j|^2(|k|^2+1)^\alpha \leq C \quad \forall j \in \mathbb{N}. \tag{3.2}$$

where c_k^j are the Fourier coefficients of u_j. Fix $\varepsilon > 0$. Then there exists $\nu_\varepsilon > 0$ such that

$$(|k|^2+1)^{-\alpha} < \varepsilon \text{ for } |k| > \nu_\varepsilon.$$

By (3.2), we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} |c_k^j|^2 &= \sum_{|k|\leq\nu_\varepsilon} |c_k^j|^2 + \sum_{|k|>\nu_\varepsilon} |c_k^j|^2 \\ &= \sum_{|k|\leq\nu_\varepsilon} |c_k^j|^2 + \sum_{|k|>\nu_\varepsilon} |c_k^j|^2(|k|^2+1)^\alpha(|k|^2+1)^{-\alpha} \\ &\leq \sum_{|k|\leq\nu_\varepsilon} |c_k^j|^2 + C\varepsilon.\end{aligned}$$

It follows from (3.1) that

$$\sum_{|k|\leq\nu_\varepsilon} |c_k^j|^2 < \varepsilon \text{ for } j \text{ large},$$

and thus $u_j \to 0$ in $L^2(\mathbb{T}^N)$ as $j \to +\infty$. This proves the desired compact embedding.

Therefore, $u_{s_n} \rightharpoonup u$ in $H^{1-\varepsilon}(\mathbb{T}^N)$ for all $\varepsilon > 0$ and $u_{s_n} \to u$ in $L^2(\mathbb{T}^N)$. Moreover, it holds that $u \in \dot{H}^1(\mathbb{T}^N)$.

Step 2. Let $(u_{s_n}) \subset \dot{H}^{s_n}(\mathbb{T}^N)$ be such that $u_{s_n} \rightharpoonup u$ in $H^{1-\varepsilon}(\mathbb{T}^N)$ for all $\varepsilon > 0$ and $u_{s_n} \to u$ in $L^2(\mathbb{T}^N)$. Then

$$\liminf_{n\to+\infty} [u_{s_n}]_{s_n}^2 \geq [u]_1^2.$$

Denote by c_k, with $k \in \mathbb{Z}^N$, the Fourier coefficients of u. We observe that the strong convergence in $L^2(\mathbb{T}^N)$ implies that

$$|c_k^{s_n}| \to |c_k| \text{ as } n \to \infty \quad \forall k \in \mathbb{Z}^N.$$

Indeed, for all $h \in \mathbb{Z}^N$, we obtain

$$\begin{aligned}||c_h^{s_n}| - |c_h||^2 &\leq \sum_{k\in\mathbb{Z}^N} ||c_k^{s_n}| - |c_k||^2 \\ &\leq \sum_{k\in\mathbb{Z}^N} |c_k^{s_n} - c_k|^2 \to 0 \text{ as } n \to +\infty.\end{aligned}$$

Then, using the above information and Fatou's Lemma we get

$$\begin{aligned}\liminf_{n\to+\infty} [u_{s_n}]_{s_n}^2 &= \liminf_{n\to+\infty} \left(\sum_{|k|\geq 1} |k|^{2s_n} |c_k^{s_n}|^2 \right) \\ &\geq \sum_{|k|\geq 1} |k|^2 |c_k|^2 = [u]_1^2.\end{aligned}$$

Step 3. It holds

$$\Gamma - \limsup_{n\to+\infty} \mathcal{J}_{s_n} \leq \mathcal{J}_1.$$

Let

$$u = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}} \in \dot{H}^1(\mathbb{T}^N)$$

and consider the constant sequence $v_n = u$ for all $n \in \mathbb{N}$. Since $c_0 = 0$ and $0 < s_n < 1$ for all $n \in \mathbb{N}$, we infer that

$$\begin{aligned}\limsup_{n\to+\infty} [v_n]_{s_n}^2 &= \limsup_{n\to+\infty} [u]_{s_n}^2 \\ &= \limsup_{n\to+\infty} \left(\sum_{|k|\geq 1} |k|^{2s_n} |c_k|^2 \right) \\ &\leq \limsup_{n\to+\infty} \left(\sum_{|k|\geq 1} |k|^2 |c_k|^2 \right) \\ &= [u]_1^2.\end{aligned}$$

Collecting Step 1, Step 2 and Step 3 we conclude that $\Gamma - \lim_{s\to 1^-} \mathcal{J}_s = \mathcal{J}_1$.

Remark 3.1. The above approach can be easily modified to prove the same convergence results for the interpolation spaces considered by Lions and Magenes [11]:

$$\mathbb{H}^s(\Omega) = \left\{ u \in L^2(\Omega) : \|u\|^2 = \sum_{j=1}^{\infty} \mu_j^s \phi_j^2 < +\infty \right\}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, (ϕ_j) denotes an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions, associated with the eigenvalues (μ_j). Clearly, similar arguments hold when $\mathbb{T}^N$ is replaced by any compact manifold without boundary.

4. APPENDIX: SOBOLEV INEQUALITY ON $\mathbb{T}^N$

In this appendix we deal with the fractional Sobolev inequality on the torus. It is well-known that the fractional Sobolev spaces $H^s(\mathcal{M})$, where $\mathcal{M}$ is a closed N-dimensional manifold of class C^∞, can be defined by use of local coordinates and partition of unity (see [1, 11, 20]). Thus the fractional Sobolev embedding on $\mathcal{M}$ can be easily deduced by the fractional Sobolev embedding in $\mathbb{R}^N$. Anyway, in this paper we prefer to give a proof of the fractional Sobolev embedding in $\mathbb{T}^N$ by using only Fourier series.

We start with the following Hardy–Littlewood–Sobolev type inequality on $\mathbb{T}^N$.

Lemma 4.1. *Let $s \in (0,1)$ and $N > 2s$. Then there exists a constant $C = C(N,s) > 0$ such that*

$$\left(\sum_{k\in\mathbb{Z}^N} (1+|k|^2)^{-s}|d_k|^2\right)^{\frac{1}{2}} = \|(1-\Delta)^{-\frac{s}{2}}v\|_{L^2(\mathbb{T}^N)} \le C\|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)}, \tag{4.1}$$

for any $v \in L^{\frac{2N}{N+2s}}(\mathbb{T}^N)$, where d_k are the Fourier coefficients of v. Here the Bessel operator $(1-\Delta)^{-\frac{s}{2}}$ is given by (for $v \in C^\infty(\mathbb{T}^N)$)

$$(1-\Delta)^{-\frac{s}{2}}v(x) = \sum_{k\in\mathbb{Z}^N} \frac{d_k}{(1+|k|^2)^{\frac{s}{2}}} \frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}}.$$

Proof. Assume that $v \in C^\infty(\mathbb{T}^N)$, $v \not\equiv 0$, and fix $p \in [1, \frac{N}{s})$. Define

$$e^{t(\Delta-1)}v(x) = \sum_{k\in\mathbb{Z}^N} e^{-t}e^{-t|k|^2} d_k \frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}} = \int_{\mathbb{T}^N} W_t(x-y)v(y)\,dy \tag{4.2}$$

with

$$W_t(x) = \sum_{k\in\mathbb{Z}^N} e^{-t}e^{-t|k|^2}\frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}} = \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{k\in\mathbb{Z}^N} e^{-t}e^{-\frac{|x+2k\pi|^2}{4t}} \ge 0, \tag{4.3}$$

where in the second equality we used the Poisson summation formula [17]. Taking into account

$$\int_{\mathbb{R}^N} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{N}{2}}}\,dy = 1 \quad \forall x \in \mathbb{R}^N, \forall t > 0,$$

and (4.3), we have that

$$\begin{aligned}\int_{\mathbb{T}^N} W_t(x-y)\,dy &= \sum_{k\in\mathbb{Z}^N} \int_{[-\pi,\pi)^N-\{2k\pi\}} e^{-t}\frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{N}{2}}}\,dy \\ &= \int_{\mathbb{R}^N} e^{-t}\frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{N}{2}}}\,dy = e^{-t}\leq 1 \quad \forall x\in\mathbb{T}^N, \forall t>0.\end{aligned} \tag{4.4}$$

Then (4.2) and (4.4) yield

$$\left\|e^{t(\Delta-1)}v\right\|_{L^p(\mathbb{T}^N)} \leq e^{-t}\|v\|_{L^p(\mathbb{T}^N)} \quad \forall t>0. \tag{4.5}$$

On the other hand, in view of integral test for series, we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}^N} e^{-t|k|^2}e^{-t} = e^{-t}\Big(\sum_{k\in\mathbb{Z}} e^{-t|k|^2}\Big)^N &= e^{-t}\Big(1+2\sum_{k\in\mathbb{N}} e^{-tk^2}\Big)^N \\ &\leq e^{-t}\Big(1+2\int_1^{+\infty} e^{-tx^2}\,dx\Big)^N \\ &= e^{-t}\Big(1+\frac{2}{\sqrt{t}}\int_{\sqrt{t}}^{+\infty} e^{-x^2}\,dx\Big)^N \\ &\leq e^{-t}\Big(1+\frac{2}{\sqrt{t}}\int_0^{+\infty} e^{-x^2}\,dx\Big)^N \\ &= e^{-t}\Big(1+\sqrt{\frac{\pi}{t}}\Big)^N \\ &\leq Ce^{-t}+\frac{C}{t^{\frac{N}{2}}} \\ &\leq \frac{C}{t^{\frac{N}{2}}} \quad \forall t>0\end{aligned}$$

where the constants C depend only on N. The above estimate together with (4.2) and (4.3) implies that

$$\left\|e^{t(\Delta-1)}v\right\|_{L^\infty(\mathbb{T}^N)} \leq Ct^{-\frac{N}{2p}}\|v\|_{L^p(\mathbb{T}^N)} \quad \forall t>0. \tag{4.6}$$

Now we observe that (see [18])

$$(1-\Delta)^{-\frac{s}{2}}v(x) = \frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^{+\infty} t^{\frac{s}{2}-1}e^{t(\Delta-1)}v(x)\,dt,$$

and that

$$\|(1-\Delta)^{-\frac{s}{2}}v\|_{L^r(\mathbb{T}^N)} \leq \|v\|_{L^r(\mathbb{T}^N)} \text{ for all } r\in[1,+\infty]. \tag{4.7}$$

Let $T>0$ to be chosen later. Thus, we write

$$(1-\Delta)^{-\frac{s}{2}}v(x) = G_T(x) + H_T(x),$$

where

$$G_T(x) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^T t^{\frac{s}{2}-1} e^{t(\Delta-1)} v(x)\,dt,$$

and

$$H_T(x) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_T^{+\infty} t^{\frac{s}{2}-1} e^{t(\Delta-1)} v(x)\,dt.$$

By (4.6), we have that

$$\begin{aligned}\|H_T\|_{L^\infty(\mathbb{T}^N)} &\leq \frac{C}{\Gamma\left(\frac{s}{2}\right)} \int_T^{+\infty} t^{\frac{s}{2}-1-\frac{N}{2p}} \|v\|_{L^p(\mathbb{T}^N)}\,dt \\ &= C\|v\|_{L^p(\mathbb{T}^N)} T^{\frac{s}{2}-\frac{N}{2p}}.\end{aligned}$$

Given $\lambda>0$, we take T such that

$$\frac{\lambda}{2} = C\|v\|_{L^p(\mathbb{T}^N)} T^{\frac{s}{2}-\frac{N}{2p}}. \tag{4.8}$$

Hence,

$$\begin{aligned}\left|\left\{x\in\mathbb{T}^N \,:\, |(1-\Delta)^{-\frac{s}{2}}v(x)| \geq \lambda\right\}\right| &\leq \left|\left\{x\in\mathbb{T}^N \,:\, |G_T(x)| \geq \frac{\lambda}{2}\right\}\right| \\ &\leq \left(\frac{\lambda}{2}\right)^{-p} \|G_T\|^p_{L^p(\mathbb{T}^N)} \\ &\leq \left(\frac{\lambda}{2}\right)^{-p} \left[\frac{2}{s\Gamma\left(\frac{s}{2}\right)} T^{\frac{s}{2}} \|v\|_{L^p(\mathbb{T}^N)}\right]^p,\end{aligned}$$

where in the last inequality we used Minkowski's integral inequality and (4.5). In view of (4.8) we obtain

$$\left|\left\{x\in\mathbb{T}^N \,:\, |(1-\Delta)^{-\frac{s}{2}}v(x)| \geq \lambda\right\}\right| \leq C\lambda^{-q}\|v\|^q_{L^p(\mathbb{T}^N)}, \tag{4.9}$$

where $q=\frac{Np}{N-sp}$.

Since $C^\infty(\mathbb{T}^N)$ is dense in $L^p(\mathbb{T}^N)$, we can see that if $v \in L^p(\mathbb{T}^N)$ then there exists a sequence $(v_n) \subset C^\infty(\mathbb{T}^N)$ such that $v_n \to v$ in $L^p(\mathbb{T}^N)$. By (4.7), we deduce that

$$(1-\Delta)^{-\frac{s}{2}} v_n \to (1-\Delta)^{-\frac{s}{2}} v \text{ in } L^p(\mathbb{T}^N)$$

which gives

$$\left|\left\{x \in \mathbb{T}^N : |(1-\Delta)^{-\frac{s}{2}} v(x)| \geq \lambda\right\}\right| \leq \liminf_{n\to+\infty} \left|\left\{x \in \mathbb{T}^N : |(1-\Delta)^{-\frac{s}{2}} v_n(x)| \geq \lambda\right\}\right|.$$

Then we have that (4.9) is verified for any $v \in L^p(\mathbb{T}^N)$. Hence, $(1-\Delta)^{-\frac{s}{2}}$ is a linear operator of weak type (p,q) for all $p \in [1, \frac{N}{s})$. In particular, $(1-\Delta)^{-\frac{s}{2}}$ is of weak type $(\frac{2N}{N+2s}, 2)$ and also of weak type $(1, \frac{N}{N-s})$. Applying the Marcinkiewicz interpolation theorem (see [17]) we deduce that $(1-\Delta)^{-\frac{s}{2}}$ is of type $(\frac{2N}{N+2s}, 2)$, that is (4.1) holds true. □

Now we are ready to give the proof of the Sobolev embeddings for the fractional Sobolev spaces on the torus.

Theorem 4.2. *Let $s \in (0,1)$ and $N > 2s$. The inclusion of $H^s(\mathbb{T}^N)$ in $L^q(\mathbb{T}^N)$ is continuous for any $q \in [1, \frac{2N}{N-2s}]$ and compact for any $q \in [1, \frac{2N}{N-2s})$.*

Proof. Let

$$u = \sum_{k\in\mathbb{Z}^N} c_k \frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}} \in C^\infty(\mathbb{T}^N),$$

where

$$c_k = \frac{1}{\sqrt{(2\pi)^N}} \int_{\mathbb{T}^N} u(x) e^{-\imath k\cdot x}\, dx \quad (k \in \mathbb{Z}^N).$$

Take

$$v = \sum_{k\in\mathbb{Z}^N} d_k \frac{e^{\imath k\cdot x}}{\sqrt{(2\pi)^N}} \in L^2(\mathbb{T}^N) \subset L^{\frac{2N}{N+2s}}(\mathbb{T}^N).$$

Applying the Cauchy-Schwartz inequality and using Lemma 4.1 we see that

$$\begin{aligned}
|(u,v)_{L^2(\mathbb{T}^N)}| &= \left|\sum_{k\in\mathbb{Z}^N} c_k \bar{d}_k\right| = \left|\sum_{k\in\mathbb{Z}^N} (1+|k|^2)^{\frac{s}{2}} (1+|k|^2)^{-\frac{s}{2}} c_k \bar{d}_k\right| \\
&\leq \left(\sum_{k\in\mathbb{Z}^N} (1+|k|^2)^s |c_k|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^N} (1+|k|^2)^{-s} |d_k|^2\right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{k\in\mathbb{Z}^N} (1+|k|^2)^s |c_k|^2\right)^{\frac{1}{2}} \|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)} \\
&\leq C\|u\|_s \|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)},
\end{aligned} \tag{4.10}$$

where we used the elementary inequality

$$(1+|k|^2)^s \leq (1+|k|^{2s}) \quad \forall k \in \mathbb{Z}^N.$$

Taking $v=|u|^{\frac{N+2s}{N-2s}-1}u$ we get

$$|(u,v)_{L^2(\mathbb{T}^N)}|=\|u\|^{\frac{2N}{N-2s}}_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}$$

and

$$\|v\|_{L^{\frac{2N}{N+2s}}(\mathbb{T}^N)}=\|u\|^{\frac{N+2s}{N-2s}}_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}$$

which combined with (4.10) yields

$$\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}\leq C\|u\|_s. \tag{4.11}$$

Since $C^\infty(\mathbb{T}^N)$ is dense in $H^s(\mathbb{T}^N)$, we see that (4.11) holds for all $u\in H^s(\mathbb{T}^N)$. This means that $H^s(\mathbb{T}^N)$ is continuously embedded in $L^{\frac{2N}{N-2s}}(\mathbb{T}^N)$. Since $L^{\frac{2N}{N-2s}}(\mathbb{T}^N)\subset L^r(\mathbb{T}^N)$ for all $r\in[1,\frac{2N}{N-2s}]$ we obtain the desired continuous embedding.

Now, fix $q\in(2,\frac{2N}{N-2s})$. By the interpolation inequality for L^p-spaces and (4.11), we deduce that

$$\begin{aligned}\|u\|_{L^q(\mathbb{T}^N)}&\leq\|u\|^\theta_{L^2(\mathbb{T}^N)}\|u\|^{1-\theta}_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}\\&\leq C\|u\|^\theta_{L^2(\mathbb{T}^N)}\|u\|^{1-\theta}_s,\end{aligned}$$

for some $\theta\in(0,1)$. Since in Section 3 we proved that $H^s(\mathbb{T}^N)$ is compactly embedded in $L^2(\mathbb{T}^N)$, we get that $H^s(\mathbb{T}^N)$ is compactly embedded in $L^q(\mathbb{T}^N)$. On the other hand, $\|u\|_{L^r(\mathbb{T}^N)}\leq C\|u\|_{L^2(\mathbb{T}^N)}$ for all $r\in[1,2]$. Thus we conclude that $H^s(\mathbb{T}^N)$ is compactly embedded in $L^q(\mathbb{T}^N)$ for all $q\in[1,\frac{2N}{N-2s})$. □

Remark 4.3. It is also possible to give an alternative proof of the above continuous embedding by combining the results in [2, 3] and [11]. To prove it, we start by identifying $\mathbb{T}^N$ by $[-\pi,\pi]^N$. Then $H^s(\mathbb{T}^N)$ can be seen as the closure of the set of functions $u\in C^\infty(\mathbb{R}^N)$ which are 2π-periodic in each variable and such that $\|u\|_s<\infty$. Denote by $X^s_{2\pi}$ the closure of the set of functions $U\in C^\infty(\mathbb{R}^N\times[0,+\infty))$ such that $U(x,y)$ is 2π-periodic in x and satisfies

$$\|U\|^2_{X^s_{2\pi}}=\int_{(-\pi,\pi)^N}\int_0^{+\infty}y^{1-2s}(|\nabla U(x,y)|^2+U^2(x,y))\,dxdy<+\infty. \tag{4.12}$$

We recall that in [2,3] has been proved that there exists a linear trace operator from $X^s_{2\pi}$ onto $H^s(\mathbb{T}^N)$.

Fix $U\in C^\infty(\mathbb{R}^N\times[0,+\infty))$ such that $U(x,y)$ is 2π-periodic in x and fulfilling (4.12). Take $\phi\in C^\infty_0(\mathbb{R}^N)$ such that $\phi(x)=1$ in $[-\pi,\pi]^N$ and $supp(\phi)\subset[-2\pi,2\pi]^N$, and let $\eta\in C^\infty([0,+\infty))$ be such that $\eta=1$ in $[0,1]$ and $\eta=0$ in $[2,+\infty)$. Then $V(x,y)=U(x,y)\phi(x)\eta(y)\in C^\infty_0(\mathbb{R}^N\times[0,+\infty))$ and using the well-known results in [11] we know that there exists a constant $C>0$ such that

$$\|V(\cdot,0)\|_{H^s(\mathbb{R}^N)}\leq C\int_{\mathbb{R}^N}\int_0^{+\infty}y^{1-2s}(|\nabla V(x,y)|^2+V^2(x,y))\,dxdy.$$

Since $H^s(\mathbb{R}^N)$ is continuously embedded in $L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ (see [11,19]) we deduce that

$$\begin{aligned}\|V(\cdot,0)\|_{H^s(\mathbb{R}^N)} &\geq C\|V(\cdot,0)\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}\\ &\geq C\|V(\cdot,0)\|_{L^{\frac{2N}{N-2s}}((-\pi,\pi)^N)}\\ &= C\|U(\cdot,0)\|_{L^{\frac{2N}{N-2s}}((-\pi,\pi)^N)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_0^{+\infty} y^{1-2s}(|\nabla V(x,y)|^2+V^2(x,y))\,dxdy\\ &=\int_{(-2\pi,2\pi)^N}\int_0^{+\infty} y^{1-2s}(|\nabla(U(x,y)\phi(x)\eta(y))|^2+(U(x,y)\phi(x)\eta(y))^2)\,dxdy\\ &\leq C\int_{(-2\pi,2\pi)^N}\int_0^{+\infty} y^{1-2s}(|\nabla U(x,y)|^2+U^2(x,y))\,dxdy\\ &= C\int_{(-\pi,\pi)^N}\int_0^{+\infty} y^{1-2s}(|\nabla U(x,y)|^2+U^2(x,y))\,dxdy.\end{aligned}$$

Consequently,

$$\|U(\cdot,0)\|_{L^{\frac{2N}{N-2s}}((-\pi,\pi)^N)} \leq C\|U\|_{X^s_{2\pi}}. \tag{4.13}$$

By density, we obtain that (4.13) holds true for any $U\in X^s_{2\pi}$.

Now, let $u\in H^s(\mathbb{T}^N)$ and denote by $U\in X^s_{2\pi}$ its periodic extension in the cylinder $(-\pi,\pi)^N\times(0,+\infty)$ (see [2,3]), that is U is the unique solution to

$$\begin{cases}-\operatorname{div}(y^{1-2s}\nabla U)+m^2\xi^{1-2s}U=0 & \text{in } (-\pi,\pi)^N\times(0,\infty),\\ U_{|\{x_i=-\pi\}}=U_{|\{x_i=\pi\}} & \text{on } \partial(-\pi,\pi)^N\times[0,\infty),\\ U(\cdot,0)=u & \text{on } (-\pi,\pi)^N\times\{0\},\end{cases}$$

Thus, U satisfies

$$\sqrt{\kappa_s}\|u\|_s=\|U\|_{X^s_{2\pi}}$$

for some constant $\kappa_s>0$, and using (4.13) we deduce that

$$\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{T}^N)}\leq C\|u\|_s.$$

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