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ENTIRE SOLUTIONS FOR SOME CRITICAL EQUATIONS IN THE HEISENBERG GROUP

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Abstract. We complete the study started in the paper [P. Pucci, L. Temperini, *On the concentration–compactness principle for Folland–Stein spaces and for fractional horizontal Sobolev spaces*, Math. Eng. 5 (2023), Paper no. 007], giving some applications of its abstract results to get existence of solutions of certain critical equations in the entire Heinseberg group. In particular, different conditions for existence are given for critical horizontal p-Laplacian equations.

Keywords: Heisenberg group, entire solutions, critical exponents.

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1. INTRODUCTION

In this paper, we consider the critical equation

$$-\Delta_{H,p}u = \lambda w(\xi)|u|^{q-2}u + K(\xi)|u|^{p^*-2}u \quad \text{in } \mathbb{H}^n, \tag{$\mathcal{E}$}$$

with $1 < p < Q$, where $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$; furthermore, $p \le q < p^*$ and

$$p^* = \frac{pQ}{Q-p}$$

is the critical exponent associated to p.

The operator $\Delta_{H,p}$ is the well known horizontal p (Kohn–Spencer) Laplacian, which is defined as

$$\Delta_{H,p}\varphi = \operatorname{div}_H(|D_H\varphi|_H^{p-2}D_H\varphi),$$

for all $\varphi \in C^2(\mathbb{H}^n)$. Here the vector

$$D_H\varphi = (X_1\varphi, \cdots, X_n\varphi, Y_1\varphi, \cdots, Y_n\varphi)$$

denotes the horizontal gradient of φ, where $\{X_j, Y_j\}_{j=1}^n$ is the basis of the horizontal left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

for $j = 1, \dots, n$.

Critical problems have been intensively studied in the last decades, starting with the pioneering paper by Brezis and Nirenberg [8] for the Dirichlet Laplacian problems in bounded domains of $\mathbb{R}^N$.

In the context of stratified groups, the study of critical equations has received a great deal of interest in the last years, due to the connection with the Yamabe problem and the Webster scalar curvature problem on CR manifolds. More precisely, when $p = 2$ and Ω is a smooth bounded domain of $\mathbb{H}^n$, the Dirichlet problem

$$\begin{cases} -\Delta_H u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has been studied by Citti in [9], using the explicit knowledge of the Sobolev minimizers for the horizontal gradient, see the paper of Jerison and Lee [22]. For other extensions to Choquard critical linearities in Dirichlet problems we refer to [17] and the references therein.

Afterwards, Loiudice studied in [24] the existence of positive and sign changing solutions for the problem in a general Carnot group $\mathbb{G}$. The key tools are the results due to Garofalo and Vassilev in [15], concerning the best constant in the Folland–Stein embedding on Carnot groups, obtained by using concentration–compactness arguments, and a deep analysis developed by Bonfiglioli and Uguzzoni in [5]. Interestingly, unlike the Euclidean case, in the Heinseberg setting the existence of positive solutions for the problem is not related to the space dimension. Indeed, in the space $\mathbb{R}^N$ it is well known that a different behavior occurs when $N = 3$ and $N \geq 4$. This phenomenon – known as "critical dimension" and observed in a wide class of elliptic critical problems – does not occur here, since the homogeneous dimension $Q = 2n + 2$, which plays the key role in this context, is always greater or equal to 4.

In [26], Molica Bisci and Repovš prove the existence of at least one nontrivial solution for a subelliptic critical equation with subcritical continuous perturbations on a smooth and bounded domain Ω of a Carnot group $\mathbb{G}$. This type of problems naturally arises in the study of the Yamabe problem for a CR manifold (M, g). Let us also mention the paper by Garofalo and Lanconelli [13] and its references, where the authors study differential problems involving subelliptic operators on stratified groups.

Concerning the study on the nonlocal equations in the Heisenberg group we refer to [16, 30].

The Folland–Stein space $S^{1,p}(\Omega)$, $1 < p < Q$, is defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|D_H\varphi\|_p = \left(\int_\Omega |D_H\varphi|_H^p d\xi \right)^{1/p}.$$

When the domain Ω is not bounded, $S^{1,p}(\Omega)$, $1 < p < Q$, may not be compactly embedded into a Lebesgue space. This lack of compactness produces several difficulties in applying the variational methods. For example, Molica Bisci and Pucci in [25] study the problem

$$\begin{cases} -\Delta_H u + u = h(\xi)f(u) + \lambda|u|^{q^*-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

in a domain Ω which satisfies some geometrical assumptions to recover compactness.

Finally, in the more general case $1 < p < Q$, Bordoni, Filippucci and Pucci in [6] establish existence and asymptotic behavior of nontrivial solutions for the following problem involving Hardy terms in bounded and unbounded domains of $\mathbb{H}^n$

$$\begin{cases} -\Delta_{H,p} u - \gamma\psi^p \cdot \dfrac{|u|^{p-2}u}{r^p} = \lambda w(\xi)|u|^{q-2}u + K(\xi)|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ and λ are real parameters, the exponent q is such that $p < q < p^*$, r is the Korányi norm $r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4}$, $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2n}$. Furthermore, the weight function ψ is defined as $\psi = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n}$, while the weights w and K satisfy

$$\begin{aligned} &(BFP)_1 \quad w > 0 \textit{ a.e. in } \mathbb{H}^n \textit{and } w \in L^{\wp}(\mathbb{H}^n), \textit{with } \wp = \frac{p^*}{p^* - q} \\ & \textit{and } p < q < p^*, \\ &(BFP)_2 \quad K \geq 0 \textit{ a.e. in } \mathbb{H}^n \textit{ and } K \in L^\infty(\mathbb{H}^n). \end{aligned}$$

Note that when $\gamma = 0$ we recover equation $(\mathcal{E})$, in the special case $p < q < p^*$. We also mention the works [27–29], concerning critical problems in $\mathbb{H}^n$, involving subelliptic operators, with nonstandard growth conditions.

In this paper, when dealing with equation $(\mathcal{E})$, we distinguish two different situations:

(1) $1 < p < q < p^*$,
(2) $1 < p = q < p^*$.

In the first case, we assume that

(w_1) $w \geq 0$, $w \in L^1_{\text{loc}}(\mathbb{H}^n)$, *is such that the embedding* $S^{1,p}(\mathbb{H}^n) \hookrightarrow L^q(\mathbb{H}^n, wd\xi)$ *is compact*;

(K_1) $K > 0$ *a.e. in* $\mathbb{H}^n$, $K \in L^\infty(\mathbb{H}^n)$ *and* $\lim_{r(\xi)\to\infty} K(\xi) = K_\infty \in \mathbb{R}^+_0$,

where $\mathbb{R}^+_0 = [0, \infty)$.

Then it is possible to prove the following result.

Theorem 1.1. *Let* $1 < p < Q$ *and* $p < q < p^*$. *Assume that* (w_1) *and* (H_1) *are satisfied. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$ *the equation* $(\mathcal{E})$ *admits at least a nontrivial solution.*

Clearly, Theorem 1.1 trivially extends Theorem 1.1 of [6], when $\gamma = 0$. Indeed, condition $(BFP)_1$ ensures, by means of Lemma 3.1 of [6], that the embedding $S^{1,p}(\mathbb{H}^n) \hookrightarrow L^q(\mathbb{H}^n, wd\xi)$ is compact or, in other words, that (w_1) holds. Theorem 1.1 is obtained via an application of the concentration–compactness results in [29].

The second case, namely when p and q are equal, is more challenging and is not treated in [6]. Following somehow [4], if $p = q$ we assume that $K(\xi) \equiv 1$, so that (K_1) is trivially satisfied, and that w verifies (w_1) and the additional request

(w_2) *$w \in L^\infty(\mathbb{H}^n)$ and there exists $\xi_0 \in \mathbb{H}^n$ such that w is continuous at ξ_0 and $w(\xi_0) > 0$.*

We are then able to prove the following theorem.

Theorem 1.2. *Let $p > 1$ be such that $p^2 < Q$. Assume that the function w satisfies (w_1) with $p = q$ and (w_2) and that $K \equiv 1$. Then, equation $(\mathcal{E})$ admits at least a nontrivial solution for any $\lambda \in (0, \lambda_1)$, where*

$$\lambda_1 = \lambda_1(w) = \inf_{v \in S^{1,p}(\mathbb{H}^n)} \frac{\|D_H v\|_p^p}{\int_{\mathbb{H}^n} w(\xi)|v|^p d\xi}. \tag{1.1}$$

The idea behind the construction of the solution in Theorem 1.2 goes back to the seminal paper by Brezis and Nirenberg [8]. The main difficulty is the unavailability of an explicit form of the extremals for the Folland–Stein embedding. Let us recall it. *If $1 < p < Q$, then there exists a positive constant $C = C(p, Q) > 0$ such that*

$$\int_{\mathbb{H}^n} |\varphi|^{p^*} d\xi \le C \int_{\mathbb{H}^n} |D_H \varphi|_H^p d\xi \quad \textit{for all } \varphi \in C_c^\infty(\mathbb{H}^n). \tag{1.2}$$

The above result is due to Folland and Stein [11] and it is valid in the more general context of Carnot groups. Unlike the Euclidean case, cf. [32] and [1], the value of the best constant in (1.2) is unknown. In the particular case $p = 2$, the problem of the determination of the best constant in (1.2) is related to the CR Yamabe problem and it has been solved by the works of Jerison and Lee [20–23]. In the general case, existence of extremal functions of (1.2) was proved by Vassilev in [33] via the concentration–compactness method of Lions, see also [19]. This method does not allow an explicit determination of the best constant C_{p^*} of (1.2). However, we know from [33] that C_{p^*} is achieved in the Folland–Stein space $S^{1,p}(\mathbb{H}^n)$ and so we can write the best constant $C_{p^*} = C_{p^*}(p, Q)$ of the Folland–Stein inequality (1.2) as

$$C_{p^*} = \inf_{\substack{u \in S^{1,p}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_H u\|_p^p}{\|u\|_{p^*}^p} \tag{1.3}$$

and clearly $C_{p^*} > 0$.

Note that the Euler–Lagrange equation of the nonnegative extremals of (1.2) leads to the critical equation

$$-\Delta_{H,p} u = |u|^{p^*-2} u \quad \text{in } \mathbb{H}^n.$$

Thus, what is known from Theorem 1.2 of [24] is that if $1 < p < Q$, then there exists an extremal $U \in S^{1,p}(\mathbb{H}^n)$ for (1.2) and the following estimate holds:

$$U(\xi) \sim r(\xi)^{\frac{p-Q}{p-1}} \quad \text{as } r(\xi) \to \infty. \tag{1.4}$$

The knowledge of the exact asymptotic behavior at infinity of Sobolev extremals turns out to be crucial in order to obtain existence results for the Brezis–Nirenberg type problems whenever the explicit form of minimizers is not known. Finally, assumption $p^2 < Q$, together with (1.4), ensures that $U \in L^p(\mathbb{H}^n)$ since otherwise, as we already noted, functions in $S^{1,p}(\mathbb{H}^n)$ may not belong to the Lebesgue space $L^p(\mathbb{H}^n)$.

The paper is divided into three sections. Section 2 contains some preliminaries and notations. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. PRELIMINARIES

We briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [13, 14, 19, 33]. Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n+1$, that is the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and whose group structure is given by the non–Abelian law

$$\xi \circ \xi' = \left(z + z', t + t' + 2\sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right)$$

for all ξ, $\xi' \in \mathbb{H}^n$, with

$$\xi = (z,t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \quad \text{and} \quad \xi' = (z', t') = (x_1', \ldots, x_n', y_1', \ldots, y_n', t').$$

The inverse is given by $\xi^{-1} = -\xi$ and so $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

The real Lie algebra of $\mathbb{H}^n$ is generated by the left–invariant vector fields on $\mathbb{H}^n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk} T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

Moreover, all the commutators of length greater than two vanish, and so $\mathbb{H}^n$ is a nilpotent graded stratified group of step two. A left invariant vector field X, which is in the span of $\{X_j, Y_j\}_{j=1}^n$, is called *horizontal*.

For each real positive number R, the dilation $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$, naturally associated with the Heisenberg group structure, is defined by

$$\delta_R(\xi) = (Rz, R^2 t) \quad \text{for all } \xi = (z,t) \in \mathbb{H}^n.$$

It is easy to verify that the Jacobian determinant of the dilatation δ_R is constant and equal to R^{2n+2}, where the natural number $Q = 2n+2$ is the *homogeneous dimension* of $\mathbb{H}^n$, which coincides with its Hausdorff dimension.

The anisotropic dilation structure on $\mathbb{H}^n$ introduces the *Korányi norm*, which is given by

$$r(\xi) = r(z,t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } \xi = (z,t) \in \mathbb{H}^n.$$

Consequently, the Korányi norm is homogeneous of degree 1, with respect to the dilations δ_R, $R > 0$, that is

$$r(\delta_R(\xi)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = R\, r(\xi) \quad \text{for all } \xi = (z,t) \in \mathbb{H}^n.$$

Clearly, $\delta_R(\eta \circ \xi) = \delta_R(\eta) \circ \delta_R(\xi)$. The corresponding distance, the so called *Korányi distance*, is

$$d_K(\xi, \xi') = r(\xi^{-1} \circ \xi') \quad \text{for all } (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

Let $B_R(\xi_0) = \{\xi \in \mathbb{H}^n \,:\, d_K(\xi, \xi_0) < R\}$ be the *Korányi open ball* of radius R centered at ξ_0. For simplicity we put $B_R = B_R(O)$, where $O = (0,0)$ is the natural origin of $\mathbb{H}^n$.

The Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under the left translations of the Heisenberg group. Thus, since the Haar measures on Lie groups are unique up to constant multipliers, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n+1)$–Lebesgue measure and by $|U|$ the $(2n+1)$–dimensional Lebesgue measure of any measurable set $U \subseteq \mathbb{H}^n$. Furthermore, the Haar measure on $\mathbb{H}^n$ is Q–homogeneous with respect to dilations δ_R. Consequently,

$$|\delta_R(U)| = R^Q |U|, \quad d(\delta_R \xi) = R^Q d\xi.$$

In particular, $|B_R(\xi_0)| = |B_1| R^Q$ for all $\xi_0 \in \mathbb{H}^n$.

We define the *horizontal gradient* of a C^1 function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$D_H u = \sum_{j=1}^{n} \left[(X_j u) X_j + (Y_j u) Y_j \right].$$

Clearly, $D_H u \in \operatorname{span}\{X_j, Y_j\}_{j=1}^n$. In $\operatorname{span}\{X_j, Y_j\}_{j=1}^n \simeq \mathbb{R}^{2n}$ we consider the natural inner product given by

$$\big(X, Y\big)_H = \sum_{j=1}^{n} \left(x^j y^j + \widetilde{x}^j \widetilde{y}^j \right)$$

for $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \widetilde{y}^j Y_j\}_{j=1}^n$. The inner product $\big(\cdot, \cdot\big)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{\big(X, X\big)_H}$$

for any horizontal vector field X.

Let $X = X(\xi)$, $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$, be a horizontal vector field function of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$. Then, the *horizontal divergence* of X is defined by

$$\operatorname{div}_H X = \sum_{j=1}^{n} [X_j(x^j) + Y_j(\widetilde{x}^j)].$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the *Kohn–Spencer Laplacian* in $\mathbb{H}^n$, or equivalently the *horizontal Laplacian*, or the *sub–Laplacian*, of u is

$$\begin{aligned}\Delta_H u &= \sum_{j=1}^n (X_j^2 + Y_j^2)u \\ &= \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2}.\end{aligned}$$

According to the celebrated Theorem 1.1 and the terminology due to Hörmander in [18], the operator Δ_H is *hypoelliptic*. In particular, $\Delta_H u = \operatorname{div}_H D_H u$ for each $u \in C^2(\mathbb{H}^n)$.

A well known generalization of the Kohn–Spencer Laplacian is the *horizontal p–Laplacian* on the Heisenberg group, $p \in (1, \infty)$, defined by

$$\Delta_{H,p}\varphi = \operatorname{div}_H(|D_H\varphi|_H^{p-2} D_H\varphi) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n).$$

3. PROOF OF THEOREM 1.1

From now on we require that the structural assumptions (w_1) and (K_1) hold and that $1 < p \le q < Q$, without further mentioning unless necessary. Observe that (weak) solutions of $(\mathcal{E})$ correspond to critical points of the associated Euler–Lagrange functional I_λ, with $I_\lambda : S^{1,p}(\mathbb{H}^n) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{p}\|D_H u\|_p^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p^*}\|u\|_{p^*,K}^{p^*},$$

where

$$\|u\|_{q,w} = \left(\int_{\mathbb{H}^n} w(\xi)|u|^q d\xi \right)^{1/q} \quad \text{and} \quad \|u\|_{p^*,K} = \left(\int_{\mathbb{H}^n} K(\xi)|u|^{p^*} d\xi \right)^{1/p^*}.$$

Note that I_λ is of class a $C^1(S^{1,p}(\mathbb{H}^n))$ by the structural assumptions and

$$\langle I_\lambda'(u), v\rangle = \langle u, v\rangle_p - \lambda\langle u, v\rangle_{q,w} - \langle u, v\rangle_{p^*,K}$$

for any $u, v \in S^{1,p}(\mathbb{H}^n)$.

From here on $\langle \cdot, \cdot \rangle$ simply denotes the dual pairing between $S^{1,p}(\mathbb{H}^n)$ and its dual space $[S^{1,p}(\mathbb{H}^n)]'$ and for brevity we put

$$\langle u, v\rangle_p = \int_{\mathbb{H}^n} \left(|D_H u|_H^{p-2} D_H u, D_H v\right)_H d\xi, \quad \langle u, v\rangle_{q,w} = \int_{\mathbb{H}^n} w(\xi)|u|^{q-2}uv\, d\xi,$$

$$\langle u, v\rangle_{p^*,K} = \int_{\mathbb{H}^n} K(\xi)|u|^{p^*-2}uv\, d\xi.$$

The simplified notation is reasonable, since for all $u \in S^{1,p}(\mathbb{H}^n)$ the functionals $\langle u, \cdot\rangle_p$, $\langle u, \cdot\rangle_{q,w}$, $\langle u, \cdot\rangle_{p^*,K}$ are linear and bounded on $S^{1,p}(\mathbb{H}^n)$.

Before studying equation $(\mathcal{E})$, let us recall the following, crucial inequality, originally proved by Simon in [31]. For all $s \in (1,\infty)$ there exists $\kappa > 0$, depending only on s, such that

$$|X - Y|_H^s \le \kappa \begin{cases} A_s(X,Y), & s \ge 2, \\ A_s(X,Y)^{s/2} \cdot (|X|^s + |Y|^s)^{(2-s)/2}, & 1 < s \le 2, \end{cases} \tag{3.1}$$

where

$$A_s(X,Y) = \big(|X|_H^{s-2}X - |Y|_H^{s-2}Y, X - Y\big)_H$$

for all X and Y in the span of $\{X_j, Y_j\}_{j=1}^n$.

For later purposes, let us observe that the eigenvalue λ_1 defined in (1.1) is strictly positive. Indeed, by (w_1), the embedding $S^{1,p}(\mathbb{H}^n) \hookrightarrow L^q(\mathbb{H}^n, wd\xi)$ is continuous. Therefore, there exists $C = C(p,q,w) > 0$ such that

$$\int_{\mathbb{H}^n} w(\xi)|v|^q d\xi \le C\|D_H v\|_p^p \quad \text{for any } v \in S^{1,p}(\mathbb{H}^n).$$

Hence, passing to the infimum over $v \in S^{1,p}(\mathbb{H}^n)$ we get

$$\lambda_1 = \inf_{v \in S^{1,p}(\mathbb{H}^n)} \frac{\|D_H v\|_p^p}{\int_{\mathbb{H}^n} w(\xi)|v|^q d\xi} \ge \frac{1}{C} > 0.$$

Let us recall that if $\lambda > 0$ is fixed, a sequence $(u_k)_k \subset S^{1,p}(\mathbb{H}^n)$ is a Palais–Smale sequence of I_λ at some real level c, if

$$I_\lambda(u_k) \to c \quad \text{and} \quad I'_\lambda(u_k) \to 0 \quad \text{in } [S^{1,p}(\mathbb{H}^n)]' \text{ as } k \to \infty. \tag{3.2}$$

Lemma 3.1. *Let $\lambda > 0$ be such that*

$$\begin{cases} \lambda > 0, & \text{if } 1 < p < q < p^*, \\ 0 < \lambda < \lambda_1, & \text{if } 1 < p = q < p^*, \end{cases} \tag{3.3}$$

where λ_1 is defined in (1.1)*, and let $(u_k)_k$ be a Palais–Smale sequence for I_λ at some level $c > 0$. Then, there exists $u \in S^{1,p}(\mathbb{H}^n)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $S^{1,p}(\mathbb{H}^n)$ and u is a (weak) solution of* $(\mathcal{E})$.

Proof. The proof is more or less classical, so we sketch it. Fix $\lambda > 0$ as in (3.3). Let $(u_k)_k \subset S^{1,p}(\mathbb{H}^n)$ be a Palais–Smale sequence of I_λ at a level $c > 0$. We start showing that $(u_k)_k$ is bounded in $S^{1,p}(\mathbb{H}^n)$. Thanks to (3.2), as $k \to \infty$

$$\begin{aligned} c + o(1) &\ge I_\lambda(u_k) - \begin{cases} \frac{1}{q}\langle I'_\lambda(u_k), u_k\rangle, & \text{if } 1 < p < q < p^* \\ \frac{1}{p^*}\langle I'_\lambda(u_k), u_k\rangle, & \text{if } 1 < p = q < p^* \end{cases} \\ &\ge \kappa\|D_H u_k\|_p^p \end{aligned}$$

by (K_1), (w_1) and the fact that $1 < p \le q < p^*$, where

$$\kappa = \begin{cases} \frac{1}{p} - \frac{1}{q}, & \text{if } 1 < p < q < p^*, \\ \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(1 - \frac{\lambda}{\lambda_1}\right), & \text{if } 1 < p = q < p^*. \end{cases}$$

Consequently, $(u_k)_k$ is bounded in $S^{1,p}(\mathbb{H}^n)$.

Thus, since $S^{1,p}(\mathbb{H}^n)$ is a reflexive Banach space, in virtue of Proposition 1.202 of [12] and (w_1), there exist $u \in S^{1,p}(\mathbb{H}^n)$ and two bounded nonnegative Radon measures μ and ν on $\mathbb{H}^n$, such that, up to a subsequence, we have

$$\begin{aligned} u_k \rightharpoonup u \text{ in } S^{1,p}(\mathbb{H}^n), &\quad |D_H u_k|_H^{p-2} D_H u_k \rightharpoonup \Theta \text{ in } L^{p'}(\mathbb{H}^n; \mathbb{R}^{2n}), \\ u_k \rightharpoonup u \text{ in } L^{p^*}(\mathbb{H}^n), &\quad u_k \to u \text{ in } L^q(\mathbb{H}^n, w d\xi), \\ |u_k|^{p^*} d\xi \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\mathbb{H}^n), &\quad |D_H u_k|_H^p d\xi \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbb{H}^n), \end{aligned} \tag{3.4}$$

for some $\Theta \in L^{p'}(\mathbb{H}^n; \mathbb{R}^{2n})$.

It remains to see that u is a (weak) solution of $(\mathcal{E})$. In order to see that, fix any $\varphi \in C_c^\infty(\mathbb{H}^n)$. Then, we have

$$o(1) = \langle I'_\lambda(u_k), \varphi \rangle = \langle u_k, \varphi \rangle_p - \lambda \langle u_k, \varphi \rangle_{q,w} - \langle u_k, \varphi \rangle_{p^*,K}.$$

Now, by Proposition A.8 in [2], it is clear from (3.4) that $|u_k|^{q-2} u_k \rightharpoonup |u|^{q-2} u$ in $L^{q'}(\mathbb{H}^n, w d\xi)$, and $|u_k|^{p^*-2} u_k \rightharpoonup |u|^{p^*-2} u$ in $L^{p^{*\prime}}(\mathbb{H}^n, K d\xi)$. Consequently, it follows straightly that

$$\langle u_k, \varphi \rangle_{q,w} \to \langle u, \varphi \rangle_{q,w}, \quad \langle u_k, \varphi \rangle_{p^*,K} \to \langle u, \varphi \rangle_{p^*,K}.$$

Let us now show that, passing to a subsequence if necessary,

$$D_H u_k \to D_H u \quad \text{a.e. in } \mathbb{H}^n. \tag{3.5}$$

The proof of the above fact follows the lines of Lemma 3.5 of [6], see also the proofs of Theorem 2.1 of [3], of Lemma 2 of [10] and of Step 1 of Theorem 4.4 of [2] in the Euclidean setting. However, in order to make the paper self contained, we report it here.

Fix $R > 0$. Let $\varphi_R \in C_c^\infty(\mathbb{H}^n)$ be such that $0 \le \varphi_R \le 1$ in $\mathbb{H}^n$ and $\varphi_R \equiv 1$ in B_R. Fix $\varepsilon > 0$ and define for $\xi \in \mathbb{H}^n$

$$v_k^\varepsilon(\xi) = \begin{cases} u_k - u, & \text{if } |u_k - u| < \varepsilon, \\ \varepsilon \dfrac{u_k - u}{|u_k - u|}, & \text{if } |u_k - u| \ge \varepsilon. \end{cases}$$

Clearly, $(\varphi_R v_k^\varepsilon)_k$ is bounded in $S^{1,p}(\mathbb{H}^n)$. Hence, by (3.2), we get

$$\begin{aligned}\int_{\mathbb{H}^n} &\varphi_R\big(|D_H u_k|_H^{p-2}D_H u_k - |D_H u|_H^{p-2}D_H u, D_H v_k^\varepsilon\big)_H d\xi \\ &= -\int_{\mathbb{H}^n} v_k^\varepsilon\big(|D_H u_k|_H^{p-2}D_H u_k, D_H\varphi_R\big)_H d\xi \\ &\quad -\int_{\mathbb{H}^n} \varphi_R\big(|D_H u|_H^{p-2}D_H u, D_H v_k^\varepsilon\big)_H d\xi \\ &\quad + \lambda\langle u_k, \varphi_R v_k^\varepsilon\rangle_{q,w} + \langle u_k, \varphi_R v_k^\varepsilon\rangle_{p^*,K} + o(1).\end{aligned} \tag{3.6}$$

Observe now that

$$\int_{\mathbb{H}^n} v_k^\varepsilon\big(|D_H u_k|_H^{p-2}D_H u_k, D_H\varphi_R\big)_H d\xi \to 0 \quad \text{as } k\to\infty,$$

since $|v_k^\varepsilon D_H\varphi_R|_H \to 0$ in $L^p(\operatorname{supp}\varphi_R)$ and $|D_H u_k|_H^{p-2}D_H u_k \rightharpoonup \Theta$ in $L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$ by (3.4). Moreover, $D_H v_k^\varepsilon \rightharpoonup 0$ in $L^p(\mathbb{H}^n;\mathbb{R}^{2n})$, since $u_k \rightharpoonup u$ in $S^{1,p}(\mathbb{H}^n)$, and so

$$\int_{\mathbb{H}^n} \varphi_R\big(|D_H u|_H^{p-2}D_H u, D_H v_k^\varepsilon\big)_H d\xi \to 0 \quad \text{as } k\to\infty,$$

being $|D_H u|_H^{p-2}D_H u \in L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$. Now, recalling that $0\le\varphi_R\le 1$ in $\mathbb{H}^n$, by the Hölder inequality, we obtain

$$\begin{aligned}|\langle u_k, \varphi_R v_k^\varepsilon\rangle_{q,w}| &\le \int_{\operatorname{supp}\varphi_R} w(\xi)|u_k|^{q-1}|v_k^\varepsilon| d\xi \\ &\le \varepsilon \int_{\operatorname{supp}\varphi_R} w(\xi)^{1/q}\cdot w(\xi)^{1/q'}|u_k|^{q-1}d\xi \\ &\le \varepsilon\left(\int_{\operatorname{supp}\varphi_R} w(\xi)d\xi\right)^{1/q}\cdot\left(\int_{\mathbb{H}^n} w(\xi)|u_k|^q d\xi\right)^{1/q'} \\ &\le \varepsilon\, C_{w,\varphi_R,q},\end{aligned}$$

since $w\in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ by (w_1) and $(u_k)_k$ is bounded in $L^q(\mathbb{H}^n, wd\xi)$. Similarly, since $K\in L^\infty(\mathbb{H}^n)\subset L^1_{\mathrm{loc}}(\mathbb{H}^n)$ by (K_1) and $(u_k)_k$ is bounded in $L^{p^*}(\mathbb{H}^n, Kd\xi)$, then

$$|\langle u_k, \varphi_R v_k^\varepsilon\rangle_{p^*,K}| \le \varepsilon\, C_{K,\varphi_R,p^*}.$$

Using the notations of (3.1), put

$$A_p(D_H u_k, D_H u) = \big(|D_H u_k|_H^{p-2}D_H u_k - |D_H u|_H^{p-2}D_H u, D_H(u_k-u)\big)_H.$$

By convexity $A_p(D_H u_k, D_H u) \geq 0$ a.e. in $\mathbb{H}^n$ and for all k. Consequently, the definitions of φ_R and v_k^ε yield

$$\varphi_R\big(|D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u, D_H v_k^\varepsilon\big)_H \geq 0$$

a.e. in $\mathbb{H}^n$. Combining all these facts with (3.6), we find that

$$\begin{aligned}
&\limsup_{k\to\infty} \int\limits_{B_R} \big(|D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u, D_H v_k^\varepsilon\big)_H d\xi \\
&= \limsup_{k\to\infty} \int\limits_{B_R} \varphi_R\big(|D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u, D_H v_k^\varepsilon\big)_H d\xi \\
&\leq \limsup_{k\to\infty} \int\limits_{\mathbb{H}^n} \varphi_R\big(|D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u, D_H v_k^\varepsilon\big)_H d\xi \\
&\leq \varepsilon\,(\lambda C_{w,\varphi_R,q} + C_{K,\varphi_R,p^*}) = \varepsilon C_R,
\end{aligned} \tag{3.7}$$

where $C_R = \lambda C_{w,\varphi_R,q} + C_{K,\varphi_R,p^*}$, being $\varphi_R \equiv 1$ in B_R. Note also that $(A_p(D_H u_k, D_H u))_k$ is bounded in $L^1(\mathbb{H}^n)$. Indeed,

$$\begin{aligned}
0 &\leq \int\limits_{\mathbb{H}^n} A_p(D_H u_k, D_H u) d\xi \\
&\leq \||D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u\|_{p'} \|D_H u_k - D_H u\|_p \\
&\leq C_0,
\end{aligned} \tag{3.8}$$

where C_0 is an appropriate constant, independent of k, since $(D_H u_k)_k$ is bounded in $L^p(\mathbb{H}^n; \mathbb{R}^{2n})$ and $(|D_H u_k|_H^{p-2} D_H u_k)_k$ is bounded in $L^{p'}(\mathbb{H}^n; \mathbb{R}^{2n})$, as shown above. Fix $\theta \in (0,1)$. Split the ball B_R into

$$S_k^\varepsilon(R) = \{\xi \in B_R : |u_k(\xi) - u(\xi)| \leq \varepsilon\}, \quad G_k^\varepsilon(R) = B_R \setminus S_k^\varepsilon(R).$$

Clearly, Hölder's inequality gives

$$\begin{aligned}
\int\limits_{B_R} A_p(D_H u_k, D_H u)^\theta d\xi &\leq \Bigg(\int\limits_{S_k^\varepsilon(R)} A_p(D_H u_k, D_H u) d\xi\Bigg)^\theta |S_k^\varepsilon(R)|^{1-\theta} \\
&\quad + \Bigg(\int\limits_{G_k^\varepsilon(R)} A_p(D_H u_k, D_H u) d\xi\Bigg)^\theta |G_k^\varepsilon(R)|^{1-\theta},
\end{aligned}$$

Finally, by (3.7), (3.8), the facts that $D_H v_k^\varepsilon = D_H(u_k - u)$ in $S_k^\varepsilon(R)$ and that $|G_k^\varepsilon(R)| \to 0$ as $k \to \infty$ by (3.4), we get

$$0 \leq \limsup_{k\to\infty} \int\limits_{B_R} A_p(D_H u_k, D_H u)^\theta\, d\xi \leq (\varepsilon C_R)^\theta |B_R|^{1-\theta}.$$

Letting ε tend to 0^+, we find that $A_p(D_H u_k, D_H u)^\theta \to 0$ in $L^1(B_R)$ and so, since $R > 0$ is arbitrary, we deduce that, up to a subsequence,

$$A_p(D_H u_k, D_H u) \to 0 \quad \text{a.e. in } \mathbb{H}^n.$$

Therefore, from Lemma 3 of [10], it follows the validity of (3.5), and the claim is proved.

In particular, $|D_H u_k|_H^{p-2} D_H u_k \to |D_H u|_H^{p-2} D_H u$ a.e. in $\mathbb{H}^n$. Hence, Proposition A.7 of [2], thanks to (3.4) and (3.5), implies $\Theta = |D_H u|_H^{p-2} D_H u$ a.e. in $\mathbb{H}^n$. A combination of all these facts yields that u is a solution of $(\mathcal{E})$. □

Let us now show that the functional I_λ satisfies the $(PS)_c$ condition at certain levels c.

Lemma 3.2. *Let $(u_k)_k \subset S^{1,p}(\mathbb{H}^n)$ be a Palais–Smale sequence of I_λ at a level c, with*

$$c < \frac{C_{p^*}^{Q/p}}{Q} \|K\|_\infty^{(p-Q)/p}.$$

Then, up to a subsequence, $u_k \to u$ in $S^{1,p}(\mathbb{H}^n)$ for some $u \in S^{1,p}(\mathbb{H}^n)$.

Proof. Fix c as in the statement and let $(u_k)_k \subset S^{1,p}(\mathbb{H}^n)$ be a Palais–Smale sequence of I_λ at level c, that is such that (3.2) holds. Without loss of generality, passing to an appropriate subsequence if necessary, from Lemma 3.3, we assume that for some $u \in S^{1,p}(\mathbb{H}^n)$ all the convergences in (3.4) hold. Then, by Theorem 1.1 and Theorem 1.2 of [29], there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$, two families of nonnegative numbers $\{\mu_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ such that if we define

$$\nu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |u_k|^{p^*} d\xi, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |D_H u_k|_H^p d\xi,$$

then,

$$\begin{aligned}
&\nu = |u|^{p^*} d\xi + \sum_{j\in J} \nu_j \delta_{\xi_j}, \quad \mu \ge |D_H u|_H^p d\xi + \sum_{j\in J} \mu_j \delta_{\xi_j},\\
&\limsup_{k\to\infty} \int_{\mathbb{H}^n} |u_k|^{p^*} d\xi = \nu(\mathbb{H}^n) + \nu_\infty,\\
&\limsup_{k\to\infty} \int_{\mathbb{H}^n} |D_H u_k|_H^p d\xi = \mu(\mathbb{H}^n) + \mu_\infty,\\
&\nu_j^{p/p^*} \le \frac{\mu_j}{C_{p^*}} \text{ for all } j \in J, \quad \nu_\infty^{p/p^*} \le \frac{\mu_\infty}{C_{p^*}},
\end{aligned} \tag{3.9}$$

where C_{p^*} is defined in (1.3), while δ_{ξ_j} is the Dirac function at the point ξ_j of $\mathbb{H}^n$.

Let us first show the following crucial estimate

$$\begin{cases} \nu_j \ge C_{p^*}^{Q/p} \|K\|_\infty^{-Q/p}, \ \mu_j = \nu_j K(\xi_j), & \text{if } K(\xi_j) > 0,\\ \nu_j = \mu_j = 0, & \text{if } K(\xi_j) = 0, \end{cases} \tag{3.10}$$

To this aim, fix a test function $\varphi \in C_c^\infty(\mathbb{H}^n)$, such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in B_1, while $\varphi \equiv 0$ in B_2^c, and $\|D_H\varphi\|_\infty \le 2$. Take $\varepsilon > 0$. Fix $j \in J$ and put $\varphi_{\varepsilon,j}(\xi) = \varphi(\delta_{1/\varepsilon}(\xi \circ \xi_j^{-1}))$, $\xi \in \mathbb{H}^n$, where $\{\xi_j\}_j$ is introduced in (3.9). Then $\varphi_{\varepsilon,j}u_k \in S^{1,p}(\mathbb{H}^n)$ and so $\langle I'_\lambda(u_k), \varphi_{\varepsilon,j}u_k\rangle = o(1)$ as $k \to \infty$ by (3.2). Therefore, as $k \to \infty$

$$\begin{aligned} o(1) &= \int_{\mathbb{H}^n} (|D_H u_k|_H^{p-2} D_H u_k, D_H(\varphi_{\varepsilon,j}u_k))_H d\xi \\ &\quad - \lambda \int_{\mathbb{H}^n} w(\xi)|u_k|^q \varphi_{\varepsilon,j} d\xi - \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*}\varphi_{\varepsilon,j} d\xi \\ &= \int_{\mathbb{H}^n} \{(|D_H u_k|_H^{p-2} D_H u_k, D_H\varphi_{\varepsilon,j})_H u_k d\xi + \int_{\mathbb{H}^n} |D_H u_k|_H^p \varphi_{\varepsilon,j} d\xi \\ &\quad - \lambda \int_{\mathbb{H}^n} w(\xi)|u_k|^q \varphi_{\varepsilon,j} d\xi - \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*}\varphi_{\varepsilon,j} d\xi. \end{aligned} \tag{3.11}$$

Now, since $p/p^* + p/Q = 1$, the Cauchy–Schwarz inequality and the Hölder inequality give

$$\begin{aligned} &\left| \int_{\mathbb{H}^n} (|D_H u_k|_H^{p-2} D_H u_k, D_H\varphi_{\varepsilon,j})_H u_k d\xi \right| \\ &\le \int_{B(\xi_j,2\varepsilon)} |D_H u_k|_H^{p-1} |u_k| \cdot |D_H\varphi_{\varepsilon,j}|_H d\xi \\ &\le \|D_H u_k\|_p^{p-1} \left(\int_{B(\xi_j,2\varepsilon)} |u_k|^p \cdot |D_H\varphi_{\varepsilon,j}|_H^p d\xi \right)^{1/p} \\ &\le \|D_H u_k\|_p^{p-1} \left(\int_{B(\xi_j,2\varepsilon)} |D_H\varphi_{\varepsilon,j}|_H^Q d\xi \right)^{1/Q} \cdot \left(\int_{B(\xi_j,2\varepsilon)} |u_k|^{p^*} d\xi \right)^{1/p^*} \\ &\le c_0 c_\varphi \left(\int_{B(\xi_j,2\varepsilon)} |u_k|^{p^*} d\xi \right)^{1/p^*}, \end{aligned}$$

where

$$c_0 = \sup_{k\in\mathbb{N}} \|D_H u_k\|_p^{p-1} \quad \text{and} \quad c_\varphi = \left(\int_{B_1} |D_H\varphi(\eta)|_H^Q d\eta \right)^{1/Q},$$

being

$$\int\limits_{B_\varepsilon(\xi_j)} |D_H\varphi_{\varepsilon,j}(\xi)|_H^Q d\xi = \int\limits_{B_\varepsilon(\xi_j)} \frac{1}{\varepsilon^Q}|D_H\varphi(\delta_{1/\varepsilon}(\xi\circ\xi_j^{-1}))|_H^Q d\xi = \int\limits_{B_1} |D_H\varphi(\eta)|_H^Q d\eta.$$

Here $\eta = \delta_{1/\varepsilon}(\xi\circ\xi_j^{-1})$ is the change of variable, with $d\eta = \varepsilon^{-Q}d\xi$. Consequently,

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0^+}\limsup_{k\to\infty}\left|\int\limits_{\mathbb{H}^n}(|D_Hu_k|_H^{p-2}D_Hu_k, D_H\varphi_{\varepsilon,j})_H u_k d\xi\right| \\ &\le \lim_{\varepsilon\to 0^+} c_0c_\varphi\left(\int\limits_{B(\xi_j,2\varepsilon)}|u|^{p^*}d\xi\right)^{1/p^*} = 0. \end{aligned} \tag{3.12}$$

Similarly, by (w_1) and (3.4), as $k\to\infty$

$$\begin{aligned} 0 \le \int\limits_{\mathbb{H}^n} w(\xi)|u_k|^q\varphi_{\varepsilon,j}d\xi &\le \int\limits_{B(\xi_j,2\varepsilon)} w(\xi)|u_k|^q d\xi \\ &\to \int\limits_{B(\xi_j,2\varepsilon)} w(\xi)|u|^q d\xi, \end{aligned}$$

since $1<q<\mathfrak{m}<m<q^*$. Therefore,

$$\lim_{\varepsilon\to 0^+}\lim_{k\to\infty}\int\limits_{\mathbb{H}^n} w(\xi)|u_k|^q\varphi_{\varepsilon,j}d\xi = 0. \tag{3.13}$$

In conclusion, passing to the limit in (3.11), using (3.4), (3.12) and (3.13), we obtain the crucial formula for all $j\in J$

$$o(1) = \int\limits_{\mathbb{H}^n}\varphi_{\varepsilon,j}d\mu - \int\limits_{\mathbb{H}^n}K(\xi)\varphi_{\varepsilon,j}d\nu$$

as $\varepsilon\to 0^+$, which in turns yields by (3.9)

$$\mu_j = K(\xi_j)\nu_j \quad \text{for all } j\in J.$$

The above equality establishes that the concentration of the measure μ cannot occur at points where $K(\xi_j)=0$. In addiction, by (3.9), we also infer that $\nu_j = 0$ if $K(\xi_j)=0$, that is also the measure ν cannot concentrate in those points. On the other hand, if $K(\xi_j)>0$, again by (3.9) we get

$$\nu_j \ge \left(\frac{C_{p^*}}{K(\xi_j)}\right)^{Q/p} \ge \left(\frac{C_{p^*}}{\|K\|_\infty}\right)^{Q/p},$$

and so (3.10) is proved.

Correspondingly, we show that

$$\begin{cases} \nu_\infty \geq C_{p^*}^{Q/p} \|K\|_\infty^{-Q/p}, \; \mu_\infty = \nu_\infty K_\infty, & \text{if } K_\infty > 0, \\ \nu_\infty = \mu_\infty = 0, & \text{if } K_\infty = 0, \end{cases} \tag{3.14}$$

where the quantity K_∞ is introduced in (K_1).

To see that we take a function $\chi \in C^\infty(\mathbb{H}^n)$ such that $0 \leq \chi \leq 1$, $\chi = 0$ in B_1 and $\chi = 1$ in B_2^c. Take $R > 0$ and put

$$\chi_R(\xi) = \chi(\delta_{1/R}(\xi)), \quad \xi \in \mathbb{H}^n.$$

Note that $\operatorname{supp} \chi_R \subset B_R^c$ and $\operatorname{supp} D_H \chi_R \subset B_{2R} \setminus B_R$. Clearly, the function $\chi_R u_k \in S^{1,p}(\mathbb{H}^n)$ for any k and so $\langle I'_\lambda(u_k), \chi_R u_k \rangle = o(1)$ as $k \to \infty$ by (3.2) and the fact that $(\chi_R u_k)_k$ is bounded in $S^{1,p}(\mathbb{H}^n)$. Hence, arguing as in (3.11) we obtain as $k \to \infty$

$$\begin{aligned} o(1) = \int_{\mathbb{H}^n} &\{(|D_H u_k|_H^{p-2} D_H u_k, D_H \chi_R)_H u_k d\xi + \int_{\mathbb{H}^n} |D_H u_k|_H^p \chi_R d\xi \\ &- \lambda \int_{\mathbb{H}^n} w(\xi) |u_k|^q \chi_R d\xi - \int_{\mathbb{H}^n} K(\xi) |u_k|^{p^*} \chi_R d\xi. \end{aligned} \tag{3.15}$$

Arguing as above, the properties of χ_R, the Cauchy–Schwarz inequality and the Hölder inequality give

$$\begin{aligned} &\left| \int_{\mathbb{H}^n} (|D_H u_k|_H^{p-2} D_H u_k, D_H \chi_R)_H u_k d\xi \right| \\ &\leq \int_{\mathbb{H}^n} |D_H u_k|_H^{p-1} |u_k| |D_H \chi_R|_H d\xi \\ &\leq \|D_H u_k\|_p^{p-1} \left(\int_{B_{2R} \setminus B_R} |u_k|^p \cdot |D_H \chi_R|_H^p d\xi \right)^{1/p} \\ &\leq \|D_H u_k\|_p^{p-1} \left(\int_{B_{2R} \setminus B_R} |D_H \chi_R|_H^Q d\xi \right)^{1/Q} \left(\int_{B_{2R} \setminus B_R} |u_k|^{p^*} d\xi \right)^{1/p^*} \\ &\leq c_0 c_\chi \left(\int_{B_{2R} \setminus B_R} |u_k|^{p^*} d\xi \right)^{1/p^*}, \end{aligned}$$

where

$$c_0 = \sup_{k \in \mathbb{N}} \|D_H u_k\|_p^{p-1} \quad \text{and} \quad c_\chi = \left(\int_{B_2 \setminus B_1} |D_H \chi(\eta)|_H^Q d\eta \right)^{1/Q},$$

being

$$\int\limits_{B_{2R}\setminus B_R} |D_H\chi_R(\xi)|_H^Q d\xi = \int\limits_{B_{2R}\setminus B_R} \frac{1}{R^Q}|D_H\chi(\delta_{1/R}(\xi))|_H^Q d\xi$$
$$= \int\limits_{B_2\setminus B_1} |D_H\chi(\eta)|_H^Q d\eta.$$

Here $\eta = \delta_{1/R}(\xi)$, so that $d\xi = R^Q d\eta$. Consequently,

$$\begin{aligned} 0 &\le \lim_{R\to\infty} \limsup_{k\to\infty} \left| \int\limits_{\mathbb{H}^n} (|D_H u_k|_H^{p-2} D_H u_k, D_H\chi_R)_H u_k d\xi \right| \\ &\le \lim_{R\to\infty} c_0 c_\chi \left(\int\limits_{B_{2R}\setminus B_R} |u|^{p^*} d\xi \right)^{1/p^*} = 0. \end{aligned} \tag{3.16}$$

Now, observe that

$$\int\limits_{B_{2R}^c} |D_H u_k|_H^p d\xi \le \int\limits_{\mathbb{H}^n} |D_H u_k|_H^p \chi_R d\xi \le \int\limits_{B_R^c} |D_H u_k|_H^p d\xi$$

and so the definition of μ_∞ implies

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int\limits_{\mathbb{H}^n} |D_H u_k|_H^p \chi_R d\xi = \mu_\infty. \tag{3.17}$$

Moreover, by (w_1) it easy to see that

$$0 \le \lim_{R\to\infty} \limsup_{k\to\infty} \int\limits_{\mathbb{H}^n} w(\xi)|u_k|^q \chi_R d\xi \le \lim_{R\to\infty} \int\limits_{B_R^c} w(\xi)|u|^q d\xi = 0. \tag{3.18}$$

Furthermore, we have

$$\int\limits_{\mathbb{H}^n} K(\xi)|u_k|^{p^*}\chi_R d\xi = K_\infty \int\limits_{\mathbb{H}^n} |u_k|^{p^*}\chi_R d\xi + \int\limits_{\mathbb{H}^n} (K(\xi)-K_\infty)|u_k|^{p^*}\chi_R d\xi,$$

where arguing as in (3.17) for ν, we see that

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int\limits_{\mathbb{H}^n} |u_k|^{p^*}\chi_R d\xi = \nu_\infty.$$

On the other hand, by (K_1) for any $\delta > 0$ there exists $\bar{R} = \bar{R}(\delta) > 0$ such that for all $\xi \in \mathbb{H}^n$, with $r(\xi) \ge \bar{R}$, we have

$$|K(\xi) - K_\infty| \le \frac{\delta}{M},$$

where $M = \sup_k \|u_k\|_{p^*}^{p^*}$.

Hence, for all $R \geq \bar{R}$ we deduce

$$\left| \int_{\mathbb{H}^n} (K(\xi) - K_\infty)|u_k|^{p^*} \chi_R d\xi \right| \leq \frac{\delta}{M} \int_{B_R^c} |u_k|^{p^*} d\xi \leq \delta$$

and so

$$\lim_{R\to\infty} \limsup_{k\to\infty} \int_{\mathbb{H}^n} (K(\xi) - K_\infty)|u_k|^{p^*} \chi_R d\xi = 0.$$

Consequently,

$$0 \leq \lim_{R\to\infty} \limsup_{k\to\infty} \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*} \chi_R d\xi = K_\infty \nu_\infty. \tag{3.19}$$

Then, recalling (3.15) and using (3.16)–(3.19), we obtain

$$\mu_\infty = K_\infty \nu_\infty,$$

which yields (3.14).

It remains to show the strong convergence of the sequence $(u_k)_k$. Now, from (3.9) it follows that

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} |D_H u_k|_H^p d\xi \geq \int_{\mathbb{H}^n} |D_H u|_H^p d\xi + \sum_{j\in J} \mu_j + \mu_\infty, \tag{3.20}$$

and

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*} d\xi = \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*} d\xi + \sum_{j\in J} \nu_j K(\xi_j) + \nu_\infty K_\infty. \tag{3.21}$$

Finally, by (3.4),

$$\lim_{k\to\infty} \int_{\mathbb{H}^n} w(\xi)|u_k|^q d\xi = \int_{\mathbb{H}^n} w(\xi)|u|^q d\xi. \tag{3.22}$$

Hence, combining (3.20)–(3.22), we have

$$\begin{aligned}
c &= \lim_{k\to\infty} I_\lambda(u_k) \\
&= \lim_{k\to\infty} \left\{ \frac{1}{p} \int_{\mathbb{H}^n} |D_H u_k|_H^p d\xi - \frac{\lambda}{q} \int_{\mathbb{H}^n} w(\xi)|u_k|^q d\xi - \frac{1}{p^*} \int_{\mathbb{H}^n} K(\xi)|u_k|^{p^*} d\xi \right\} \\
&\geq \frac{1}{p} \int_{\mathbb{H}^n} |D_H u|_H^p d\xi - \frac{\lambda}{q} \int_{\mathbb{H}^n} w(\xi)|u|^q d\xi - \frac{1}{p^*} \int_{\mathbb{H}^n} K(\xi)|u|^{p^*} d\xi \\
&\quad + \frac{1}{p} \left(\sum_{j\in J} \mu_j + \mu_\infty \right) - \frac{1}{p^*} \left(\sum_{j\in J} \nu_j K(\xi_j) + \nu_\infty K_\infty \right) \\
&= I_\lambda(u) + \frac{1}{p} \left(\sum_{j\in J} \mu_j + \mu_\infty \right) - \frac{1}{p^*} \left(\sum_{j\in J} \nu_j K(\xi_j) + \nu_\infty K_\infty \right).
\end{aligned}$$

Now, u is a solution of $(\mathcal{E})$ thanks to Lemma 3.1 and so

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{q}\langle I'_\lambda(u), u\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|D_H u\|_p^p + \left(\frac{1}{q} - \frac{1}{p^*}\right)\int\limits_{\mathbb{H}^n} K(\xi)|u|^{p^*} d\xi \ge 0. \end{aligned}$$

We claim that $\nu_j = 0$ for all $j \in J \cup \{\infty\}$. Denote by

$$\widetilde{J} = \{j \in J \cup \{\infty\} \ : \ \nu_j > 0 \text{ and } K(\xi_j) > 0\},$$

where for simplicity we put $\xi_\infty = \infty$. As already noted, if $K(\xi_j) = 0$, then from (3.10) and (3.14) we know that $\nu_j = 0$. Hence, if we show that $\widetilde{J} = \emptyset$, then the claim is proved. Assume by contradiction that $\widetilde{J} \neq \emptyset$. Therefore, using (3.10) and (3.14)

$$\begin{aligned} c &\ge \frac{1}{p}\left(\sum_{j\in J}\mu_j + \mu_\infty\right) - \frac{1}{p^*}\left(\sum_{j\in J}\nu_j K(\xi_j) + \nu_\infty K_\infty\right) \\ &= \frac{1}{Q}\sum_{j\in\widetilde{J}}\nu_j K(\xi_j) \ge \frac{C_{p^*}^{Q/p}}{Q}\|K\|_\infty^{(p-Q)/p}, \end{aligned}$$

which contradicts the main assumption on c. Consequently, $\widetilde{J} = \emptyset$, that is $\nu_j = \mu_j = 0$ for all $j \in J \cup \{\infty\}$.

In particular, the fact that $\nu_j = 0$ for all $j \in J$ implies that $|u_k|^{p^*} d\xi \overset{*}{\rightharpoonup} \nu = |u|^{p^*} d\xi$ in $\mathcal{M}(\mathbb{H}^n)$ by (3.9). Moreover, (3.9) and $\nu_\infty = 0$ yield that

$$\lim_{k\to\infty}\int\limits_{\mathbb{H}^n} |u_k|^{p^*} d\xi = \nu(\mathbb{H}^n) + \nu_\infty = \int\limits_{\mathbb{H}^n} |u|^{p^*} d\xi.$$

On the other hand, $u_k \to u$ a.e. in $\mathbb{H}^n$, so that an application of the Brezis–Lieb lemma [7] gives

$$\lim_{k\to\infty}\int\limits_{\mathbb{H}^n} |u_k - u|^{p^*} d\xi = 0.$$

Now, since $K \in L^\infty(\mathbb{H}^n)$, we get as $k \to \infty$

$$\begin{aligned} 0 \le \langle u_k, u_k - u\rangle_{p^*,K} &= \int\limits_{\mathbb{H}^n} K(\xi)|u_k|^{p^*-1}|u_k - u| d\xi \\ &\le \|K\|_\infty \|u_k\|_{p^*}^{p^*-1}\left(\int\limits_{\mathbb{H}^n} |u_k - u|^{p^*} d\xi\right)^{1/p^*} \to 0. \end{aligned} \tag{3.23}$$

A similar argument shows that

$$
\begin{aligned}
0 &\le \langle u, u_k - u\rangle_{p^*,K} = \int\limits_{\mathbb{H}^n} K(\xi)|u|^{p^*-1}|u_k - u|d\xi \to 0,\\
0 &\le \langle u_k, u_k - u\rangle_{q,w} = \int\limits_{\mathbb{H}^n} w(\xi)|u_k|^{q-1}|u_k - u|d\xi \to 0,\\
0 &\le \langle u, u_k - u\rangle_{q,w} = \int\limits_{\mathbb{H}^n} w(\xi)|u|^{q-1}|u_k - u|d\xi \to 0
\end{aligned} \tag{3.24}
$$

as $k \to \infty$.

Recall that $\langle I'_\lambda(u_k), u_k - u\rangle = o(1)$ as $k \to \infty$ by (3.23) and (3.24). Moreover, $\langle I'_\lambda(u), u_k - u\rangle = 0$ for all k, since u is a solution of $(\mathcal{E})$ by Lemma 3.1. Consequently,

$$
\begin{aligned}
o(1) &= \langle I'_\lambda(u_k), u_k - u\rangle - \langle I'_\lambda(u), u_k - u\rangle\\
&= \int\limits_{\mathbb{H}^n} \big(|D_H u_k|_H^{p-2} D_H u_k - |D_H u|_H^{p-2} D_H u, D_H u_k - D_H u\big)_H d\xi\\
&= \int\limits_{\mathbb{H}^n} A_p(D_H u_k, D_H u)d\xi,
\end{aligned} \tag{3.25}
$$

using the notation introduced in (3.1). Let us now distinguish two different cases.

Case $p \ge 2$. By (3.1) and (3.25), there exists $\kappa > 0$, depending only on p, such that

$$
\frac{1}{\kappa}\int\limits_{\mathbb{H}^n} |D_H u_k - D_H u|_H^p d\xi \le \int\limits_{\mathbb{H}^n} A_p(D_H u_k, D_H u)d\xi = o(1)
$$

as $k \to \infty$.

Case $1 < p < 2$. Again, by (3.1), (3.25) and the Hölder inequality, with $\mathfrak{p} = 2/p$ and $\mathfrak{p}' = 2/(2-p)$, there exists $\kappa > 0$, depending only on p, such that

$$
\begin{aligned}
&\frac{1}{\kappa}\int\limits_{\mathbb{H}^n} |D_H u_k - D_H u|_H^p d\xi\\
&\le \left\{\int\limits_{\mathbb{H}^n} A_p(D_H u_k, D_H u)d\xi\right\}^{p/2} \cdot \left\{\|D_H u_k\|_p^p + \|D_H u\|_p^p\right\}^{(2-p)/2}\\
&\le M^{(2-p)/2}\left\{\int\limits_{\mathbb{H}^n} A_p(D_H u_k, D_H u)d\xi\right\}^{p/2} = o(1)
\end{aligned}
$$

as $k \to \infty$, where $\|D_H u_k\|_p^p + \|D_H u\|_p^p \le M$ for some nonnegative constant M.

Therefore, in all cases we get

$$\int_{\mathbb{H}^n} |D_H u_k - D_H u|_H^p d\xi \to 0$$

as $k \to \infty$. Thus, $u_k \to u$ in $S^{1,p}(\mathbb{H}^n)$, as required. This concludes the proof. □

The next lemma shows that the functional I_λ satisfies the geometry of the mountain pass lemma. The proof is standard, see, e.g., Theorem 3.8 of [4] and Lemma 3.2 of [6]. However, for completeness we present it.

Lemma 3.3. *If $\lambda > 0$, then there exists a nonnegative function $e \in S^{1,p}(\mathbb{H}^n)$, independent of λ, such that $\|D_H e\|_p \geq 2$ and $I_\lambda(e) < 0$. If furthermore λ satisfies* (3.3)*, then there exist $\rho = \rho(\lambda) \in (0,1]$ and $\jmath = \jmath(\lambda) > 0$ such that $I_\lambda(u) \geq \jmath$ for any $u \in S^{1,p}(\mathbb{H}^n)$, with $\|D_H u\|_p = \rho$.*

Proof. Fix $\lambda > 0$ and take a nonnegative function $v \in S^{1,p}(\mathbb{H}^n)$, such that $\|D_H v\|_p = 1$. Since $1 < p \leq q < p^*$, we get as $\tau \to \infty$

$$\begin{aligned} I_\lambda(\tau v) &\leq \frac{1}{p}\tau^p - \lambda \frac{\|v\|_{q,w}^q}{q}\tau^q - \frac{\|v\|_{p^*,K}^{p^*}}{p^*}\tau^{p^*} \\ &\leq \frac{1}{p}\tau^p - \frac{\|v\|_{p^*,K}^{p^*}}{p^*}\tau^{p^*} \to -\infty, \end{aligned}$$

by (K_1). Hence, taking $e = \tau_* v$, with $\tau_* > 0$ large enough, we obtain that $\|D_H e\|_p \geq 2$ and $I_\lambda(e) < 0$. In particular, being $K > 0$ a.e. in $\mathbb{H}^n$ by (K_1), the function e is independent of λ.

Now, fix any $u \in S^{1,p}(\mathbb{H}^n)$, with $\|D_H u\|_p \leq 1$. By (1.1) and (K_1) we have

$$I_\lambda(u) \geq \frac{1}{p}\|D_H u\|_p^p - \frac{\lambda}{q\lambda_1}\|D_H u\|_p^q - \frac{1}{p^*}\|K\|_\infty^{p^*}\|D_H u\|_p^{p^*}.$$

Thus, setting

$$\psi_\lambda(\tau) = \begin{cases} \frac{1}{p}\tau^p - \frac{\lambda}{q\lambda_1}\tau^q - \frac{1}{p^*}\|K\|_\infty^{p^*}\tau^{p^*}, & \text{if } 1 < p < q < p^*, \\ \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\tau^p - \frac{1}{p^*}\|K\|_\infty^{p^*}\tau^{p^*}, & \text{if } 1 < p = q < p^*, \end{cases}$$

we find some $\rho \in (0,1]$ so small that $\max_{\tau \in [0,1]} \psi_\lambda(\tau) = \psi_\lambda(\rho) > 0$, since $1 < p \leq q < p^*$. It follows $I_\lambda(u) \geq \jmath = \psi_\lambda(\rho) > 0$ for any $u \in S^{1,p}(\mathbb{H}^n)$, with $\|D_H u\|_p = \rho$. □

Lemma 3.3 arises the positive special level

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} I_\lambda(\gamma(\tau)) \tag{3.26}$$

of I_λ, where

$$\Gamma = \big\{\gamma \in C([0,1], S^{1,p}(\mathbb{H}^n)) \,:\, \gamma(0) = 0,\ I_\lambda(e) < 0\big\}$$

for all $\lambda > 0$ in the prescribed range (3.3). Moreover, Lemma 3.3 and the mountain pass lemma yield that there exists a Palais–Smale sequence $(u_k)_k \subset S^{1,p}(\mathbb{H}^n)$ of I_λ at the level c_λ for all $\lambda > 0$ satisfying (3.3).

Now we recall an asymptotic property of the levels c_λ as $\lambda \to \infty$ in the case $p < q < p^*$, which is crucial in the proof of the Theorem 1.1. This result was obtained in a slightly more general context in [6], cf. Lemma 3.3.

Lemma 3.4. *Let $p < q < p^*$. Assume that* (w_1) *and* (H_1) *are satisfied. Then,*

$$\lim_{\lambda\to\infty} c_\lambda = 0,$$

where c_λ is defined (3.26)*.*

For the proof of Lemma 3.4 we refer to Lemma 3.3 of [6]. Finally, we are ready to prove the first main result of the paper, that is Theorem 1.1.

Proof of Theorem 1.1. First, Lemma 3.3 guarantees that the functional I_λ has the geometry of the mountain pass lemma. Thus, I_λ admits a Palais–Smale sequence $(u_k)_k$ at level c_λ which, in virtue of Lemma 3.1, up to a subsequence, weakly converges to some limit $u \in S^{1,p}(\mathbb{H}^n)$, which is also a critical point of I_λ. Now, from Lemma 3.4 there exists $\lambda^* > 0$ such that $c_\lambda < C_{p^*}^{Q/p} Q^{-1} \|K\|_\infty^{(p-Q)/p}$ for all $\lambda \geq \lambda^*$. Hence, for such λs the functional I_λ satisfies the $(PS)_{c_\lambda}$ condition by Lemma 3.2. Therefore, up to a subsequence, $u_k \to u$ in $S^{1,p}(\mathbb{H}^n)$ as $k \to \infty$ and so u is a nontrivial solution of equation $(\mathcal{E})$. □

Let us now turn to the case $p = q$.

Proof of Theorem 1.2. Here $K \equiv 1$ and so $\|K\|_\infty^{(p-Q)/p} = 1$. Therefore, by Lemmas 3.3–3.2, it is sufficient now to show that there exists $v \in S^{1,p}(\mathbb{H}^n)$ such that

$$\sup_{\tau>0} I_\lambda(\tau v) < \frac{C_{p^*}^{Q/p}}{Q}.$$

To this purpose, let us consider, as explained in the Introduction, an extremal $U \in S^{1,p}(\mathbb{H}^n)$ for (1.2). Then, Theorem 1.2 of [24] yields the validity of (1.4), that is

$$U(\xi) \sim r(\xi)^{\frac{p-Q}{p-1}} \quad \text{as } r(\xi) \to \infty.$$

Clearly, $U \in L^p(\mathbb{H}^n)$ by (1.4) and the assumption $p^2 < Q$. We can assume, up to a normalization, that U is such that

$$\|D_H U\|_p^p = C_{p^*}^{Q/p}, \quad \|U\|_{p^*}^{p^*} = C_{p^*}^{Q/p}. \tag{3.27}$$

Indeed, if $\|D_H U\|_p^p = C_{p^*}^{Q/p}$, then (1.3) and the fact that U is an extremal imply that

$$\|U\|_{p^*}^{p^*} = \left(\frac{\|D_H u\|_p^p}{C_{p^*}}\right)^{p^*/p} = (C_{p^*}^{\frac{Q}{p}-1})^{p^*/p} = C_{p^*}^{Q/p}.$$

Fix $\varepsilon > 0$ and define

$$U_\varepsilon(\xi) = \varepsilon^{-\frac{Q-p}{p}} U(\delta_{1/\varepsilon}(\xi_0^{-1} \circ \xi)) \quad \text{for all } \xi \in \mathbb{H}^n, \tag{3.28}$$

where ξ_0 is the point introduced in (w_2). It is easy to see that the norms in the Folland–Stein inequality (1.2) and the functionals in the variational problem (1.3) are invariant under the translations and the rescaling (3.28). Hence, from (3.27) we also infer that

$$\|D_H U_\varepsilon\|_p^p = C_{p^*}^{Q/p}, \quad \|U_\varepsilon\|_{p^*}^{p^*} = C_{p^*}^{Q/p}. \tag{3.29}$$

We aim to prove that there exists $\varepsilon_* > 0$ such that

$$\sup_{\tau>0} I_\lambda(\tau U_\varepsilon) < \frac{C_{p^*}^{Q/p}}{Q}$$

for all $\varepsilon \le \varepsilon_*$. Now, using (3.29)

$$I_\lambda(\tau U_\varepsilon) = \left(\frac{\tau^p}{p} - \frac{\tau^{p^*}}{p^*}\right) C_{p^*}^{Q/p} - \frac{\lambda \tau^p}{p} \int_{\mathbb{H}^n} w(\xi)|U_\varepsilon(\xi)|^p d\xi,$$

and a simple change of variables, $\eta = \delta_{1/\varepsilon}(\xi_0^{-1} \circ \xi)$, so that $d\xi = \varepsilon^Q d\eta$, shows that

$$\begin{aligned}\int_{\mathbb{H}^n} w(\xi)|U_\varepsilon(\xi)|^p d\xi &= \varepsilon^p \int_{\mathbb{H}^n} w(\xi_0 \circ \delta_\varepsilon(\eta))|U(\eta)|^p d\eta \\ &= \varepsilon^p \left[w(\xi_0)\|U\|_p^p + \int_{\mathbb{H}^n} \Big(w(\xi_0 \circ \delta_\varepsilon(\eta)) - w(\xi_0)\Big)|U(\eta)|^p d\eta \right].\end{aligned}$$

Hence, from the fact that $U \in L^p(\mathbb{H}^n)$, we obtain by (w_2)

$$\int_{\mathbb{H}^n} w(\xi)|U_\varepsilon(\xi)|^p d\xi = \varepsilon^p w(\xi_0)\|U\|_p^p + o(\varepsilon^p).$$

Put

$$\psi(\tau) = \frac{A}{p}\tau^p - \frac{B}{p^*}\tau^{p^*},$$

where

$$A = C_{p^*}^{Q/p} - \lambda \varepsilon^p w(\xi_0)\|U\|_p^p + o(\varepsilon^p), \quad B = C_{p^*}^{Q/p}$$

and choose $\varepsilon > 0$ so small that $A > 0$. Obviously, $\psi(\tau) = I_\lambda(\tau U_\varepsilon)$. A straightforward calculation shows that

$$\max_{\tau>0} \psi(\tau) = \frac{1}{Q}\left(\frac{A}{B^{(Q-p)/Q}}\right)^{Q/p}.$$

Now, choosing ε even smaller so that

$$C_{p^*} - C_{p^*}^{1-Q/p}\lambda\varepsilon^p w(\xi_0)\|U\|_p^p + o(\varepsilon^p) < C_{p^*},$$

from what we observed above we get

$$\sup_{\tau>0} I_\lambda(\tau U_\varepsilon) = \frac{1}{Q}\Big(C_{p^*} - C_{p^*}^{1-Q/p}\lambda\varepsilon^p w(\xi_0)\|U\|_p^p + o(\varepsilon^p)\Big)^{Q/p} < \frac{C_{p^*}^{Q/p}}{Q},$$

as required. This completes the proof. □

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