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POWER IN MODEL OF TWO – DIMENSIONAL PERCOLATION ON HEXAGONAL ELECTRICAL LATTICE

Abstract

Introduction and aims: This paper presents a power in some model of two-dimensional percolation on hexagonal lattice for various frequencies of force voltage in matrix notation. Main aim is some determination of current characteristics for created model of percolation in dependence of shorted bounds in accordance with a right algorithm.
Material and methods: Taking into account the current characteristics and other parameters some phase characteristics of percolation model have been determined for various frequencies. Analytical and numerical methods in MathCAD program were shown in the paper.
Results: Percolation current increases together with some increase of number of shorted-bounds. The characteristics of percolation current for frequency from 50Hz to 5000Hz have the similar form and increasing trend. The value of active power of percolation model increases during some increase of the number of shorted-bounds and has zero value in percolation threshold. The characteristics of active and reactive power for frequency from 50 Hz to 5000 Hz have the similar form. For frequency 10 Hz the graphs of reactive power are symmetrically placed in relation to x-axis.
Conclusion: Presented percolation model on hexagonal lattice has been verified taking using numerical values of percolation threshold.

Keywords: Power, two-dimensional model, percolation, phase characteristic.
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MOC W MODELU DWUWYMIAROWEJ PERKOLACJI NA SZEŚCIOKĄTNEJ SIECI ELEKTRYCZNEJ

Streszczenie

Wstęp i cele: *W artykule przedstawiono w zapisie macierzowym moc w modelu dwuwymiarowej perkolacji określonej na sześciokątnej sieci dla różnych częstotliwości napięcia. Głównym celem jest wyznaczenie charakterystyk prądu dla utworzonego modelu perkolacji w zależności od zwierania wiązań sieci zgodnie z przyjętym algorytmem.*
Materiał i metody: *Biorąc pod uwagę charakterystyki prądowe i wartości parametrów niektóre cechy fazowe modelu perkolacji wyznaczone zostały dla różnych częstotliwości prądu. Zastosowano metodę analityczno i numeryczną programie MathCAD.*
Wyniki: *Prąd perkolacji wzrasta równocześnie ze wzrostem liczby zrywanych wiązań. Charakterystyki prądu perkolacji dla częstotliwości od 50Hz do 5000Hz mają podobne grafy i trend wzrastający. Wartość mocy czynnej w modelu perkolacji wzrasta równocześnie ze wzrostem liczby zrywanych wiązań oraz ma wartość zero w progu perkolacji. Charakterystyki mocy czynnej i biernej dla częstotliwości od 50 Hz do 5000 Hz mają podobne grafy. Natomiast dla częstotliwości 10 Hz charakterystyka mocy biernej jest położona symetrycznie względem osi OX.*
Wniosek: *Pokazany model perkolacji na heksagonalnej sieci został zweryfikowany z uwzględnieniem wartości liczbowych progu perkolacji.*

Słowa kluczowe: *Moc, model dwuwymiarowy, perkolacja, charakterystyka faz.*
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1. Introduction

Percolation theory *(lat. percolare – to percolate)* contains some statistical and geometrical models. It was created by mathematician J.M. Hammersley in 1957 [7]. Percolation theory is used for description of very disordered systems and situations with stochastic geometry.

That theory is very interesting because it has some incidental elements in mathematical modelling and good defines a model of random surface processes.

In practice aspect percolation theory is concerned with some effects of changeable range of reciprocal interactions in disordered topological systems. Moreover in disordered systems with interactions, density, packing or concentration increasing suddenly occur some long-term ranges.

Sudden occurring of long-term ranges is defined as some percolation transition. There are two kinds of percolation on lattice structures. There is some percolation on bounds and percolation on nodes.

The bound is some connection between two nodes. Bound occurring is defined by some probability p, where $0 \le p \le 1$. Moreover when there is not any bound, than a probability is defined in the form $(1 – p)$.

Increase of some concentration p means some sudden occurring of percolation threshold p_c. Occurring of percolation threshold p_c means some existing of unlimited and expanded percolation cluster. In the other hand percolation cluster means a set of bounds or nodes connected with adjacent ones.

The models of two-dimensional percolation are created on some lattices. As a rule, that kind of model is defined by percolation threshold using bounds p_c and sites p_c^1.

The percolation thresholds for selected lattices, which create above models, are shown in the table 1 [16].

Table 1. Percolation thresholds for bounds and sites of selected lattices

No.	Kind of lattice	Dimension d	Co-ordinating number q	Percolation threshold p_c	Percolation threshold p_c^1
1	Triangular	2	6	0,3473	0,5000
2	Square	2	4	0,5000	0,5930
3	Cagomé	2	4	0,4500	0,6527
4	Hexagonal	2	3	0,6527	0,698

The authors did not find in literature a problem of power for two-dimensional percolation model on lattice with series bounds R and C in matrix form by using the complex numbers. Moreover, the authors did not find in literature some current and phase characteristics of percolation model on some lattices in complex notation.

Thus the main aims of this paper are:

- modelling of two-dimensional percolation on hexagonal lattice in matrix notation with bounds, which include some series connection of elements R and C,
- determination of current characteristics for created percolation model,
- determination of percolation threshold in percolation model on hexagonal lattice,
- determination of power characteristics for created percolation model.

2. Physical interpretation of power in percolation model on hexagonal electrical lattice

Model of percolation *(stochastic and geometric)* on some hexagonal lattice was created on the base of surface of VH polymer insulators (Fig. 1). The VH polymer insulators surface erode in acting of electric filed in determined by the surroundings conditions.

The VH polymer, which is some space structure of polymer chains, may be modelled by square lattice [6], [11].

The authors decided to make the right modelling using the hexagonal lattice. Taking into account fact that surface conductivity of insulator is more bigger from the inner conductivity – the polymer insulator in cylinder form may be modelled by a lattice in a ring form in 3D system (Fig. 2). Evolving that ring, we obtain a net with bounds on hexagonal lattice in 2D system.

The bounds situation and simulation of their destruction (i.e. shorted-bound) is defined in the following form:

- polymer bounds represent some real dielectrics, which additional scheme may be used as a series connections $\langle Z_{k,k} = R + 1/(j\omega C) \rangle$ of the elements R and C [3], [4],
- shorting of insulator polymer bounds means some impurities occurring with big conductance and also carbonized places on surface [9], [14], i.e. shorted-bound has some impedance $\langle Z_{k,k} = 0 \rangle$,
- bound destroying occurs as a uniform process.

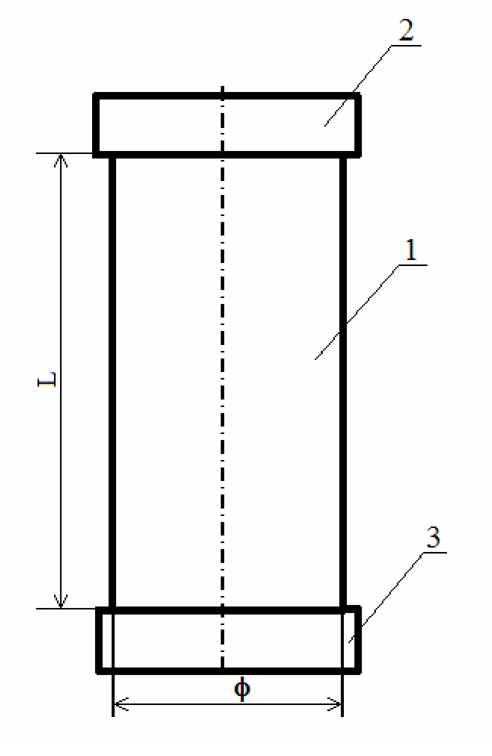

Fig. 1. Model of polymer insulator:
1 - insulator cylinder surface,
2 - upper electrode, 3 - lower electrode,
L- insulator length, ϕ - insulator diameter
Source: Elaborated by the Authors

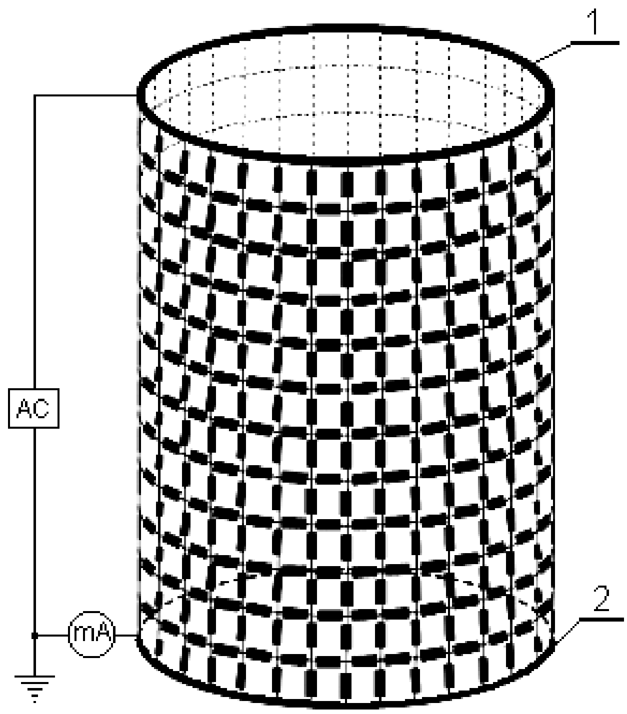

Fig. 2. Model of polymer insulator with square lattice:
$\boxed{\text{AC}}$ - insulator electric circuit,
1 - upper electrode, 2 - lower electrode
Source: Elaborated by the Authors

3. Analytical form of power in percolation model on hexagonal lattice in matrix notation

3.1 Definition of percolation threshold

Percolation threshold p_c of two-dimensional percolation model created on hexagonal lattice during short-bounding is defined by the following formula:

$$p_c = \frac{\sum_{i=1}^{m} Nz_i}{Zw_1 + \sum_{i=1}^{n} N_i} \tag{1}$$

where the symbol N_i means the number of lattice bounds ($1 \le i \le n$), Nz_i - the number of lattice short-bounds ($1 \le i \le m < n$), Zw_1 – one bound of inner impedance of voltage source, for $m, n \in N$.

During the shorting bounds of lattice with applicable forced voltage sudden occurs a percolation threshold. The specific quality of percolation threshold (1) is a sudden increase of current, which tends to infinity.

3.2 Power characteristics of percolation model on hexagonal lattice

Model of two-dimensional percolation on hexagonal lattice contains twenty one meshes *(i.e. unit cells).* Mesh structure of lattice is created by some branches *(i.e. bounds),* which refer to polymer chains. But bounds of meshes are created by some real dielectrics presented by some series connections of the elements R and C. The analysed model can be described by some method of Maxwell mesh currents [1], [2]. The figure 3 shows some structure of two-dimensional percolation model.

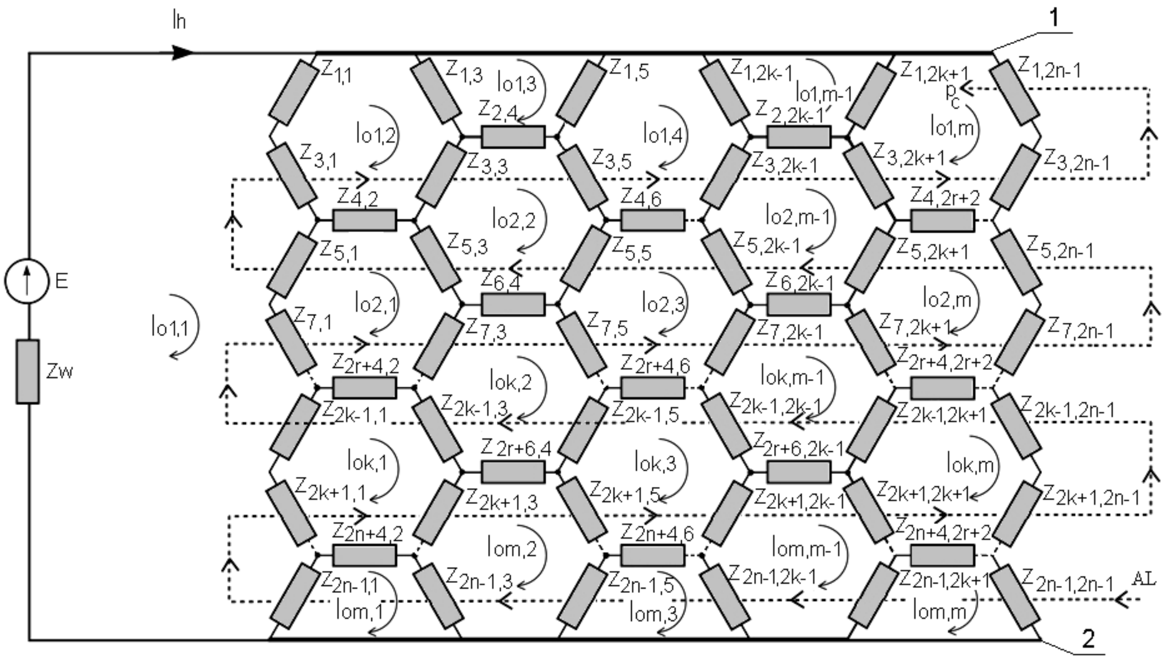

Fig. 3. Model of percolation on hexagonal lattice with algorithm of bounds destruction:
1 – upper electrode, 2 – lower electrode, E – electromotive force, AL – algorithm of bounds destruction Ih – percolation current, p_c - percolation threshold, $I_{ok,m}$ mesh currents for k=1,2,..., m≤n, n∈N, Zw – impedance of polymer bounds
⟨ i.e. series connections of elements R and C, $Z_{k,k} = R + 1/(j\omega C)$ for k = 0, 2, 4, 6, 8, ... ⟩
Source: Elaborated by the Authors

The structure of two-dimensional percolation model is described by the matrix equation:

$$\mathbf{Zo} \cdot \mathbf{Io} = \mathbf{Eo}\ , \tag{2}$$

where the symbol **Zo** means matrix of mesh impedance for percolation model, which describes some structure of bounds on square lattice, **Io** - one-column matrix, which is created by the vector of mesh current of percolation model on hexagonal lattice, **Eo** - one-column matrix, which is created by the vector of electromotive mesh forces of percolation model on hexagonal lattice. Matrices **Zo**, **Io** and **Eo** are defined by the following formulae:

$$\mathbf{Ih1} = \begin{bmatrix} Z_{1,1} & -Z_{1,2} & \cdots & -Z_{1,i} & \cdots & -Z_{1,n} \\ -Z_{2,1} & Z_{2,2} & \cdots & -Z_{2,i} & \cdots & -Z_{2,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -Z_{i,1} & -Z_{i,2} & \cdots & -Z_{i,i} & \cdots & -Z_{i,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -Z_{n,1} & -Z_{n,2} & \cdots & -Z_{n,i} & \cdots & -Z_{n,n} \end{bmatrix}, \tag{3}$$

$$\mathbf{Io} = \begin{bmatrix} Io_1 \\ Io_2 \\ \vdots \\ Io_i \\ \vdots \\ Io_n \end{bmatrix}, \tag{4}$$

$$\mathbf{Eo} = \begin{bmatrix} Eo_1 \\ Eo_2 \\ \vdots \\ Eo_i \\ \vdots \\ Eo_n \end{bmatrix}. \tag{5}$$

Left-sided multiplying the equation (2) by the inverse matrix $(\mathbf{Zo})^{-1}$ to impedance mesh matrix **Zo** we obtain the following matrix equation:

$$(\mathbf{Zo})^{-1} \cdot \mathbf{Zo} \cdot \mathbf{Io} = (\mathbf{Zo})^{-1} \cdot \mathbf{Eo}\ . \tag{6}$$

Taking into account the formula (6) and following matrix properties

$$(\mathbf{Zo})^{-1} \cdot \mathbf{Zo} = \mathbf{I} \quad \text{and} \quad \mathbf{I} \cdot \mathbf{Io} = \mathbf{Io} \tag{7}$$

where $\mathbf{I}$ - identity matrix. We obtain some one-column matrix of mesh currents in the form:

$$\mathbf{Io} = (\mathbf{Zo})^{-1} \cdot \mathbf{Eo}\ . \tag{8}$$

In the case of shorted bounds for hexagonal lattice in the sequence defined by AL algorithm, shown on the figure 1, the one-column matrix of mesh currents describes the following matrix equation:

$$\mathbf{Io}(\mathbf{Nh}) = [\mathbf{Zo}(\mathbf{Nh})]^{-1} \cdot \mathbf{Eo} \tag{9}$$

where the symbol **Nh** means some vector of shorted-bounds number of lattice.
The current of two-dimensional percolation model **Is**, created on hexagonal lattice (Fig. 1), is equal to mesh current Io_1. The mesh current Io_1 refers to the first row of mesh current vector.

For one-column matrix **X**:

$$\mathbf{X} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

we obtain the one-row transpose matrix $\mathbf{X}^\mathbf{T}$, which has the following form:

$$\mathbf{X}^\mathbf{T} = [1,0,\ldots,0] \tag{11}$$

Left-sided multiplying the matrix equation (9) by matrix (11), we obtain the current **Is** in the following matrix notation:

$$\mathbf{Ih} = \mathbf{X}^\mathbf{T} \cdot [\mathbf{Zo}(\mathbf{Nh})]^{-1} \cdot \mathbf{Eo} \tag{12}$$

where

$$\mathbf{Ih} = \mathbf{X}^\mathbf{T} \cdot \mathbf{Io}(\mathbf{Nh}) \tag{13}$$

and the symbol $\mathbf{X}^\mathbf{T}$ means, in the other hand, some neutralization vector of mesh currents.

3.3 Power for percolation model on hexagonal electrical lattice

One-column matrix of impedance for two-dimensional percolation model created on hexagonal lattice (i.e. series structure of bounds R and C) is defined from the 2-nd Kirchoff's law in the following matrix form:

$$\mathbf{Ih} \cdot (\mathbf{Zh} + \mathbf{Zw}) = \mathbf{E} \tag{14}$$

where the symbol **Zh** – means a one-column matrix of impedance for percolation model created on hexagonal lattice, **Ih** – one-column matrix, which creates a current vector of percolation model on hexagonal lattice, **E** – one-column matrix, which rows are some values of electromotive force of percolation model created on hexagonal lattice, **Zw** – one-column matrix, which creates some inner impedance of electromotive force of percolation model.

Left-sided multiplying the matrix equation (14) by transverse matrix of the matrix $\mathbf{Ih}_\mathbf{f}$ to the current matrix of percolation model we obtain the following matrix equation:

$$(\mathbf{Ih})^{-1} \cdot \mathbf{Ih} \cdot (\mathbf{Zh} + \mathbf{Zw}) = (\mathbf{Ih})^{-1} \cdot \mathbf{E}\ . \tag{15}$$

Thus we obtain:

$$\mathbf{Zh} + \mathbf{Zw} = (\mathbf{Ih})^{-1} \cdot \mathbf{E}\ . \tag{16}$$

After both-sided subtraction of the matrix **Zw**, we totally obtain a one-column impedance matrix of percolation model in the following form:

$$\mathbf{Zh} = (\mathbf{Ih})^{-1} \cdot \mathbf{E} - \mathbf{Zw}\ . \tag{17}$$

Power, in matrix notation, for percolation model on hexagonal lattice is determined from the following matrix equation [3]:

$$\mathbf{Sh}(\mathbf{Nh}) = \mathbf{Ih1} \cdot \mathbf{Zh1} \cdot \mathbf{Ih^*} \equiv \mathbf{Ph} + j\mathbf{Qh} \tag{18}$$

where

$$\mathbf{Ih1} = \begin{bmatrix} \mathrm{Ih}_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mathrm{Ih}_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathrm{Ih}_i & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathrm{Ih}_n \end{bmatrix} \tag{19}$$

means a diagonal matrix of currents for percolation model, and

$$\mathbf{Zh1} = \begin{bmatrix} \mathrm{Zh}_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mathrm{Zh}_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathrm{Zh}_i & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathrm{Zh}_n \end{bmatrix} \tag{20}$$

shows a diagonal matrix of impedance, and a one-column matrix

$$\mathbf{Ih*} = \begin{bmatrix} \mathrm{Re}(\mathrm{Ih}_1) + (-j)\,\mathrm{Im}(\mathrm{Ih}_1) \\ \mathrm{Re}(\mathrm{Ih}_2) + (-j)\,\mathrm{Im}(\mathrm{Ih}_2) \\ \vdots \\ \mathrm{Re}(\mathrm{Ih}_i) + (-j)\,\mathrm{Im}(\mathrm{Ih}_i) \\ \vdots \\ \mathrm{Re}(\mathrm{Ih}_n) + (-j)\,\mathrm{Im}(\mathrm{Ih}_n) \end{bmatrix} \tag{21}$$

is some vector of coupling currents.

Taking into account the matrix equation (18) it is possible to determine the active power **Ph** of percolation model on hexagonal lattice from the following relation:

$$\mathbf{Ph} = \mathrm{Re}[\mathbf{Sh}(\mathbf{Nh})] \tag{22}$$

where

$$\mathbf{Ph} = \begin{bmatrix} \mathrm{Re}(\mathrm{Sh}_1) \\ \mathrm{Re}(\mathrm{Sh}_2) \\ \vdots \\ \mathrm{Re}(\mathrm{Sh}_i) \\ \vdots \\ \mathrm{Re}(\mathrm{Sh}_n) \end{bmatrix} \tag{23}$$

is the one-column matrix as a vector of active power.

Also using the matrix equation (18) it is possible to determine the reactive power **Qh** of percolation model from the following relation:

$$\mathbf{Qh} = \mathrm{Im}[\mathbf{Sh}(\mathbf{Nh})] \tag{24}$$

where

$$\mathbf{Qh} = \begin{bmatrix} \mathrm{Im}(\mathrm{Sh}_1) \\ \mathrm{Im}(\mathrm{Sh}_2) \\ \vdots \\ \mathrm{Im}(\mathrm{Sh}_i) \\ \vdots \\ \mathrm{Im}(\mathrm{Sh}_n) \end{bmatrix} \tag{25}$$

is the one-column matrix as a vector of reactive power.

4. Numerical analysis of power and currents for percolation model on hexagonal lattice

4.1 Characteristics of current in complex notation

Obtained current characteristics of percolation model on hexagonal lattice for series bounds of the elements R *(i.e. resistors)* and C *(i.e. condensers)* in dependence from method of shorted-bounds for frequency f10 = 10 Hz, f50 = 50 Hz, f100 = 100 Hz, f200 = 200 Hz, f5000 = 5000 Hz, calculated by using the formula (21) are shown on the figure 4.

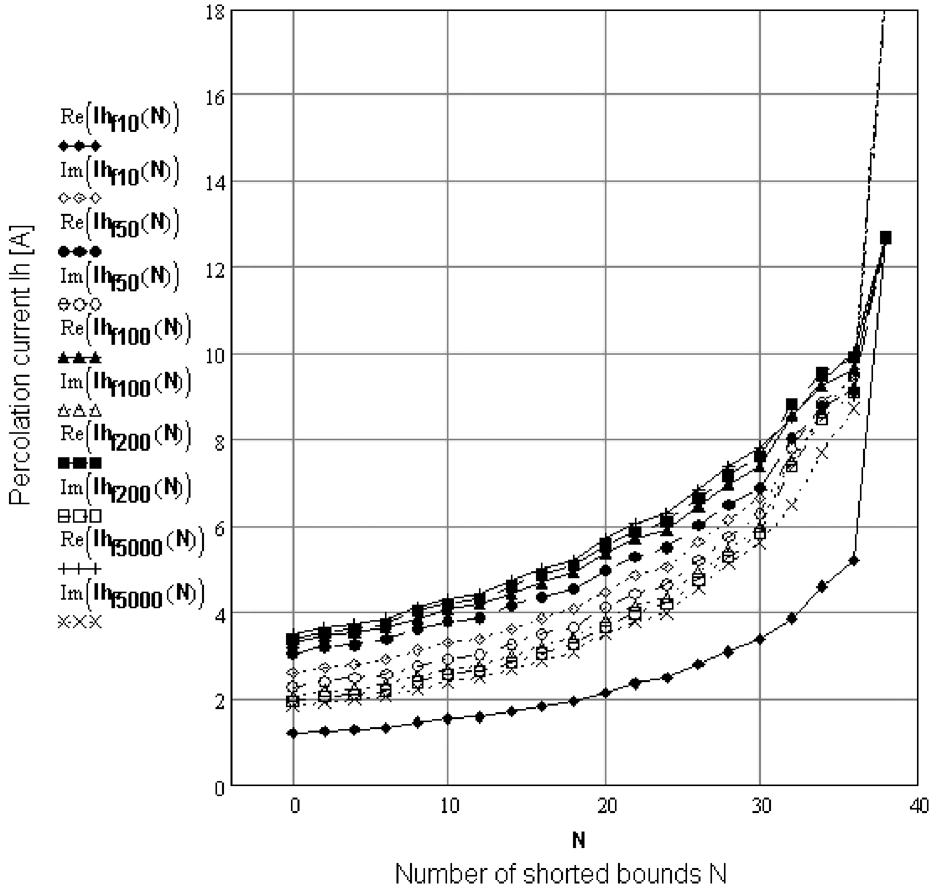

Fig. 4. Current characteristic **Ih** in [A] of percolation model on hexagonal lattice for frequency of forced voltage 10 [Hz], 50 [Hz], 100 [Hz], 200 [Hz] and 5000 [Hz] vs. Number of shorted bounds N

Source: Elaborated by the Authors

4.2 Characteristics of active and reactive power in complex notation

Obtained characteristics of active and reactive power of percolation model on hexagonal lattice for shorted-bounds of the elements R *(i.e. resistors)* and C *(i.e. condensers)* in dependence from method of shorted-bounds for frequency $f10 = 10$ Hz, $f50 = 50$ Hz, $f100 = 100$ Hz, $f200 = 200$ Hz, $f5000 = 5000$ Hz calculated by using the formulae (23) and (25) are shown on the figure 5.

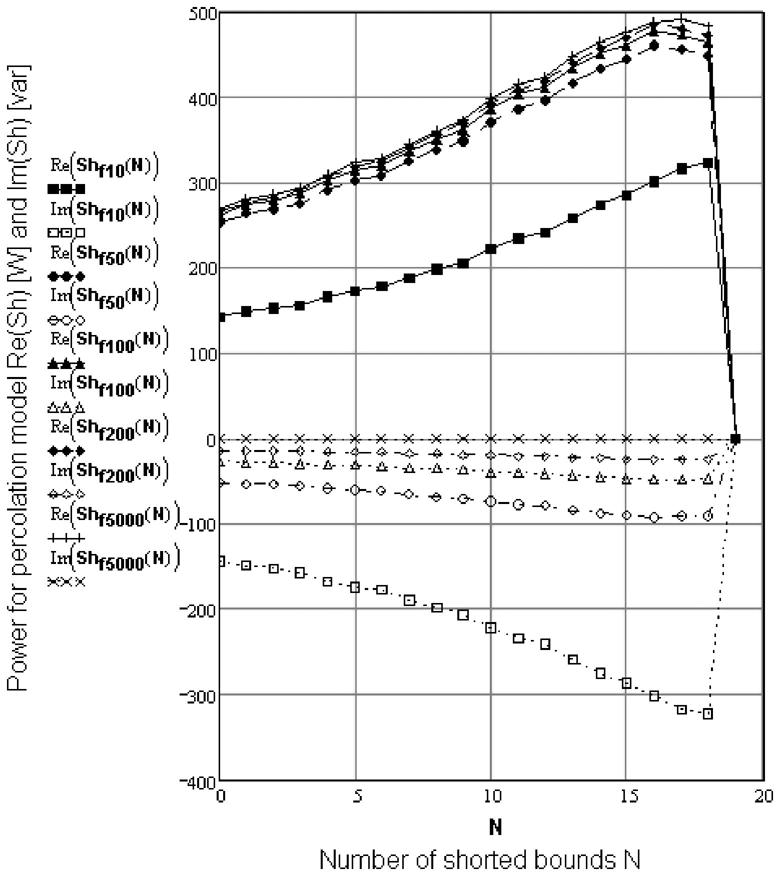

Fig. 5. Characteristics of active power $Re(Sh_f)$ in [W] and reactive power $Im(Sh_f)$ in [var] of percolation model on hexagonal lattice for frequency of forced voltage 10 [Hz], 50 [Hz], 100 [Hz], 200 [Hz] and 5000 [Hz] vs. Number of shorted bounds N

Source: Elaborated by the Authors

5. Verification of simulation results

Taking into account the simulation results of created percolation model on various lattices were determined percolation thresholds for bounds using the formula (1). The calculation results are shown on the table 2.

Table 2. Numerical values of percolation thresholds determined for selected lattices by the formula (1)

No.	Kind of lattice	Dimension d	Co-ordinating number q	Percolation threshold p_c
1	Triangular	2	6	0,3333
2	Square	2	4	0,5000
3	Hexagonal	2	3	0,6720

6. Conclusions

- Percolation current increases together with some increase of number of shorted-bounds. It impetuously increases in percolation threshold. The characteristics of percolation current for frequency from 50Hz to 5000Hz have the similar form and increasing trend.
- The value of active power of percolation model increases during some increase of the number of shorted-bounds and has zero value in percolation threshold. The characteristics of active and reactive power for frequency from 50 Hz to 5000 Hz have the similar form. But for frequency 10 Hz the characteristics of reactive power are symmetrically placed in relation to x-axis.

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