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SEVEN LARGEST TREES PACK

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Abstract. The Tree Packing Conjecture (TPC) by Gyárfás states that any set of trees $T_2,\ldots,T_{n-1},T_n$ such that T_i has i vertices pack into K_n. The conjecture is true for bounded degree trees, but in general, it is widely open. Bollobás proposed a weakening of TPC which states that k largest trees pack. The latter is true if none tree is a star, but in general, it is known only for $k=5$. In this paper we prove, among other results, that seven largest trees pack.

Keywords: tree, packing, tree packing conjecture.

Mathematics Subject Classification: 05C35, 05C05, 05C70.

1. INTRODUCTION

Graphs $G_1,\ldots,G_k$ are said to *pack into a complete graph* K_n if $G_1,\ldots,G_k$ can be found as edge-disjoint subgraphs in K_n. In other words, there exist mappings $f_i : V(G_i)\to[1,n]$, $i=1,\ldots,k$, such that $f_i^*[E(G_i)]\cap f_j^*[E(G_j)]=\emptyset$ if $i\neq j$, where the map $f_i^* : E(G_i)\to E(K_n)$ is induced by f_i, that means $f_i^*(uv)=f_i(u)f_i(v)$ for every $uv\in E(G_i)$.

A famous tree packing conjecture (TPC) posed by Gyárfás [7] states that any set of $n-1$ trees $T_2,\ldots,T_{n-1},T_n$ such that T_i has i vertices pack into K_n. A number of partial results concerning the TPC are known. In particular Gyárfás and Lehel [7] showed that the TPC is true if each tree is either a path or a star. Böttcher *et al.* [4] proved an asymptotic version of the TPC for trees with bounded maximum degree (see also [10] for generalizations on other families of graphs). The exact version of the TPC for large enough n and for trees with bounded maximum degree was proved in [9]. In [5, 6] Bollobás suggested the following weakening of TPC.

Conjecture 1.1. *For every $k\geq 1$ there is an $n_0(k)$ such that if $n>n_0(k)$, then any set of k trees $T_{n-k+1},\ldots,T_n$ such that T_{n-k+i} has $n-k+i$ vertices pack into K_n.*

Bourgeois, Hobbs and Kasiraj [3] showed that any three trees T_{n-2}, T_{n-1}, T_n pack into K_n. Recently, Balogh and Palmer [2] proved that any set of $k=\frac{1}{10}n^{1/4}$ trees $T_{n-k+1},\ldots,T_n$ such that no tree is a star pack into K_n. Żak [11] proved Conjecture 1.1 if every tree T_{n-k+i} has at least $i-1$ leaves or a pending path of order $i-1$ (a pending path is an induced subgraph isomorphic to a path and such that one endvertex of this path has exactly one nieghbor outside the path while the remaining vertices of the path do not vave any neighbors outside the path). In particular, he confirmed the conjecture for $k = 5$. In this paper we prove the conjecture for new sets of trees. As a corollary, we confirm Conjecture 1.1 for $k = 7$ (Theorem 1.4). While preparing the final version of this paper, we have learned that Conjecture 1.1 has been proved in a strong form in [8].

Throughout the paper k will be given, and $n \geq n_0(k)$ will be large enough. Since our main result is technical we need to introduce some notation. Let

$$\mu = \left\lceil 2\sqrt{kn} \right\rceil = \Theta(\sqrt{n}). \tag{1.1}$$

We call a tree T *starlike* if $\Delta(T) \geq 2n/3$. We call T *pathlike* if $\Delta(T) < 2\mu+1$. Otherwise, we call T *intermediate*. Let $\mathcal{S} = \{i : T_{n-k+i} \text{ is starlike}\}$, $\mathcal{P} = \{i : T_{n-k+i} \text{ is pathlike}\}$ and $\mathcal{I} = \{i : T_{n-k+i} \text{ is intermediate}\}$.

Given a graph G we say that $V(G)$ has a *1-degenerate ordering* if the vertices of G can be ordered in such a way that every vertex has at most one neighbor earlier in the ordering. It is well known that every tree has a 1-degenerate ordering. Let T^i, $i = 1, \ldots, q$, be trees and $W \subseteq [1, n]$. Let $W_i \subseteq W$, $i = 1, \ldots, q$. We say that T^i, $i = 1, \ldots, q$, have a *partial 1-degenerate packing with respect to* W_i if there are $X_i \subset V(T^i)$ and mappings $h_i : V(T^i) \to [1, n]$ such that

$$h_i(X_i) = W_i \text{ and } h_i(V(T^i)) \cap W = W_i, \tag{1.2}$$

$$h_i|_{X_i}, i = 1, \ldots, q, \text{ is a packing of } T^1[X_1], \ldots, T^q[X_q], \tag{1.3}$$

$$h_i\left(V(T^i)\right) \text{ is a 1-degenerate ordering of } V(T^i), i = 1, \ldots, q. \tag{1.4}$$

See Figure 1 for an example.

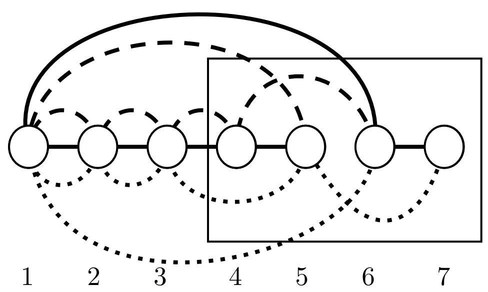

Fig. 1. Partial 1-degenerate packing of $T^1 = P_7, T^2 = P_6$ and $T^3 = P_6$ with respect to $W_1 = \{4, 5, 6, 7\}, W_2 = \{4, 5, 6\}, W_3 = \{5, 6, 7\}$, where $W = \{4, 5, 6, 7\}$

Our main theorem states that it is possible to pack the trees from Conjecture 1.1 if one can find in every tree a small (up to k vertices) special subset of vertices and pack all the subtrees induced by these subsets into a small complete graph in a special way.

Theorem 1.2. *Let k be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k. Let $n > n_0(k)$ and let T^i, $i = 1, \dots, k$, be trees of orders $n - k + i$, respectively. Let s be the number of starlike trees and for a pathlike tree T^i let s_i be the number of starlike trees T^j with $j < i$. Let $W = [n - s + 1, n]$ and $W_i = [n - s + 1, n - s + s_i]$, $i \in \mathcal{P}$. If there exist a partial 1-degenerate packing of the pathlike trees T^i, $i \in \mathcal{P}$, with respect to the sets W_i, then there is a packing of all trees T^i, $i = 1, \dots, k$, into K_n.*

While the statement of Theorem 1.2 initially appears to necessitate the existence of a packing comprising entire pathlike trees, in reality, within a 1-degenerate packing, the packing property need only be preserved by the smaller components of these trees. Therefore, it suffices to find a specialized packing strategy for the smaller subgraphs of the pathlike trees. Regrettably, such a specialized packing does not exist for all conceivable instances of Conjecture 1.1 (of course this absence does not disprove it). This limitation is due to the nature of pathlike trees. When their quantity is relatively high and they have very few leaves, a 1-degenerate ordering forces a significant number of edges within each small subgraph. Moreover, these subgraphs must be packed in a specific manner, which may be impossible even for eight trees. Further explanation of this matter will be provided in the Concluding Remarks section. On the other hand, Theorem 1.2 permits packing of any instance where each pathlike tree has at least as many leaves as is the number of starlike trees of smaller order. In such cases, a 1-degenerate packing is trivial, as the sets X_i may consits of leaves only, rendering all induced subgraphs edgeless.

Our second theorem will follow from Theorem 1.2 and the following theorem proved in [11].

Theorem 1.3 ([11])**.** *Let k be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k. Let $n > n_0(k)$ and let T^i, $i = 1, \dots, k$, be trees of orders $n - k + i$, respectively. If every T^i has at least $i - 1$ leaves or a pending path of order $i - 1$, then T^i, $i = 1, \dots, k$, pack into K_n.*

In Section 3 we will show that (for sufficiently large n) any instance of 7 trees satisfies the assumptions of Theorem 1.2 or Theorem 1.3, and so we will prove our second result:

Theorem 1.4. *Let n_0 be a sufficiently large constant. If $n > n_0$, then any set of seven trees $T_{n-6}, \dots, T_{n-1}, T_n$, such that T_{n-7+i} has $n - 7 + i$ vertices, $i = 1, \dots, 7$, pack into K_n.*

2. PROOF OF THEOREM 1.2

Given a graph $G = (V, E)$ and $v \in V$, recall that

$$N_G(v) = \{u \in V : uv \in E\}.$$

Furthermore, for $S \subseteq V$, the closed and open neighborhood is denoted, respectively,

$$N_G[S] = \bigcup_{v \in S} N_G(v),$$
$$N_G(S) = N_G[S] \setminus S.$$

We will use the following simple fact.

Proposition 2.1. *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \ldots, v_n\}$ with $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$. Then $d_G(v_j) \leq \frac{2m}{j}$.*

Proof. The proposition is true because

$$jd_G(v_j) \leq \sum_{t=1}^{j} d_G(v_t) \leq \sum_{t=1}^{n} d_G(v_j) \leq 2m. \qquad \square$$

The following technical lemma from [11] is the main tool in the proof. This is a more general form of a lemma, which first appeared in [1].

Lemma 2.2. [11] *Let G be a graph with n vertices and at most m edges and let $a \geq 1$. Let $V(G) = \{v_1, \ldots, v_n\}$ with $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$. Let A_j, $j = 1, \ldots, n$, be any subsets of $V(G)$ with the additional requirement that if $u \in A_j$ then $d_G(u) < a$. For $j = 1, \ldots, n$ let B_j be a random subset of A_j where each vertex of A_j is independently selected to B_j with probability $p < 1/a$. Let*

$$C_j = \left(\bigcup_{t=1}^{j-1} B_t \right) \cap N(v_j),$$
$$D_j = B_j \setminus \left(\bigcup_{t=1}^{j-1} N[B_t] \right).$$

Then:

(1) $\Pr\left[|C_j| \geq 4mp\right] \leq \exp(-2mp/3)$ *for* $j = 1, \ldots, n$,
(2) $\Pr\left[|D_j| \leq \frac{p|A_j|}{2e}\right] \leq \exp\left(\frac{-p|A_j|}{8e}\right)$ *for* $j = 1, \ldots, \lfloor 1/(ap) \rfloor$.

Proof of Theorem 1.2. Let T^i, $i = 1, \ldots, k$, be given trees of orders $n - k + i$, respectively, and let h_i satisfy the assumptions of Theorem 1.2. We will construct a packing

$$f_i : V(T^i) \longrightarrow [1, n], i = 1, \ldots, k,$$

iteratively, at iteration i we will construct f_i and, if needed, slightly modify f_j, $j = 1, \ldots, i - 1$.

Recall that $\mathcal{S} = \{i : T^i \text{ is starlike}\}$, $\mathcal{P} = \{i : T^i \text{ is pathlike}\}$ and $\mathcal{I} = \{i : T^i \text{ is intermediate}\}$. For $i \in \mathcal{I} \cup \mathcal{S}$ let $u_i \in V(T^i)$ with $d_{T^i}(u_i) = \Delta(T^i)$. Let

$$U_i = \{u_j : j \leq i, i \in \mathcal{S}\}. \tag{2.1}$$

If T^i is starlike or intermediate then let $F = T^i - u_i$. Since $\Delta(T^i) \geq 2\mu + 1$, F is a forest with at least $2\mu + 1$ components. Let $L_i = \{l_1, l_2, \ldots, l_{2\mu+1}\}$ such that every l_t, $t \in [1, 2\mu+1]$, is a leaf of T^i and the elements of L_i are from pairwise different components of F. Additionally, we assume that leaf-neighbors of u_i occupy as many as possible starting places of L_i. Define

$$X_i = \{l_1, \ldots, l_{|U_{i-1}|}\} \text{ if } i \in \mathcal{I} \cup \mathcal{S}. \tag{2.2}$$

Note that if T^i is starlike then $d_{T^i}(u_i) \geq 2n/3$. Hence, u_i has at least $n/3$ neighbors that are leaves. Thus, X_i consists of leaves of T^i adjacent to u_i. For a pathlike tree,

$$X_i \text{ is chosen according to (1.2)–(1.4) if } i \in \mathcal{P}. \tag{2.3}$$

Finally,

$$Z_i \coloneqq \{n - k + j : j \in \mathcal{S} \cap [1, i]\}. \tag{2.4}$$

Let G_i be the graph with $V(G_i) = [1, n]$ and with $E(G_i) = \bigcup_{j=1}^{i} f_j^*(E(T^j))$. At the end of step i the mapping f_i and the graph G_i will satisfy:

$$f_i(u_i) = n - k + i \text{ for } i \in \mathcal{I} \cup \mathcal{S}, \tag{2.5}$$

$$f_i^{-1}(Z_i) \subseteq L_i \cup \{u_i\} \text{ if } i \in \mathcal{S} \text{ and } f_i^{-1}(Z_i) \subseteq L_i \text{ if } i \in \mathcal{I}, \tag{2.6}$$

$$f_i(X_i) = Z_{i-1}, \tag{2.7}$$

$$n - k + i + 1, \ldots, n \text{ are isolated in } G_i, \tag{2.8}$$

$$d_{G_i}(v) < 2n/3 + i \cdot (2\mu + 1) \text{ for every } v \in [1, n] \setminus Z_i. \tag{2.9}$$

Note that we may have to modify f_j, $j < i$, while constructing f_i in order to ensure that $\{f_1, \ldots, f_i\}$ is a packing of $\{T^1, \ldots, T^i\}$. However, we shall ensure that the modifications will satisfy the following properties for all $j < i$:

$$f_j^{-1}(Z_i) \subseteq L_j \cup \{u_j\} \text{ if } j \in \mathcal{S} \text{ and } f_j^{-1}(Z_j) \subseteq L_j \text{ if } i \in \mathcal{I}, \tag{2.10}$$

$$N_{G_i}(f_j(u_j)) \cap Z_i \subset [1, n - k + j - 1] \text{ for every } j \in \mathcal{I} \cup \mathcal{S}. \tag{2.11}$$

So suppose that we have constructed f_j with $j < i$, and we now construct f_i. Let $G = G_{i-1}$ and $Z = Z_{i-1}$. Clearly

$$|E(G)| < kn. \tag{2.12}$$

□

2.1. PACKING A STARLIKE OR AN INTERMEDIATE TREE

Suppose first that T^i is starlike or intermediate. We construct a packing f_i in six stages. At each point of the construction, some vertices of $V(T^i)$ are *matched* to some vertices of $V(G)$, while the other vertices of $V(T^i)$ and $V(G)$ are unmatched yet. Initially, all vertices are unmatched. Let $M_q(T^i)$ denotes the vertices of T^i that are matched at the end of stage q. Similarly, we denote $M_q(G)$.

Stage 1. We match u_i with $n-k+i$, i.e. $f_i(u_i) = n-k+i$. Trivially, (2.5) is satisfied. Moreover,

$$d_{T^i}(u) \le n/2 \quad \text{for every } u \in V(T^i) \setminus M_1(T^i). \tag{2.13}$$

Stage 2. Let $\{w_1, \dots, w_{n'}\} = [1, n-k+i-1]$ with

$$d_G(w_1) \ge d_G(w_2) \ge \dots \ge d_G(w_{n'}).$$

Note that by Proposition 2.1, $Z \subset \{w_1, \dots, w_\mu\}$. We will match $X'_i := \{l_1, \dots, l_\mu\}$ with $\{w_1, \dots, w_\mu\}$ almost arbitrarily, just carrying that $f_i(X_i) = Z$, i.e. to maintain (2.7). If T^i is starlike then $Z_i = Z \cup \{f_i(u_i)\}$, and so $f_i^{-1}(Z_i) = X_i \cup \{u_i\}$. If T^i is intermediate then $Z_i = Z$, so $f_i^{-1}(Z_i) = X_i$. In both cases, by (2.2), the property (2.6) holds. By (2.8) and since X'_i are leaves of T^i, the packing property is trivially preserved. Furthermore,

$$d_{G_i}(w_j) = d_{G_{i-1}}(w_j) + 1, \quad j = 1, \dots, \mu, \tag{2.14}$$

hence (2.9) is satisfied for $v \in \{w_1, \dots, w_n\} \setminus Z$. By (2.12), Proposition 2.1 and (1.1),

$$d_G(w) < \frac{2kn}{\mu} \le \sqrt{kn} \quad \text{for every } w \in V(G) \setminus M_2(G), \tag{2.15}$$

which together with (2.13) assure that from now on the property (2.9) will be satisfied for any extension of so far defined f_i. Clearly, (2.5)–(2.7) will be maintained, as well, while in order to preserve (2.8) we will not use vertices $[n-k+i+1, n] \subset V(G)$ in iteration i. The maintenance of (2.11) and (2.11) will be explained in the next stage.

Stage 3. Let $Y_i := N_{T^i}(X_i)$. We match $Y_i \setminus M_2(T^i)$. If this set is empty, then we skip this stage. In particular we skip it when T^i is starlike – this will preserve (2.10) and (2.11) as we will not modify any f_j, $j < i$, in Stages 4–6. Otherwise, let $y \in Y_i$ and T_i be intermediate. Thus, y is the neighbor of some $l \in X_i \subseteq \{l_1, \dots, l_\mu\}$. Hence, $f_i(l) = f_j(u_j) =: z \in Z$ for some $j < i$. Recall that L_j consists of $2\mu + 1$ leaves of T^j and only at most $|M_2(G)| + |Y_i| \le \mu + 1 + k < 2\mu + 1$ of them are already matched in iteration i. Hence, we can find a leaf $l^j \in L_j$ such that $f_j(l^j)$ is yet unmatched. By the definition of L_j, since T^j is starlike, $l^j \in N_{T^j}(u_j)$. We set $f_i(y) = f_j(l^j)$. This however spoils the packing property since $f_i(l)f_i(y) = f_j(u_j)f_j(l^j)$. In order to maintain the packing property, we modify f_j a bit, namely, $f_j(l^j) = n-k+i$ (note that this modification preserves (2.10) as well as (2.11) for T^i is not starlike, and so $n-k+i$ does not belong to any Z_ι, $\iota \ge i$). Indeed, since l is a leaf in T^i, y is its only neighbor in T^i and so $f_i^{-1}(n-k+i)f_i^{-1}(z) = u_i l$ is not an edge of T^i. Moreover, by (2.5), $f_j(u_j) \ne f_{j'}(u_{j'})$ for $j \ne j'$. Thus, such modifications of f_j's for different j's maintain the packing property inside $Z \cup \{n-k+i\}$. Furthermore, since in particular $l^j \in L_j$, this modification does not violate (2.5)–(2.9). Finally, since Y_i has elements from pairwise different components of $T^i - u_i$, there are no edges in $T^i[Y_i]$. Thus, the packing property is maintained in this stage.

Stage 4. In this stage we match the neighbors of $X_i' \setminus X_i$. Let $Y_i' = N_{T^i}(X_i' \setminus X_i)$. To see that this is possible, consider an iteration of Stage 4 where $y \in Y'$ is a yet unmatched neighbor of some already matched leaf $l \in \{l_1, \dots, l_\mu\} \setminus X_i$. Note that there are at most $|M_3(G)| + \mu = 1 + \mu + |Y_i| + \mu \le k + 1 + 2\mu$ matched vertices in G. Let $w = f_i(l)$. Hence, by (2.9) w has at least $(n/3 - (k-i) - i(2\mu+1)) - (k+1+2\mu) > 0$ yet unmatched non-neighbors in $[1, n-k+i]$. We match y with one of them. Since elements of Y' are from pairwise different components of $T^i - u_i$, l is the only matched neighbor of y and so the packing property is preserved.

Stage 5. In this stage we match the vertices of $V(T^i) \setminus (L_i \cup M_4(T^i))$ with some yet unmatched vertices of $[1, n-k+i]$. By the construction of Stages 1–4, every unmatched vertex of T^i has at most one matched neighbor in $T^i - u_i$. It is easy to order the remaining vertices of T^i in such a way that this additional property is maintained throughout Stage 5. Hence, consider some iteration of Stage 5 where y is a yet unmatched vertex of $V(T^i) \setminus L_i$ having at most one matched neighbor in $T^i - u_i$. Let Q be the set of yet unmatched vertices of $[1, n-k+i]$. Clearly $|Q| \ge |\{l_{\mu+1}, \dots, l_{2\mu+1}\}| = \mu + 1$. If y does not have any matched neighbor in $T^i - u_i$ then a valid choice for $f_i(y)$ would be any vertex of $Q \setminus N_G(f_i(u_i))$ (or, simply, Q if y is not adjacent to u_i in T^i). By (2.11), $N_G(f_i(u_i)) \subseteq Z \subset V(G) \setminus Q$. Thus, $Q \setminus N_G(f_i(u_i)) = Q$ and we can match y with an arbitrary vertex of Q. Otherwise, let x be the already matched neighbor of y in $T^i - u_i$. Thus, a valid choice for $f_i(y)$ would be a vertex from $Q \setminus (N_G(f_i(x)) \cup N_G(f_i(u_i))) = Q \setminus N_G(f_i(x))$. By the construction of Stages 2–4, $f_i(x) \notin M_2(G)$. Hence, by (2.15),

$$|Q \setminus N_G(f_i(x))| \ge \mu + 1 - d_G(f_i(x)) > \mu + 1 - \frac{2kn}{\mu} \ge 2\sqrt{kn} + 1 - \sqrt{kn} > 0,$$

and so Stage 5 can be completed.

Stage 6. It remains to match the vertices $L := \{l_{\mu+1}, \dots, l_{2\mu+1}\}$ with $Q := V(G) \setminus M_5(G)$. Consider the bipartite graph B whose sides are L and Q. For $l \in L$ and $u \in Q$ let $lu \in E(B)$ if and only if it is possible to match l with u. Let x be the already matched neighbor of l. If $x = u_i$ then by (2.8), y can be matched with any vertex $v \in Q$, and so $N_B(l) = Q$. Otherwise, a valid choice for $f_i(l)$ would be any vertex $v \in Q \setminus N_G(f_i(x))$. Again by the construction of Stages 2–4, $f_i(x) \notin M_2(G)$. Hence, by (2.15),

$$\begin{aligned} d_B(l) > |Q| - \frac{2kn}{\mu} = \mu + 1 - \frac{2kn}{\mu} \ge 2\sqrt{kn} + 1 - \sqrt{kn} \\ = \sqrt{kn} + 1 \ge \frac{\lceil 2\sqrt{kn} \rceil + 1}{2} = \frac{\mu+1}{2} = \frac{|Q|}{2}. \end{aligned}$$

We will now estimate $d_B(u)$. Let W' be the set of already matched neighbors of u. By (2.8), $n-k+i = f_i(u_i) \notin W'$. Let $S = \bigcup_{w' \in W'} N_{T^i}\left(f_i^{-1}(w')\right)$. Thus, a valid choice for $f_i^{-1}(u)$ would be any vertex from $L \setminus S = L \setminus (S \cap L)$. Recall that elements of L

are leaves of T^i from pairwise different components of $T^i - u_i$. Hence, $|S \cap L| \leq |W'|$. Thus, by (2.15),

$$d_B(u) = |L| - |S \cap L| \geq |L| - |W'| \geq |L| - d_G(u) \geq \mu + 1 - \frac{2kn}{\mu} \geq \frac{\mu+1}{2} = \frac{|L|}{2},$$

as well. Therefore, by Hall's Theorem there is a perfect matching M in B. We complete this stage by setting $f_i(y) = v$ for every $yv \in M$. □

2.2. PACKING A PATHLIKE TREE

Note that, by the definition of a pathlike tree, (2.9) will be trivially maintained throughout this section.

Stage 1. In this stage we match X_i with Z in the way that is assured by the assumed existence of a partial 1-degenerate packing of all pathlike trees. Hence, let h_i be one part of this partial 1-degenerate packing, that corresponds to T^i. Recall that then the corresponding set $W_i = [n - s + 1, n - s + s_i]$ where s denotes the number of all starlike trees and s_i denotes the number of those starlike trees T^j that satisfy $j < i$. Thus, $|Z| = |W_i|$. Note that Z_i gets a new vertex whenever we pack a starlike tree. Thus, we can enumerate its elements according to their appearance in consecutive sets Z_j, $j = 1, \ldots, k$, obtaining $Z_k = \{z_1, \ldots, z_s\}$ and $Z_i = Z_{i-1} = \{z_1, \ldots, z_{s_i}\}$. Then, for $x \in X_i$ we set $f_i(x) = z_{j_i}$ where $j_i = h_i(x) - n + s \in [1, s_i]$. Note that the partial 1-degenerate packing guarantees that there is no conflict inside Z between T^i and any pathlike tree T^j for all $j < i$. However, such an embedding may cause a conflict between T^i and a starlike or an intermediate tree T^j, $j < i$. Suppose that $f_i(x)f_i(x') = f_j(u)f_j(v)$ where $xx' \in E(T^i)$ and $uv \in E(T^j)$. Since the conflict appeared inside Z, by (2.10) and by definition of L_j, T^j is starlike, and $u = u_j$ and $v \in L_j$ (or vice versa). Without loss of generality we may assume that $u = u_j$ and $f_i(x) = f_j(u_j) = n - k + j$ by (2.5). Hence, by (2.11), $f_j(v) < n - k + j$ which means that $f_i(x') < f_i(x)$. Therefore, $h_i(x') < h_i(x)$ as well. Note that by the property of h_i, other neighbors $\tilde{x}$ of x are mapped with $h_i(\tilde{x}) > h_i(x)$. So $f_i(\tilde{x}) = z_{h_i(\tilde{x})-n+s} > z_{h_i(x)-n+s} = f_i(x)$. Due to a 1-degenerate ordering, this is the only conflict between the edges of T^i and T^j inside Z. In order to remove it, we set $f_j(v) = n - k + i$. By (2.8), this modification does not create any new conflict neither in G nor between T^j and T^i. Note that (2.11) is maintained as T^i is not starlike and so $n - k + i \notin Z_\iota$ for any $\iota = 1, \ldots, k$. Moreover, since $f_j(u_j) \neq f_{j'}(u_{j'})$ if $j \neq j'$ and remembering that $v \in L_j$ we can repeat such a modification for every $j < i$, for which it is needed, without creating any conflict neither in G nor between T^j and T^i.

Stage 2. Let $Y_i' \coloneqq N_{T^i}(X_i')$ and let $Y_i \coloneqq N_{T^i}(X_i)$. Let $P \coloneqq \{x \in V(T^i) : d_{T^i}(x) \leq 3\}$. Let $X_i' \subseteq P \setminus (X_i \cup Y_i)$ be an independent set of size $27\mu k - |X_i|$. In order to define the matching of Stage 2, we need some auxiliary claims. Let $\{w_1, \ldots, w_{n'}\} = [1, n-k+i-1]$ with

$$d_G(w_1) \geq d_G(w_2) \geq \cdots \geq d_G(w_{n'}).$$

Let $Z' = \{w_1, \ldots, w_{27k\mu}\}$. Note that by Proposition 2.1, $Z \subset Z'$. In Stage 2 we will map X_i' to $Z' \setminus Z$ together with $Y_i \cup Y_i'$.

If $w_t \in Z$ then $w_t = f_j(u_j)$ for some $j < i$ with T^j is starlike. In such case let

$$A_t \coloneqq \{f_j(l) : l \text{ is a leaf of } T^j \text{ adjacent to } u_j\} \setminus Z' \tag{2.16}$$

with additional requirement that if $u \in A_t$ then $d_G(u) < 26k$. Otherwise, let

$$A_t \coloneqq [1, n-k+i] \setminus (N_G(w_t) \cup Z') \tag{2.17}$$

with additional requirement that if $u \in A_t$ then $d_G(u) < 26k$.

In the following we will match w_t with a low degree vertex, say l', of T^i, and the unmatched neighbors of l' with some random subset of A_t. For $w_t \notin Z$, the property (2.17) will ensure the packing property. On the other hand, when $w_t \in Z$ we choose such A_t intentionally despite the fact that this choice will create a conflict with the edge $w_t f_j(l)$ in G. Instead, we get a large set A_t (which is crucial) to select the candidate random subset from and for which f_j can easily by modified.

Claim 2.3. *For each $t = 1, \ldots, 27k\mu$, $|A_t| \geq \frac{n}{4}$.*

Proof. If $w_t \in Z$ then $w_t = f_j(u_j)$ for some $j < i$. By (2.1) and by the definition u_j, $d_{T^j}(u_j) \geq 2n/3$. Hence, u_j has at least $n/3$ neighbors that are leaves in T^j. Let S be the set of the images of those leaves with (eventual) exclusion of $\{w_1, \ldots, w_{27k\mu}\}$. Thus, $|S| \geq n/3 - 27k\mu$.

If $w_t \notin Z$ then let S be the set of non-neighbors of w_t in $[1, n-k+i] \setminus \{w_1, \ldots, w_{27k\mu}\}$. In this case, by (2.9), $|S| \geq n/3 - k - i \cdot (2\mu + 1)$. In both cases, in order to obtain A_t we have to exclude from $f_j(S)$ vertices that have degree greater than $26k$. Suppose that α vertices of G have degree greater than or equal to $26k$. Thus,

$$2kn \geq 2|E(G)| = \sum_{i=1}^{n} d(v_i) \geq \alpha \cdot 26k,$$

and so $\alpha \leq \frac{n}{13}$. Therefore, by (1.1),

$$|A_t| \geq n/3 - k - 27k\mu - i \cdot (2\mu + 1) - n/13 \geq n/4. \qquad \square$$

For $t = 1, \ldots, 27k\mu$ let B_t be a random subset of A_t where each vertex of A_t is independently selected to B_t with probability

$$p = \frac{1}{26 \cdot 27k\mu} \tag{2.18}$$

Let

$$C_t = \left(\bigcup_{s=1}^{t-1} B_s \right) \cap N_G(w_t),$$

$$D_t = B_t \setminus \left(\bigcup_{s=1}^{t-1} N_G[B_s] \right).$$

Claim 2.4. *The following hold simultaneously with positive probability:*

(1) $|C_t| \le \frac{n}{8\mu}$ *for* $t = 1, \dots, 27k\mu$,
(2) $|D_t| \ge 3$ *for* $t = 1, \dots, 27k\mu$.

Proof. Recall that $m = |E(G)| \le kn$. Thus, by Lemma 2.2, the probability that $|C_t| > \frac{n}{8\mu}$, is exponentially small in $n/\mu = \Theta(\sqrt{n})$. Hence, for sufficiently large n

$$\Pr\left[|C_t| > \frac{n}{8\mu}\right] < \frac{1}{54k\mu}.$$

Therefore, by the union bound, the first statement holds with probability greater than 1/2. Furthermore, by Claim 2.3,

$$3 < \frac{p|A_t|}{2e}.$$

Hence, by Lemma 2.2 (with $a = 26k$), for each $t \in [1, 27k\mu]$ the probability that $|D_i| < 3$ is exponentially small in n/μ, as well. Hence, for sufficiently large n,

$$\Pr\left[|D_t| < 3\right] < \frac{1}{54k\mu}.$$

Therefore, by the union bound, the second statement holds with probability greater than 1/2, and so both statements hold with positive probability. □

Therefore, we may fix sets $B_1, \dots, B_{27k\mu}$ satisfying all the conditions of Claim 2.4 with respect to the cardinalities of the sets C_t and D_t. Now we are in a position to carry out Stage 2. This stage is done in $27\mu k$ steps, where in each step t we will proceed as follows:

(a) If $w_t \notin Z$, we match w_t with some vertex $x \in X'_i \setminus M_1(T^i)$,
(b) Match $N_{T^i}(x) \setminus M_2(T^i) \subseteq Y_i \cup Y'_i$ to vertices in D_t.

In particular, all neighbors of x are matched to vertices of $\bigcup_{s=1}^{t} B_s$. We also preserve the packing property but with few exceptions. Note that due to invariant b) the packing property can be temporarily violated for $w_t \in Z$ – we will fix it at the end of iteration t by modifying some of f_j with $j < i$. To see that this is possible, consider the t-th iteration of Stage 2 where $w_t \in Z$ or w_t is some yet unmatched vertex of $[1, n-k+i]$. Let Q be the set of all yet unmatched vertices of T^i having degree less than or equal to 3. Note that, by Proposition 2.1, the number of vertices of degree less than or equal to 3 in T^i is at least $n/2$. Hence,

$$|Q| \ge n/2 - 4(t-1) \ge n/2 - 108k\mu \ge n/3.$$

Let X' be the set of already matched neighbors (in G) of w_t and let

$$Y' = \bigcup_{x' \in X'} N_{T^i}(f_i^{-1}(x')).$$

Thus, if $w_t \notin Z$ (i.e. w_t is still unmatched) a choice for $f_i^{-1}(w_t)$ that preserves the packing property would be any vertex of $Q \setminus Y'$. By invariant (b),

$$Y' \cap Q \subseteq \bigcup_{x' \in C_t} N_{T^i}(f_i^{-1}(x')).$$

Hence,

$$|Q \setminus Y'| \geq n/3 - \Delta(T^i)|C_t| \geq n/3 - \frac{n}{8\mu} \cdot (2\mu + 1) > 0.$$

In order to maintain the second invariant it remains to match the yet unmatched neighbors of $f_i^{-1}(w_t)$ with vertices from B_t. Let R' be the set of neighbors of $f_i^{-1}(w_t)$ in T^i that are still unmatched. Recall that $|R'| \leq 3$. We match R' with a $|R'|$-subset of D_t. If $w_t \notin Z$ then by the definition of D_t and by invariant (b), the packing property is preserved. If $w_t \in Z$, then by the choice of A_t the packing property may be violated by some edge $f_j(u_j)f_j(l)$, where $w_t = f_j(u_j)$, $l \in L_j$ and $f_j(l) \notin Z'$, and so $f_j(l) \notin Z$. Note that in a 1-degenerate ordering given by h_i, the vertices from Z are the last ones in this ordering. Hence, due to the 1-degeneracy of h_i, $|R'| \leq 1$. Thus, there is at most one conflict between T^j and T^i. In order to save the packing property, we modify f_j by setting $f_j(l) = n - k + i$. Note that in such case, again by the 1-degeneracy of h_i and by (2.11), there was not any conflict between T^i and T^j while performing Stage 1, so f_j has not been modified in Stage 1. Hence, $n - k + i$ is still available, and this modification can be made. Moreover, it preserves (2.8). Finally, by Proposition 2.1, at the end of Stage 2

$$d_{G_i}(w) \leq \frac{2kn}{27k\mu} = \frac{2n}{27\mu} \text{ for every } w \in V(G) \setminus Z'. \tag{2.19}$$

Stage 3. Note that $|M_2(G)| = |M_2(T^i)| \leq 108k\mu < n/9$. Hence $T^i - M_2(T^i)$ has an independent set I with $|I| \geq 4n/9$. Let $K = V(T^i) \setminus (M_2(T^i) \cup I)$. In Stage 3 we match vertices of K one by one, with arbitrary yet unmatched vertices of $[1, n-k+i]$. Suppose that $y \in K$ is still unmatched. Let Q be the set of all yet unmatched vertices of $[1, n-k+i]$. Clearly, $|Q| \geq |I| \geq 4n/9$. Let X' be the set of already matched neighbors (neighbors in T^i) of y. Recall that $|X'| \leq \Delta(T^i) \leq 2\mu + 1$. Let

$$Y = \bigcup_{x' \in X'} N_G(f_i(x')).$$

Thus, a valid choice for $f_i(y)$ would be any vertex of $Q \setminus Y$. By invariant (b) of Stage 2, $f_i(X') \cap Z' = \emptyset$. Hence, by (2.19),

$$|Y| \leq |X'| \cdot \frac{2n}{27\mu} \leq (2\mu + 1) \cdot \frac{2n}{27\mu} \leq 2n/9.$$

Therefore, $|Q \setminus Y| > 0$, and so an appropriate choice for $f_i(y)$ is possible.

Stage 4. In order to complete a packing of G and T^i, it remains to match the vertices of $J := [1, n-k+i] \setminus M_3(G)$ with the vertices of I. Consider a bipartite graph B whose sides are J and I. For two vertices $u \in J$ and $y \in I$, we place an edge $uy \in E(B)$ if and only if it is possible to match u with y (by this we mean that mapping u to y will not violate the packing property). Thus u is not allowed to be matched to at most $d_G(u)\Delta(T^i)$ vertices of I. Hence, by (2.19),

$$d_B(u) \geq |I| - \frac{2n}{27\mu} \cdot (2\mu + 1) \geq |I| - |I|/2 \geq |I|/2.$$

On the other hand, since there is no edge from y to Z' (by invariant (b) of Stage 2), y is not allowed to be matched to at most $\Delta(T^i)\frac{2n}{27\mu}$ vertices of J, too. Hence, analogously

$$d_B(y) \geq |J|/2.$$

Therefore, by Hall's Theorem there is a perfect matching M in B. We complete this stage by setting $f_i(y) = u$ for every $yu \in M$.

3. PROOF OF THEOREM 1.4

We say that a tree T has a *pending path* of order t if there exists $e \in E(T)$ such that one component of $T - e$ is a path P of order t and $d_T(v) \leq 2$ for every $v \in V(P)$.

Proof of Theorem 1.4. We want to show that for any set of 7 trees there is a partial 1-degenerate packing of the pathlike trees. Clearly, we can skip the intermediate trees, because they do not affect the existence of a 1-degenerate packing of pathlike trees (they do not need to be packed and they do not affect the sets W_i). Thus, without a loss of generality we may assume that each of the 7 trees is pathlike or starlike.

We start with two preliminary observations. Note that since n is large enough then the following result holds:

Claim 3.1. *If a tree T_{n-k+i} has no pending path of length 6 then T_{n-k+i} has at least 4 leaves.*

Furthermore, we get:

Claim 3.2. *If a pathlike tree T_{n-k+i} has at least $|W_i|$ leaves then we can skip that tree in a partial 1-degenerate packing of pathlike trees.*

Proof. If T_{n-k+i} has at least as many leaves as the number of vertices in W_i then for X_i we choose a $|W_i|$-set of leaves of T_{n-k+i}. Furthermore, note that one can find such a 1-degenerate ordering σ of $V(T_i)$ in which the leaves take the last places in the ordering (just modify an arbitrary 1-degenerate ordering by moving the leaves at the end). Then let $h_i\,\big|_{V(T_i)\setminus X_i} = \sigma\,\big|_{V(T_i)\setminus X_i}$ and let $h_i\,|_{X_i}$ be an arbitrary bijection from X_i to W_i. Clearly, (1.2) and (1.4) hold. Furthermore, since $T_i[X_i]$ is edgeless, $h_i\,|_{X_i}$ does not affect the packing property (1.3). □

Now we are in a position to prove Theorem 1.4. Consider 4 cases:

Case 1. $s \leq 2$. In this case we have $|W| = 2$. Since each tree has at least two leaves, we are done by Claim 3.2.

Case 2. $s \geq 5$. In this case $|W| \geq 5$ and we have to find a partial 1-degenerate packing for at most two pathlike trees. When we have only 1 pathlike tree, or there is no pathlike tree, then the existence of a packing is trivial. So, assume that $|W| = 5$ and that there are two pathlike trees. If both of them have a pending path on six vertices, then $T_{n-6}, \ldots, T_n$ pack into K_n by Theorem 1.3. If one pathlike tree has at least five leaves then by Claim 3.2, it may be skipped, so the existense of a partial 1-degenerate packing is trivial, again. Thus, by Claim 3.1, we may assume that one pathlike tree has a pending path of length six and the other one has exactly four leaves, or each pathlike tree has exactly four leaves. Suppose first that $W_i = W$ for every $i \in \mathcal{P}$, i.e. the starlike trees are those of the smallest orders. Thus, for the sets X_i, $i \in \mathcal{P}$, we choose either the set of vertices of the pending path from one pathlike tree and four leaves and one of its neighbors from the other, or four leaves and one of its neighbors from both of them. In the latter situation we map the chosen vertices as in Figure 2 and in the former, we map the chosen vertices as in Figure 3 (note that the figures presents the worst situations with the maximal number of edges inside W).

If on the other hand, a pathlike tree T_{n-k+i} has four leaves and $|W_i| \leq 4$, then by Claim 3.2, it may be skipped. Therefore it remains to show the existense of a partial 1-degenerate packing in the case when the T_{n-k+i} without four leaves has the corresponding set W_i with $|W_i| \leq 4$. Such a packing may be obtained from the one presented in Figure 3 by shifting the vertices of the pending path left.

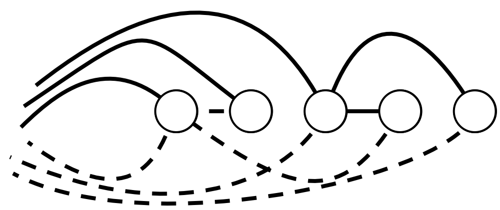

Fig. 2. Partial 1-degenerate packing of two pathlike trees each of them having 4 leaves

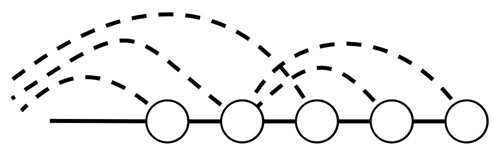

Fig. 3. Partial 1-degenerate packing of two pathlike trees, one of them having at least 4 leaves

Case 3. $s = 4$. In this case $|W| = 4$ and we have three pathlike trees. If all pathlike trees have a pending path on six vertices, then again $T_{n-6}, \dots, T_n$ pack into K_n by Theorem 1.3. Hence, by Claim 3.1, at least one pathlike tree has four leaves (or more). Thus, by Claim 3.2, one pathlike tree can be skipped, and it is enough to show a partial 1-degenerate packing of two pathlike trees each of them having exactly two or three leaves. In particular, by Claim 3.1, each of those pathlike trees has a long pending path. For the sets of vertices that will be mapped on W we choose either

(a) a pending path of length four from one tree, two pending paths of length two from the other pathlike tree, or

(b) a pending path of length four from one tree, two leaves and a pending path of length two from the other pathlike tree.

Again, assume first that $|W_i| = 4$ for every $i \in \mathcal{P}$. In subcase (a) we map the chosen vertices as in Figure 4. In subcase (b), we map the chosen vertices as in Figure 5. In both cases, if $|W_i| \leq 3$ for any $i \in \mathcal{P}$, then a partial 1-degenerate packing may be obtained from the ones presented in Figures 4 and 5 by shifting the vertices of a pending path left.

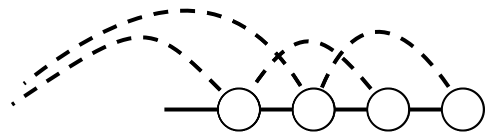

Fig. 4. Partial 1-degenerate packing in case 3(a)

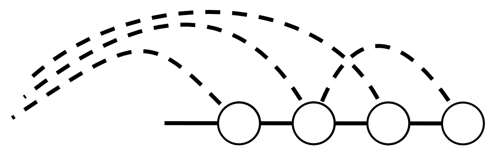

Fig. 5. Partial 1-degenerate packing in case 3(b)

Case 4. $s = 3$. In this case $|W| = 3$ and we have four pathlike trees. If each pathlike tree has a pending path on 6 vertices, then $T_{n-6}, \dots, T_n$ pack into K_n by Theorem 1.3. Hence, by Claim 3.1, at least one pathlike tree has 3 leaves (or more). On the other hand, if a pathlike tree has at least $4 \geq s \geq |W_i|$ leaves, and thus can be skipped by Claim 3.2. Therefore, it is enough to show a partial 1-degenerate packing of (at most) three paths each of them having exactly 2 leaves (or again they can be skipped by Claim 3.2) and therefore consist of long paths. Thus each of those long paths contain long pending paths. If $|W_i| = 3$ for every $i \in \mathcal{P}$ then such a packing is presented in Figure 6. On the other hand if $|W_i| \leq 2$ for some i, then by Claim 3.2, the corresponding pathlike tree may be skipped.

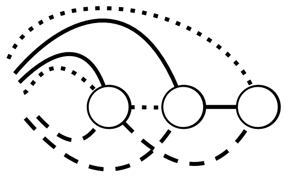

Fig. 6. Partial 1-degenerate packing in case 4

□

4. CONCLUDING REMARKS

To establish Conjecture 1.1 without imposing any constraints on the structures of the trees, our method encounters its limits at $k = 7$. Consider, for instance, the following set of eight trees:

$$\{S_{n-7}, S_{n-6}, S_{n-5}, P_{n-4}, P_{n-3}, P_{n-2}, P_{n-1}, I_n\}$$

comprising three stars, four paths, and the largest tree I_n, derived from P_{n-4} by appending two pendant edges to each of its endvertices. Clearly, I_n lacks both seven leaves and a pending path of order 7, rendering Theorem 1.3 inapplicable. Conversely, to utilize Theorem 1.2, we must designate, for each X_i, $i = 4, \ldots, 7$, the last three vertices of P_{n-8+i} within some 1-degenerate ordering of them. However, since each path has only two leaves, the last three vertices of each path contain at least one vertex of degree greater than or equal to 2. Due to 1-degeneracy, the subgraph induced by these three vertices in each path necessitates at least one edge. Consequently, the four subgraphs collectively entail at least four edges, making their packing within K_3 impossible. Consequently, Theorem 1.2 also remains inapplicable. Furthermore, multiplying the number of starlike trees (if a larger instance pack then the smaller one pack as well) only worsens the issue. Indeed, then a 1-degenerate ordering forces more edges in $T^i[X_i]$, and the preservation of the ordering in any embedding increases the number of conflicts (i.e. the number of edges that are common for at least two trees).

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