Analysis of Harmonic Distortion in Analog Circuits with the Use of Volterra Series and Kronecker Products

Andrzej Borys

***Abstract*—This paper was inspired by an article entitled "An approach to model high-frequency distortion in negative-feedback amplifiers" by S. O. Cannizzaro, G. Palumbo, and S. Pennisi. The objective of this presentation is to point out that some results presented therein are not so novel as argued. First, we point out here that an idea of partition of a nonlinear circuit into interconnected smaller basic blocks, used in the above paper under a name of an analytical approach, is not new. For the first time, it has been used in the literature by S. Narayanan, pioneer of the Volterra series usage in calculations of nonlinear distortion in electronic circuits, and afterwards by many others. Second, we show that descriptions of the basic blocks mentioned above follow from their more general representations by the Volterra series, specialized for harmonic inputs. Third, we recall references in which the joint and complementary elements as well as some invariants occurring in modelling of op amp inverting and non-inverting configurations for the purpose of nonlinear distortion evaluation have been reported before publication of some similar results by S. O. Cannizzaro, G. Palumbo, and S. Pennisi. Finally, we show that an operator *o* that was introduced by the above authors in their paper can lead to calculation errors. Alternative approach to this point is presented.**

***Index Terms*—Harmonic distortion modelling and calculation, unified model for inverting and noninverting nonlinear op amp based circuits, Volterra series**

I. Introduction

IN [1], S.O. Cannizzaro, G. Palumbo, and S. Pennisi claim to develop a novel method, alternative to the Volterra series approach, that simplifies calculation of harmonic distortion in analog weakly nonlinear circuits. We show in this paper that their method cannot be regarded as an alternative because it follows directly form the descriptions by the Volterra series, which simplify for single harmonic signals.

Moreover, we point out here that the concept of partition of a mildly nonlinear circuit into interconnected smaller basic blocks, used in [1] under a name of an analytical approach, is not novel. It was used in the literature for the first time by S. Narayanan, pioneer of the Volterra series usage in calculations of nonlinear distortion in electronic circuits, in his papers [2-4] published at the late of 1960's and the beginning of 1970's. In 1974, J.J. Bussgang, L. Ehrman, and J.W. Graham applied this methodology, amongst other seminal ideas, in their nonlinear analyses based on the use of Volterra series. These analyses were presented in their paper [5]. Also, the author of this paper used the above concept, in the 1980's in many articles [6-9].

Further, we show here that descriptions of the basic nonlinear blocks as given in [1] can be obtained via the so-called nonlinear transfer functions [5]. Then, these transfer functions assume simpler forms because of kind of the circuit input signal, being a single harmonic in this case.

Next, we recall some references [6], [7], [9], [22] in which the joint and complementary elements as well as some invariants occurring in modelling of op amp inverting and non-inverting configurations for the purpose of nonlinear distortion evaluation have been reported. Note that these results have been obtained before publication of some similar findings in [1].

Finally, we show here that an operator *o* introduced and used in papers [1], [11-13] can lead to calculation errors. A mathematically correct derivation of an alternative formula to that containing the above operator is presented in this article. It involves the usage of the Kronecker products [27].

The remainder of the paper is organized as follows. In Section II, we present the usage of basic nonlinear cascade and feedback structures in the analysis of more complicated nonlinear circuit topologies. In the next section, the topic of evaluation of harmonic and intermodulation distortion is discussed. Further, in section IV, some common and complementary elements in modelling of op amp inverting and non-inverting configurations are described. The next section is devoted to an operator *o* introduced in [1]. Finally, concluding remarks are summarized in section VI.

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II. Nonlinear Transfer Functions of Cascade and Feedback Structures and Their Usage in Description of More Complicated Topologies

Let us take into account two basic connections of weakly nonlinear elements with memory: cascade and feedback structures, as shown schematically in Figs. 1 and 2, respectively.

The variables v_i, v_1, v_2, and v_f in the circuit schemes in Figs. 1 and 2 stand for the appropriate input and output signals of the circuit basic blocks *H* and *K*, and of the resulting circuit. They can mean voltage as well as current type signals, depending upon the type of transfer characteristics modeled. If, for example, the blocks *H* and *K* model voltage amplifiers, all the above variables will have meanings of voltages. Further, we assume here that the circuit basic elements (blocks) in Figs. 1 and 2 are weakly (mildly) nonlinear ones. Moreover, we assume that they are of strictly transferring type. That is they can be fully described by input-output type relations.

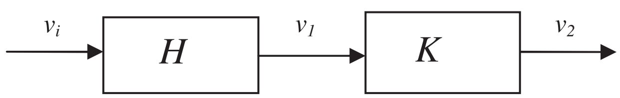

Fig. 1. Cascade connection of two nonlinear circuit basic blocks, *H* and *K*.

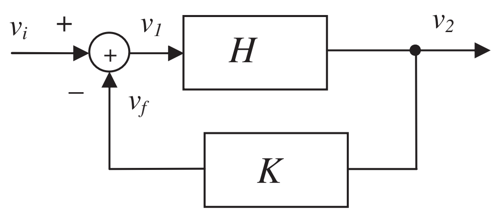

Fig. 2. Feedback structure consisting of two nonlinear circuit basic blocks, *H* and *K*.

Thus, taking into account the above two assumptions, we can describe the basic elements *H* and *K* by the Volterra series [5] as

$$y = H\left(x\right) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h^{(n)}(\tau_1,\dots,\tau_n) \prod_{k=1}^{n} x(t-\tau_k) d\tau_k \quad \text{(1a)}$$

for *H*, and similarly

$$y = K\left(x\right) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} k^{(n)}(\tau_1,\dots,\tau_n) \prod_{k=1}^{n} x(t-\tau_k) d\tau_k \quad \text{(1b)}$$

for *K*. Obviously for a linear system, the equations (1a) and (1b) are reduced to

$$y = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \quad \text{(2a)}$$

and

$$y = \int_{-\infty}^{\infty} k(\tau) x(t-\tau) d\tau \ , \quad \text{(2b)}$$

which is well known formula which can be found in many textbooks.

In (1a) and (2a), *x* and *y* correspond to v_i and v_1 in Figs. 1, and to v_1 and v_2 in Figs. 2, respectively. Similarly in (1b) and (2b), *x* and *y* correspond to v_1 and v_2 in Fig. 1, and to v_2 and v_f in Fig. 2, accordingly. $H\left(x\right)$ and $K\left(x\right)$ in (1a) and (1b) are the nonlinear operators with memory describing the basic elements *H* and *K*, respectively; they are expanded in the Volterra series in (1a) and (1b). Moreover, the variable *t* is a real time variable, but $\tau_1,\dots,\tau_n$ are artificial auxiliary ones. Furthermore, the functions $h^{(n)}(\tau_1,\dots,\tau_n)$ in (1a) and $k^{(n)}(\tau_1,\dots,\tau_n)$ in (1b) are the so-called nonlinear impulse responses of the *n*-th order [5] of the block *H* and *K*, respectively. For the linear case considered in (2a) and (2b), a simplified notation $h(\tau) = h^{(1)}(\tau_1)$ and $k(\tau) = k^{(1)}(\tau_1)$ is used.

Note that the above functions can be transferred into the multidimensional frequency domains by using the multidimensional Fourier transforms defined as

$$G^{(n)}(f_1,\dots,f_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g^{(n)}(\tau_1,\dots,\tau_n) \cdot \\ \cdot \exp(-j2\pi(f_1\tau_1 + \dots + f_n\tau_n)) d\tau_1 \cdots d\tau_n \ , \quad (3)$$

where $G^{(n)}(f_1,\dots,f_n)$ means the *n*-dimensional Fourier transform of a function $g^{(n)}(\tau_1,\dots,\tau_n)$ having *n* arguments. The transforms $H^{(n)}(f_1,\dots,f_n)$ and $K^{(n)}(f_1,\dots,f_n)$ obtained in such a way are the *n*-th order nonlinear transfer functions [5] of the circuit basic elements *H* and *K*. The variables $f_1,\dots,f_n$ therein are the subsequent frequencies in the *n*-dimensional frequency space.

Let us now describe the resulting circuit in Fig. 1 or in Fig. 2 by nonlinear operators $L\left(x\right)$ and $M\left(x\right)$, respectively. Obviously, these operators can be expanded in the Volterra series, similarly as $H\left(x\right)$ and $K\left(x\right)$ in (1a) and (1b). Further, nonlinear impulse responses $l^{(n)}(\tau_1,..,\tau_n)$ and $m^{(n)}(\tau_1,..,\tau_n)$ associated with the Volterra series expansions of $L\left(x\right)$ and $M\left(x\right)$, respectively, can be transferred into the multi-dimensional frequency domains. Then, they will be called the nonlinear transfer functions of the corresponding orders of the cascade and feedback structures of Figs. 1 and 2, respectively. Let us denote them here as $L^{(n)}(f_1,\dots,f_n)$ and $M^{(n)}(f_1,\dots,f_n)$, accordingly.

Knowing the nonlinear transfer functions $H^{(n)}(f_1,\dots,f_n)$ and $K^{(n)}(f_1,\dots,f_n)$, one can derive the resulting transfer functions for the structures in Figs. 1 and 2. This was done, with the use of different methods, by many authors. We mention here some earlier publications [2-7], [9-10] on this topic.

Let us now present the final results of the above derivations

$$L^{(1)}(f_1) = K^{(1)}(f_1)H^{(1)}(f_1) \ , \tag{4a}$$

$$\begin{aligned} L^{(2)}(f_1, f_2) = {} & K^{(1)}(f_1 + f_2)H^{(2)}(f_1, f_2) + \\ & + K^{(2)}(f_1, f_2)H^{(1)}(f_1)H^{(1)}(f_2) \ , \end{aligned} \tag{4b}$$

$$\begin{aligned} L^{(3)}(f_1, f_2, f_3) = {} & K^{(1)}(f_1 + f_2 + f_3)H^{(3)}(f_1, f_2, f_3) + \\ & + K^{(2)}(f_1, f_2 + f_3)H^{(1)}(f_1)H^{(2)}(f_2, f_3) + \\ & + K^{(2)}(f_1 + f_2, f_3)H^{(2)}(f_1, f_2)H^{(1)}(f_3) + \\ & + K^{(3)}(f_1, f_2, f_3)H^{(1)}(f_1)H^{(1)}(f_2)H^{(1)}(f_3) \end{aligned} \tag{4c}$$

for the first three nonlinear transfer functions $L^{(1)}(f_1)$, $L^{(2)}(f_1, f_2)$, and $L^{(3)}(f_1, f_2, f_3)$ of the cascade connection, and

$$M^{(1)}(f_1) = \frac{H^{(1)}(f_1)}{1 + H^{(1)}(f_1)K^{(1)}(f_1)} \ , \tag{5a}$$

$$\begin{aligned} M^{(2)}(f_1, f_2) = {} & \frac{H^{(2)}(f_1, f_2)}{\left[1 + H^{(1)}(f_1 + f_2)K^{(1)}(f_1 + f_2)\right]} \cdot \\ & \cdot \frac{1}{\left[1 + H^{(1)}(f_1)K^{(1)}(f_1)\right]\left[1 + H^{(1)}(f_2)K^{(1)}(f_2)\right]} \ , \end{aligned} \tag{5b}$$

$$\begin{aligned} M^{(3)}(f_1, f_2, f_3) = {} & \frac{1}{\left[1 + H^{(1)}(f_1 + f_2 + f_3)K^{(1)}(f_1 + f_2 + f_3)\right]} \cdot \\ & \cdot \frac{1}{\left[1 + H^{(1)}(f_1)K^{(1)}(f_1)\right]\left[1 + H^{(1)}(f_2)K^{(1)}(f_2)\right]} \cdot \\ & \cdot \frac{1}{\left[1 + H^{(1)}(f_3)K^{(1)}(f_3)\right]} \Big\{ H^{(3)}(f_1, f_2, f_3) - \\ & - \frac{H^{(2)}(f_1 + f_2, f_3)H^{(2)}(f_1, f_2)K^{(1)}(f_1 + f_2)}{\left[1 + H^{(1)}(f_1 + f_2)K^{(1)}(f_1 + f_2)\right]} - \\ & - \frac{H^{(2)}(f_1, f_2 + f_3)H^{(2)}(f_2, f_3)K^{(1)}(f_2 + f_3)}{\left[1 + H^{(1)}(f_2 + f_3)K^{(1)}(f_2 + f_3)\right]} \Big\} \end{aligned} \tag{5c}$$

for the first three nonlinear transfer functions $M^{(1)}(f_1)$, $M^{(2)}(f_1, f_2)$, and $M^{(3)}(f_1, f_2, f_3)$ of the feedback structure. Moreover, note that $L^{(1)}(f_1)$ and $M^{(1)}(f_1)$ above mean standard linear transfer functions. That is the ones for which we usually use the following notation: $L(f)$ and $M(f)$. Finally, we point out at this point that, because a lack of space, the expressions (5b) and (5c) are provided for the operator $K(x)$ being strictly linear. When this operator is nonlinear, the aforementioned expressions are notably longer.

Consider now some mixed circuit structures consisting of both cascade and feedback type connections, which are shown in Figs. 3, 4, 5, 6, and 7.

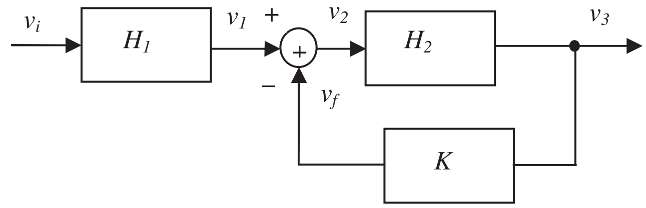

Fig. 3. Cascade connection of two nonlinear circuit blocks, in which the first of them is a basic one and the second made up of a feedback structure connecting two nonlinear basic blocks.

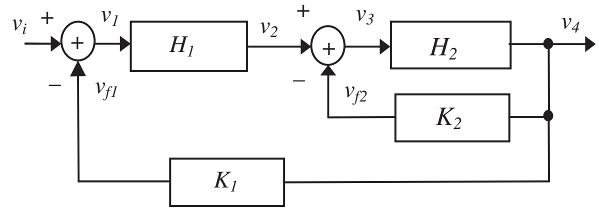

Fig. 4. Feedback structure made up of the resulting nonlinear circuit block shown in Fig. 3 and a nonlinear basic block K_1.

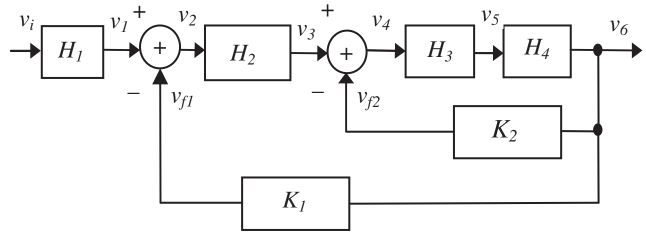

Fig. 5. Cascade structure made up of the resulting nonlinear circuit block shown in Fig. 4 and a nonlinear basic block H_1.

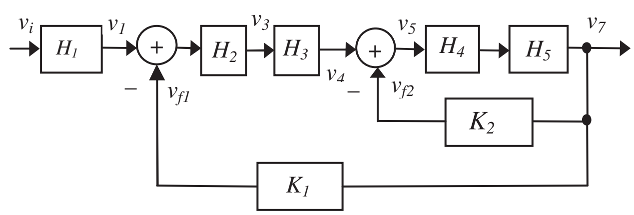

Fig. 6. Extended structure of Fig. 5 with a cascade connection of two nonlinear circuit basic blocks denoted as H_2 and H_3 instead of a basic block denoted as H_2 in Fig. 5.

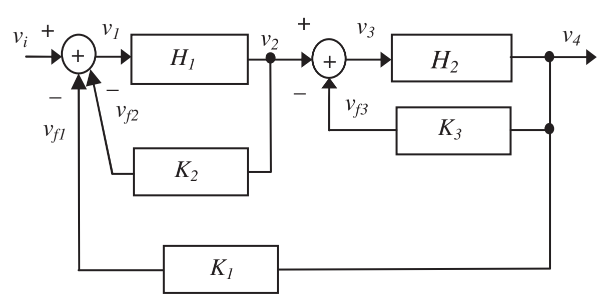

Fig. 7. Modified structure of Fig. 4 with a feedback connection of two nonlinear circuit basic blocks denoted as H_1 and K_2 instead of a basic block denoted as H_1 in Fig. 4.

Observe first that the nonlinear transfer functions of all the structures shown in Figs. 3, 4, 5, 6, and 7, and of any of their derivatives, can be easily evaluated in a systematic way using the expressions evoked above for the basic cascade and feedback structures of Figs. 1 and 2.

Next, note that the structure shown in Fig. 3 is that which models the nonlinear behavior of negative-feedback amplifiers considered in [1]. Further, see that the structure of Fig. 5, which is the derivative of that in Fig. 4, was used in [13] to describe the weakly nonlinear two-stage Miller OTAs. Observe also that the extended structure of Fig. 6 was applied in [14] to model the nonlinear behavior of the Miller-compensated three-stage amplifiers.

The structure of Fig. 7 is an extension of that shown in Fig. 4 with an additional inner loop; it can be also useful in modelling mildly nonlinear multi-stage amplifiers [15].

The above examples from the recent publications and also those cited in this section before show clearly that all practical mildly nonlinear circuits can be partitioned into interconnected smaller basic blocks. On the other way, these interconnected basic blocks form larger entities, interconnections of cascade and feedback structures. And the latter can be described with the use of the expressions determining their nonlinear transfer functions that are well known since works of Narayanan [2-4].

III. Evaluation of Intermodulation and Harmonic Distorsion Using Nonlinear Transfer Functions

Denote now the circuit nonlinear transfer functions of the first, second, and third order, related with the signal transmission from its input to its output, as $H_o^{(1)}(f_1)$, $H_o^{(2)}(f_1,f_2)$, and $H_o^{(3)}(f_1,f_2,f_3)$, respectively. As shown in the previous section, expressions determining these transfer functions can be evaluated for the most of practical nonlinear circuits in a simplified manner. This approach means:

1. partition of a circuit scheme into smaller basic blocks;
2. carrying out an analysis of interconnections between these blocks to find out all the cascade and feedback type connections;
3. successive application of the expressions presented in section II to arrive finally at the nonlinear transfer functions of the whole circuit scheme.

Since Narayanan [2-4], it is well known that the nonlinear transfer functions of a weakly nonlinear circuit can be used to calculate intermodulation distortion it generates, when a two tone harmonic signal $v_i(t)$

$$v_i(t) = AMP_a\cos(2\pi f_a t) + AMP_b\cos(2\pi f_b t) \tag{6}$$

with the frequencies f_a and f_b, where $f_a / f_b \neq m/n$, $f_a > f_b$, and the amplitudes AMP_a and AMP_b is applied to its input. In this case, when we are interested, for example, in the output intermodulation products: of the second order at the frequency $f_a - f_b$ and of the third order at the frequency $2f_a \pm f_b$, we must substitute $f_1 = f_a$, $f_2 = f_a$, and $f_3 = -f_b$ in the expressions determining the nonlinear transfer functions of a given circuit. For more explanation, see, for instance, [16] or [17].

In consideration of weakly nonlinear circuits, we usually restrict ourselves to taking into account only nonlinear transfer functions of the first three orders (including the linear one). Then, referring to our aforementioned example, we have the following expressions [6], [17]

$$IM2 = 20\log\left[\frac{AMP_o(f_a - f_b)}{AMP_a}\right] \cong 20\log\left[\left|\frac{H_o^{(2)}(f_a,-f_b)}{H_o^{(1)}(f_a)}\right| AMP_b\right] \tag{7a}$$

and

$$\begin{aligned} IM3 = 20\log\left[\frac{AMP_o(2f_a - f_b)}{AMP_a}\right] \cong \\ \cong 20\log\left[\frac{3}{4}\left|\frac{H_o^{(3)}(f_a,f_a,-f_b)}{H_o^{(1)}(f_a)}\right| AMP_a\cdot AMP_b\right], \end{aligned} \tag{7b}$$

for the intermodulation distortion factors of the second order *IM2* and of the third order *IM3*, respectively. In (7), $AMP_o(\cdot)$ means the amplitude of the circuit output signal component at the corresponding frequency, and $H_o^{(1)}(f_a)$, $H_o^{(2)}(f_a,-f_b)$ and $H_o^{(3)}(f_a,f_a,-f_b)$ are the circuit nonlinear transfer functions of the first, second and third order, respectively, relating its output with its input and calculated for the corresponding values of their arguments.

By applying a single tone harmonic signal $v_i(t)$ of the form

$$v_i(t) = AMP_s\cos(2\pi f_s t) \tag{8}$$

to the input of a mildly nonlinear circuit, we concentrate on harmonic distortion generated by this circuit. Such an approach to quantify nonlinear distortion is represented in papers [1] and [11-14]; the authors of these articles concentrate exclusively on the harmonic distortion factors.

In (8), AMP_s and f_s mean the amplitude of a single tone harmonic signal and its frequency, respectively.

With the input signal given by (8), the expressions determining the circuit harmonic distortion factors of the second order *H2* and of the third order *H3* have the following form

$$H2 = 20\log\left[\frac{AMP_o(2f_s)}{AMP_s}\right] \cong 20\log\left[\frac{1}{2}\left|\frac{H_o^{(2)}(f_s,f_s)}{H_o^{(1)}(f_s)}\right| AMP_s\right] \tag{9a}$$

and

$$H3 = 20\log\left[\frac{AMP_o(3f_s)}{AMP_s}\right] \cong 20\log\left[\frac{1}{4}\left|\frac{H_o^{(3)}(f_s,f_s,f_s)}{H_o^{(1)}(f_s)}\right| AMP_s^2\right], \tag{9b}$$

respectively. In (9), $AMP_s(\cdot)$ stands for the amplitude of the circuit output signal component at the corresponding frequency. Moreover, note that the expressions (9) indicate that we have to substitute $f_1 = f_s$, $f_2 = f_s$, and $f_3 = f_s$ in the expressions determining the nonlinear circuit input-output transfer functions.

Looking at the expressions (7) and (9), we see that we need to know the circuit nonlinear transfer functions for evaluation of its nonlinear distortion factors *IM2*, *IM3*, *H2*, and *H3*. This is crucial here: the knowledge of the expressions determining the aforementioned functions. More generally, knowing them, we can evaluate any other measure of the nonlinear distortion as, for example, the cross-modulation distortion factor [18], the third order input intercept point IIP3 [19], the so-called 1 dB compression point (in short, 1 dBc point) [20] and others used in the literature.

In [1], the harmonic distortion factors *H2* and *H3* defined by the left-hand side equalities in (9) (strictly saying, their non-logarithmic versions) were expressed by coefficients $a_1(j\omega_s)$, $a_2(j\omega_s)$, and $a_3(j\omega_s)$, named "the first (linear), second-, and third-order nonlinearity transfer functions", respectively. In the above definition, the word "nonlinearity" is generic and can mean a nonlinear circuit basic block, a cascade connection, and a feedback connection. Moreover, $\omega_s = 2\pi f_s$. Therefore, we will use, in what follows, the following notation: $a_{1x}(f_s)$, $a_{2x}(f_s)$, and $a_{3x}(f_s)$ with the index x standing for a generic name to denote a particular basic circuit block or a particular cascade connection or a particular feedback connection. The same notational convention will also regard the Volterra series based nonlinear transfer functions $H_x^{(1)}(f_s)$, $H_x^{(2)}(f_s, f_s)$, and $H_x^{(3)}(f_s, f_s, f_s)$ that we will use in further considerations.

The coefficients $a_{1x}(f_s)$, $a_{2x}(f_s)$, and $a_{3x}(f_s)$ were evaluated in [1] using the elements of phasor and harmonics balance theories, without referring to the Volterra series theory. However, it has been shown in [21] that these coefficients can be expressed by the Volterra series based nonlinear transfer functions. Then, the following equalities: $a_{1x}(f_s) = H_x^{(1)}(f_s)$, $a_{2x}(f_s) = H_x^{(2)}(f_s, f_s)$, and $a_{3x}(f_s) = H_x^{(3)}(f_s, f_s, f_s)$ hold.

Assume now that the frequency f_s in the coefficients $a_{1x}(f_s)$, $a_{2x}(f_s)$, and $a_{3x}(f_s)$ changes its value. So, it becomes a variable; denote it as f. Then, we can write $a_{2x}(f)$ and $a_{3x}(f)$, indicating clearly that they are functions of only one variable. They are conceptually functions of only one variable, see the theory presented in [1], [11-14]. On the contrary, the Volterra series based nonlinear transfer functions $H_x^{(2)}(f_1, f_2)$ and $H_x^{(3)}(f_1, f_2, f_2)$ [5], [16] are functions of two variables f_1, f_2 or of three variables f_1, f_2, f_2, respectively. Further, note that the above makes the fundamental difference between the latter and former ones. And the following statement is true.

Statement 1. It is not possible to determine the Volterra series based nonlinear transfer functions $H_x^{(2)}(f_1, f_2)$ and $H_x^{(3)}(f_1, f_2, f_2)$ of a nonlinear circuit element (block) from the functions $a_{2x}(f)$ and $a_{3x}(f)$ known for this circuit (block). However, the opposite is true.

Having in mind the previous discussions and taking into account also the Statement 1, we can formulate the next statement.

Statement 2. The method developed in [1], [11-14] for mildly nonlinear circuits can be viewed as a simplified Volterra series based analysis that is restricted to evaluation of the harmonic distortion.

Finally, it is clear from the above that most of the nonlinear distortion measures, as for example the intermodulation and cross-modulation distortion factors, cannot be calculated within the approach described in [1], [11-14]. For their evaluation, we need to use more general tools as, for example, the Volterra series descriptions of nonlinear circuit elements.

IV. Joint and Complementary Elements in Modelling of Op Amp Inverting and Non-inverting Configurations

In articles [6], [7], [9], [22], the nonlinear distortions in form of harmonics and/or intermodulation products and/or basic harmonic compression in single-amplifier active filters have been measured and evaluated with the use of the Volterra series. Op amp with resistive feedback circuitry was applied to build the filter amplifier. This amplifier worked in a nonlinear region of op amp characteristics. And this was the region of dominance of the so-called slew-rate nonlinearity. Both the inverting and non-inverting configurations of op amp with the resistive feedback loop were investigated. To model the filter amplifier, that is to calculate its nonlinear transfer functions, the structure of Fig. 2 was used. Further, to calculate the nonlinear transfer functions of the whole filter, the model shown in Fig. 3 was applied. It has been found that some components occurring in the expressions determining the *IM2*, *IM3*, *H2*, and *H3* for the whole filter, which depend exclusively upon the nonlinear transfer functions of the filter amplifier, are approximately independent of the kind of op amp configuration used. So, these components are the invariants for the above class of filters. Certainly, similar invariants can be found for other active filter topologies.

Additionally, it has been shown in [22] that the nonlinear transfer functions of the aforementioned op amp configurations are related to each other for the complementary single-amplifier filters through the complementary relations derived therein.

Active filters are often designed with the use of both the op amp inputs, inverting and non-inverting one. Each of them is then connected with the filter output via a feedback loop. We point here that there is in this case a more useful structure than that used in [1] as well as in [6], [7], [9], [22], which can be applied for the analysis. On the other hand, it is also more general than the aforementioned ones. It was already exploited by the author of this paper in [23] and is presented in Fig. 8.

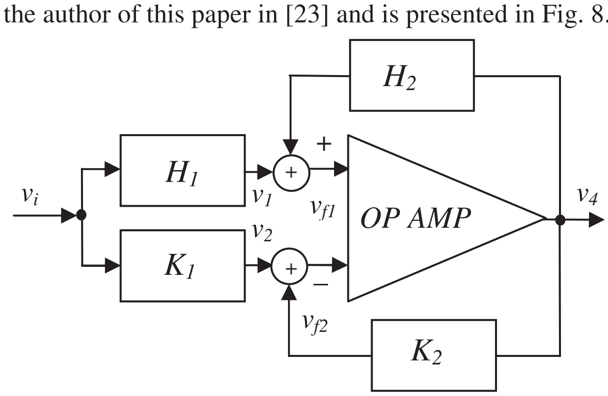

Fig. 8. Circuit structure using op amp as a basic element three-terminal element and with two feedback loops associated with its inverting and non-inverting inputs.

V. Operator *o* Revisited

In [1], an operator *o* has been introduced. Its definition was formulated therein in such a way: "Let

$$x(t)=X_1\exp(j2\pi f_st)+X_2\exp(j2\pi 2f_st)+X_3\exp(j2\pi 3f_st) \quad (10)$$

be the complex valued signal consisting of three harmonics: the fundamental of frequency f_s, the second, and third one that is applied to a weakly nonlinear circuit. In (10), X_1, X_2, and X_3, mean generally complex amplitudes of the above harmonics. Then, the signal at its output will be given by

$$y(t)=x(t)o\left[a_{1x}(f_s)+a_{2x}(f_s)x(t)+a_{3x}(f_s)(x(t))^2\right], \quad (11)$$

where the operator "*o*" means that the functions which appear within the square brackets must be evaluated at the frequency of the incoming signal. This operator must be used whenever we evaluate the output of a nonlinear block."

We see that the above definition is not mathematically clear and highly imprecise. So, its usage can lead to misleading results.

Consider now this problem in more detail. To this end, we will write the Volterra series representation modelling a weakly nonlinear circuit, as for example given by (1a), using the operator terminology. That is we will assume that an operator $H(x(t))$ stands for this representation, similarly as in [24] or [25]. Moreover, we will assume that it can be expressed in form of a sum of operators working on the input signal $x(t)$. That is $H(x(t))$ will be given by

$$\begin{aligned} y(t)&=H(x(t))=(H)(x)(t)= \\ &=\left((H_1)+(H_2)(\cdot)^2+(H_3)(\cdot)^3+\ldots\right)(x)(t) \end{aligned}, \quad (12)$$

where the meaning of H_1, H_2, and H_3 differs now from that used in the previous sections. Namely, they mean now the first three components of the operator series. They can be also viewed as the operators of the first, second, and third order (and of higher orders regarding the next ones in (12)) in the equivalent Volterra series (12); for more details, see [25].

Substituting $x(t)$ given by (10) into (12) leads to getting the following

$$\begin{aligned} y(t)&=(H_1)\big(X_1\exp(j2\pi f_st)+X_2\exp(j2\pi 2f_st)+ \\ &+X_3\exp(j2\pi 3f_st)\big)+(H_2)\big(X_1\exp(j2\pi f_st)+ \\ &+X_2\exp(j2\pi 2f_st)+X_3\exp(j2\pi 3f_st)\big)^2+ \\ &+(H_3)\big(X_1\exp(j2\pi f_st)+X_2\exp(j2\pi 2f_st)+ \\ &+X_3\exp(j2\pi 3f_st)\big)^3+\ldots\ . \end{aligned} \quad (13)$$

In the next step, after performing the operations of multiplication indicated in (13) and carrying out also the convolutions associated with the operators H_1, H_2, and H_3, we get

$$\begin{aligned} y(t)&=H^{(1)}(f_s)X_1\exp(j2\pi f_st)+H^{(1)}(2f_s)\cdot \\ &\cdot X_2\exp(j2\pi 2f_st)+H^{(1)}(3f_s)X_3\exp(j2\pi 3f_st)+ \\ &+H^{(2)}(f_s,f_s)X_1X_1\exp(j2\pi f_st)\cdot \\ &\cdot\exp(j2\pi f_st)+H^{(2)}(f_s,2f_s)X_1X_2\exp(j2\pi f_st)\cdot \\ &\cdot\exp(j2\pi 2f_st)+H^{(2)}(2f_s,f_s)X_2X_1\exp(j2\pi 2f_st)\cdot \\ &\cdot\exp(j2\pi f_st)+ \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s + \\ &+H^{(3)}(f_s,f_s,f_s)X_1X_1X_1\exp(j2\pi f_st)\exp(j2\pi f_st)\cdot \\ &\cdot\exp(j2\pi f_st)+ \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s\ , \end{aligned} \quad (14)$$

where the nonlinear transfer functions $H^{(1)}$, $H^{(2)}$, and $H^{(3)}$, calculated for the corresponding sets of frequencies, correspond with the operators H_1, H_2, and H_3, respectively.

Observe now that (14) derived with the use of the Volterra series is an alternative expression to (11). The next remark is the following: it is, however, difficult to recognize in (14) the form of the formula represented by (11) of an operator working on $x(t)$.

In what follows, we will try to write down (14) in a shorter form by applying the vectors, matrices, and Kronecker products. And for this task, let us define the following vectors

$$\mathbf{H}^{(1)}=\left[H^{(1)}(f_s)\ \ H^{(1)}(2f_s)\ \ H^{(1)}(3f_s)\right], \quad (15a)$$

$$\begin{aligned}\mathbf{H}^{(2)}=\big[&H^{(2)}(f_s,f_s)\ \ H^{(2)}(f_s,2f_s)\ \ H^{(2)}(f_s,3f_s) \\ &H^{(2)}(2f_s,f_s)\ \ H^{(2)}(2f_s,2f_s)\ \ H^{(2)}(2f_s,3f_s) \\ &H^{(2)}(3f_s,f_s)\ \ H^{(2)}(3f_s,2f_s)\ \ H^{(2)}(3f_s,3f_s)\big]\end{aligned}, \quad (15b)$$

$$\begin{aligned}\mathbf{H}^{(3)}=\big[&H^{(3)}(f_s,f_s,f_s)\ \ H^{(3)}(f_s,f_s,2f_s)\ \ H^{(3)}(f_s,f_s,3f_s) \\ &H^{(3)}(f_s,2f_s,f_s)\ \ H^{(3)}(f_s,2f_s,2f_s)\ \ H^{(3)}(f_s,2f_s,3f_s) \\ &H^{(3)}(f_s,3f_s,f_s)\ \ H^{(3)}(f_s,3f_s,2f_s)\ \ H^{(3)}(f_s,3f_s,3f_s) \\ &H^{(3)}(2f_s,f_s,f_s)\ \ H^{(3)}(2f_s,f_s,2f_s)\ \ H^{(3)}(2f_s,f_s,3f_s) \\ &H^{(3)}(2f_s,2f_s,f_s)\ \ H^{(3)}(2f_s,2f_s,2f_s)\ \ H^{(3)}(2f_s,2f_s,3f_s) \\ &H^{(3)}(2f_s,3f_s,f_s)\ \ H^{(3)}(2f_s,3f_s,2f_s)\ \ H^{(3)}(2f_s,3f_s,3f_s) \\ &H^{(3)}(3f_s,f_s,f_s)\ \ H^{(3)}(3f_s,f_s,2f_s)\ \ H^{(3)}(3f_s,f_s,3f_s) \\ &H^{(3)}(3f_s,2f_s,f_s)\ \ H^{(3)}(3f_s,2f_s,2f_s)\ \ H^{(3)}(3f_s,2f_s,3f_s) \\ &H^{(3)}(3f_s,3f_s,f_s)\ \ H^{(3)}(3f_s,3f_s,2f_s)\ \ H^{(3)}(3f_s,3f_s,3f_s)\big]\end{aligned}, \quad (15c)$$

and

$$\mathbf{x}=\begin{bmatrix} X_1\exp(j2\pi f_st) \\ X_2\exp(j2\pi 2f_st) \\ X_3\exp(j2\pi 3f_st) \end{bmatrix}. \quad (15d)$$

The vectors $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$, and $\mathbf{H}^{(3)}$ given by (15a), (15b), and (15c), respectively, are the row vectors, but the vector $\mathbf{x}$ given by (15d) is a column one. The former gather the linear and nonlinear transfer functions of the second and third order calculated at all the possible products of frequencies occurring in the input signal (10), taking into account also their positions as arguments in the aforementioned transfer functions. Further, $\mathbf{x}$ is a vector description of the input signal given by (10).

In the next step, observe that all the products of the components of the input signal (10), which occur in (14), can be expressed in a compact form with the use of the Kronecker formalism [27]. That is as the Kronecker products [27] of the vector $\mathbf{x}$ given by(15d). For example, for the products of the second order of the vector elements, we get

$$\mathbf{x}\otimes\mathbf{x}=\begin{bmatrix}X_1\exp(j2\pi f_s t)\\X_2\exp(j2\pi 2f_s t)\\X_3\exp(j2\pi 3f_s t)\end{bmatrix}\otimes\begin{bmatrix}X_1\exp(j2\pi f_s t)\\X_2\exp(j2\pi 2f_s t)\\X_3\exp(j2\pi 3f_s t)\end{bmatrix}=$$
$$=\begin{bmatrix}X_1\exp(j2\pi f_s t)\begin{bmatrix}X_1\exp(j2\pi f_s t)\\X_2\exp(j2\pi 2f_s t)\\X_3\exp(j2\pi 3f_s t)\end{bmatrix}\\X_2\exp(j2\pi 2f_s t)\begin{bmatrix}X_1\exp(j2\pi f_s t)\\X_2\exp(j2\pi 2f_s t)\\X_3\exp(j2\pi 3f_s t)\end{bmatrix}\\X_3\exp(j2\pi 3f_s t)\begin{bmatrix}X_1\exp(j2\pi f_s t)\\X_2\exp(j2\pi 2f_s t)\\X_3\exp(j2\pi 3f_s t)\end{bmatrix}\end{bmatrix}, \tag{16}$$

where the symbol $\otimes$ stands for the right Kronecker product [27]. And finally, after carrying out all the multiplications of the vectors by functions indicated on the most right-hand side of (16), we arrive at

$$\mathbf{x}\otimes\mathbf{x}=\begin{bmatrix}X_1X_1\exp(j2\pi f_s t)\cdot\exp(j2\pi f_s t)\\X_1X_2\exp(j2\pi f_s t)\cdot\exp(j2\pi 2f_s t)\\X_1X_3\exp(j2\pi f_s t)\cdot\exp(j2\pi 3f_s t)\\X_2X_1\exp(j2\pi 2f_s t)\cdot\exp(j2\pi f_s t)\\X_2X_2\exp(j2\pi 2f_s t)\cdot\exp(j2\pi 2f_s t)\\X_2X_3\exp(j2\pi 2f_s t)\cdot\exp(j2\pi 3f_s t)\\X_3\exp(j2\pi 3f_s t)\cdot\exp(j2\pi f_s t)\\X_3X_2\exp(j2\pi 3f_s t)\cdot\exp(j2\pi 2f_s t)\\X_3X_3\exp(j2\pi 3f_s t)\cdot\exp(j2\pi 3f_s t)\end{bmatrix}. \tag{17}$$

Similarly, by calculating $\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}$, one gets all the third order products of the elements of the vector $\mathbf{x}$ put into one vector. That is she/he gets the following

$$\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}=$$
$$=\begin{bmatrix}X_1X_1X_1\exp(j2\pi f_s t)\cdot\exp(j2\pi f_s t)\cdot\exp(j2\pi f_s t)\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\\X_1X_3X_2\exp(j2\pi f_s t)\cdot\exp(j2\pi 3f_s t)\cdot\exp(j2\pi 2f_s t)\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\\X_3X_3X_3\exp(j2\pi 3f_s t)\cdot\exp(j2\pi 3f_s t)\cdot\exp(j2\pi 3f_s t)\end{bmatrix}. \tag{18}$$

Now, observe that using (15a), (15b), (15c), (15d), (17), and (18) we can express (14) as a sum of the scalar products of these vectors. So, the formula will be the following

$$y(t)=\mathbf{H}^{(1)}\cdot\mathbf{x}(t)+\mathbf{H}^{(2)}\cdot\big((\mathbf{x}\otimes\mathbf{x})(t)\big)+\mathbf{H}^{(3)}\cdot\big((\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x})(t)\big). \tag{19}$$

Note that this formula is a compact representation of (14). Its form resembles the form of the expression (11). However, it is still difficult to deduce from it a mathematically correct definition of the operator o occurring in (11).

The analyses presented in [1] and [11-13] were restricted to considering only the first three harmonic components in $y(t)$, occurring at the frequencies f_s, $2f_s$, and $3f_s$. Note that this corresponds to approximating $y(t)$ in (14) by

$$\begin{aligned}\hat{y}(t)&=H^{(1)}(f_s)X_1\exp(j2\pi f_s t)+H^{(1)}(2f_s)\cdot\\&\cdot X_2\exp(j2\pi 2f_s t)+H^{(1)}(3f_s)X_3\exp(j2\pi 3f_s t)+\\&+H^{(2)}(f_s,f_s)X_1X_1\exp(j2\pi f_s t)\cdot\exp(j2\pi f_s t)+\\&+H^{(2)}(f_s,2f_s)X_1X_2\exp(j2\pi f_s t)\cdot\exp(j2\pi 2f_s t)+\\&+H^{(2)}(2f_s,f_s)X_2X_1\exp(j2\pi 2f_s t)\cdot\exp(j2\pi f_s t)+\\&+H^{(3)}(f_s,f_s,f_s)X_1X_1X_1\exp(j2\pi f_s t)\exp(j2\pi f_s t)\cdot\\&\cdot\exp(j2\pi f_s t)\end{aligned}, \tag{20}$$

where $\hat{y}(t)$ means the approximated value of $y(t)$ in the sense given above. Further, see that using the filtering terminology we can interpret this as filtering out the first three harmonics from the signal $y(t)$. Obviously, this procedure can be also applied to the compact description (19). See that we achieve this goal by setting to zero all the elements in the vectors given by (15a), (15b), and (15c) whose sums of arguments are greater than $3f_s$. (Note that these elements are the nonlinear transfer functions calculated for given sets of frequencies.)

Then, the modified vectors (15a), (15b), and (15c), denoted here by $\hat{\mathbf{H}}^{(1)}$, $\hat{\mathbf{H}}^{(2)}$, and $\hat{\mathbf{H}}^{(3)}$, respectively, will have the following form

$$\hat{\mathbf{H}}^{(1)}=\mathbf{H}^{(1)}=\left[H^{(1)}(f_s)\ \ H^{(1)}(2f_s)\ \ H^{(1)}(3f_s)\right], \tag{21a}$$

$$\hat{\mathbf{H}}^{(2)} = \big[H^{(2)}(f_s, f_s) \;\; H^{(2)}(f_s, 2f_s) \;\; 0 \;\; H^{(2)}(2f_s, f_s) \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \big], \tag{21b}$$

and

$$\mathbf{H}^{(3)} = \big[H^{(3)}(f_s, f_s, f_s) \;\; 0 \big]. \tag{21c}$$

And finally, with the use of the row vectors $\hat{\mathbf{H}}^{(1)}$, $\hat{\mathbf{H}}^{(2)}$, and $\hat{\mathbf{H}}^{(3)}$, defined above, it will be possible to write (20) in the following compact form

$$\hat{y}(t) = \hat{\mathbf{H}}^{(1)} \cdot \mathbf{x}(t) + \hat{\mathbf{H}}^{(2)} \cdot \big((\mathbf{x} \otimes \mathbf{x})(t)\big) + \hat{\mathbf{H}}^{(3)} \cdot \big((\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(t)\big). \tag{22}$$

VI. Concluding Remarks

It has been shown in this paper that some recently published results concerning evaluation of the harmonic distortion in weakly nonlinear circuits need reviewing in view of the older ones. The basic lines for such a reviewing have been sketched here. Detailed analyses, in opinion of the author of this paper, are still desired and could be a subject, for example, of a Ph. D. E. E. thesis.

Finally, we note also that this paper was based on the material presented at the conference MIXDES'2016 [26]. Extension of the material presented here regards derivations and discussions of section V. Among others, a new correct formula for calculation of the output signal of a mildly nonlinear circuit driven by an input signal of the form given by (10) was derived. It was derived with the use of the Volterra series and Kronecker products, and possesses a compact form.

References

[1] S. O. Cannizzaro, G. Palumbo, and S. Pennisi, "An approach to model high-frequency distortion in negative-feedback amplifiers," Journal of Circuit Theory and Applications, vol. 36, pp. 3-18, 2008.

[2] S. Narayanan, "Transistor distortion analysis using Volterra series representation," The Bell Syst. Tech. Journal, vol. 46, pp. 991-1024, May-June 1967.

[3] S. Narayanan, "Intermodulation distortion of cascaded transistors," IEEE Journal of Solid-State Circuits, vol. SC-4, pp. 97-106, June 1969.

[4] S. Narayanan, "Application of Volterra series to intermodulation distortion analysis of a transistor feedback amplifier," IEEE Trans. Circuit Theory, vol. CT-17, pp. 518-527, Nov. 1970.

[5] J. J. Bussgang, L. Ehrman, and J. W. Graham, "Analysis of nonlinear systems with multiple inputs," Proceedings of the IEEE, vol. 62, pp. 1088-1119, 1974.

[6] A. Borys, "On intermodulation and harmonic distortion in single-amplifier active filters," Journal of the Audio Engineering Society, vol. 28, Oct. 1980.

[7] A. Borys, "Harmonic and intermodulation distortion analysis in single-amplifier active filters using the Volterra-Wiener series approach," Archiwum Elektrotechniki, z. 4, 1981.

[8] A. Borys, "Nonlinear distortion in active RC filters," Proceedings of the Sixth Summer Symposium on Circuit Theory, Prague, Czechoslovakia, pp. 207-212, 1982.

[9] A. Borys, "An analysis of slew-induced distortion in single-amplifier active filters using the Volterra-Wiener series technique," Int. J. Cir. Theor. Appl., vol. 10, pp. 81-94, 1982.

[10] L. O. Chua and C.-Y Ng, "Frequency domain analysis of nonlinear systems: formulation of transfer functions," IEE J. Electron. Circuits and Systems, vol. 3, pp. 257-269, 1979.

[11] S. O. Cannizzaro, G. Palumbo, and S. Pennisi, "Effects of nonlinear feedback in the frequency domain," IEEE Trans. Circuits and Systems-I: Fundamental Theory and Applications, vol. 53, pp. 225-234, February 2006 .

[12] G. Palumbo, M. Pennisi, and S. Pennisi, "Miller theorem for weakly nonlinear feedback circuits and application to CE amplifier," IEEE Trans. Circuits and Systems-II: Express Briefs, vol. 55, pp. 991-995, October 2008 .

[13] S. O. Cannizzaro, G. Palumbo, and S. Pennisi, "Accurate estimation of high-frequency harmonic distortion in two-stage Miller OTAs", Proc. IEE, vol. 152, Oct. 2005, pp. 417-424.

[14] S. O. Cannizzaro, G. Palumbo, and S. Pennisi, "Distortion analysis of Miller-compensated three-stage amplifier", IEEE Trans. Circuits and Systems-I: Regular Papers, vol. 53, pp. 961–976, May 2006.

[15] B. Hernes and W. Sansen, "Distortion in single-, two- and three-stage amplifiers," IEEE Trans. Circuits and Systems-I: Regular Papers, vol. 52, pp. 846–856, May 2005.

[16] L. O. Chua and C.-Y. Ng, "Frequency domain analysis of nonlinear systems: general theory, IEE J. Electron. Circuits and Systems, vol. 3, pp. 165-185, 1979.

[17] A. Borys, "A simplified analysis of nonlinear distortion in analog electronic circuits using the Volterra-Wiener series," Scientia Electrica, vol. 30, pp. 78-103, 1984.

[18] R. G. Meyer, M. J. Shensa, and R. Eschenbach, "Crossmodulation and intermodulation in amplifiers at high frequencies," IEEE Journal of Solid-State Circuits, vol. SC-7, pp. 16-23, Feb. 1972.

[19] S. Ganesan, E. Sánchez-Sinencio, and J. Silva-Martinez, "A highly linear low-noise amplifier," IEEE Trans. Microwave Theory and Techniques, vol. 54, pp. 4079-4085, Dec. 2006.

[20] F. J. Casas, J. P. Pascual, M. L. de la Fuente, E. Artal, and J. Portilla, "Simple nonlinearity evaluation and modeling of low-noise amplifiers with application to radio astronomy receivers", Review of Scientific Instruments, vol. 81, pp. 1-6, 2010.

[21] A. Borys, "On correctness of the basics of Palumbo and Pennisi's means of harmonic distortion calculation in analog integrated circuits," Zesz. Nauk. WSInf w Łodzi: Teoria i zastosowania informatyki, vol. 8, pp. 5-16, 2009.

[22] A. Borys and N. Fliege, "Complementary transformation in mildly nonlinear feedback systems," Proceedings of the ECCTD'89, Brighton, United Kingdom, pp. 445-447, 1989.

[23] A. Borys, "Effect of op amp output resistance on intermodulation distortion of single amplifier filters," Proceedings of the ECCTD'81, The Hague, The Netherlands, pp. 988-992, 1981.

[24] A. Borys, „Exploiting admittance formalism in the nonlinear analysis," Zeszyty Naukowe Wyższej Szkoły Informatyki (WSInf) w Łodzi, Teoria i zastosowania informatyki, vol. 8, pp. 17-39, 2009.

[25] M. Schetzen, The Volterra and Wiener Theories of Nonlinear Systems, New York: John Wiley & Sons, 1980.

[26] A. Borys, "On analysis of harmonic distortion in op amps based circuits via Volterra series," Proceedings of the 23rd International Conference Mixed Design of Integrated Circuits and Systems, Łódź, Poland, pp. 330-335, 2016.

[27] A. A. M. Saleh, "Matrix analysis of mildly nonlinear, multiple-input, multiple-output systems with memory," The Bell System Technical Journal, vol. 61, pp. 2221-2243, Nov. 1982.

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