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AN APPROACH TO GENERALIZATION OF THE INTUITIONISTIC FUZZY TOPSIS METHOD IN THE FRAMEWORK OF EVIDENCE THEORY

Ludmila Dymova[1], Krzysztof Kaczmarek[1], Pavel Sevastjanov[1,*],
Łukasz Sułkowski[2,3], Krzysztof Przybyszewski[4,5]

[1] *Department of Computer Science,
Czestochowa University of Technology, Poland*

[2] *Faculty of Management and Social Sciences, Jagiellonian University, Cracow, Poland*

[3] *Management Department, University of Social Sciences, Lodz, Poland*

[4] *Information Technology Institute, University of Social Sciences, Lodz, Poland*

[5] *Clark University, Worcester, USA*

[*] *E-mail: sevast@icis.pcz.pl*

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Abstract

A generalization of technique for establishing order preference by similarity to the ideal solution ($TOPSIS$) in the intuitionistic fuzzy setting based on the redefinition of intuitionistic fuzzy sets theory ($A-IFS$) in the framework of Dempster-Shafer theory (DST) of evidence is proposed. The use of DST mathematical tools makes it possible to avoid a set of limitations and drawbacks revealed recently in the conventional Atanassov's operational laws defined on intuitionistic fuzzy values, which may produce unacceptable results in the solution of multiple criteria decision-making problems. This boosts considerably the quality of aggregating operators used in the intuitionistic fuzzy $TOPSIS$ method. It is pointed out that the conventional $TOPSIS$ method may be naturally treated as a weighted sum of some modified local criteria. Because this aggregating approach does not always reflects well intentions of decision makers, two additional aggregating methods that cannot be defined in the framework of conventional $A-IFS$ based on local criteria weights being intuitionistic fuzzy values, are introduced. Having in mind that different aggregating methods generally produce different alternative rankings to obtain the compromise ranking, the method for aggregating of aggregation modes has been applied. Some examples are used to illustrate the validity and features of the proposed approach.

Keywords: TOPSIS, intuitionistic fuzzy sets, Dempster-Shafer theory, aggregating modes

1 Introduction

An approach to Order Preference based on the Similarity to Ideal Solution ($TOPSIS$) proposed by Hwang and Yoon [1] with its different modifications nowadays is probably most frequently used to solve multiple criteria decision making ($MCDM$) problems. A comprehensive review of 266 papers devoted to applications and methodology of this approach presented in 103 scientific journals since 2000 is made in [2]. A more fresh survey of the conventional $TOPSIS$ method applications in the crisp (real-valued) environment is presented in [3]. In [4], the fuzzy $TOPSIS$ applications in the last decade are analyzed and systematized. There are many approaches to the fuzzy extension of the $TOPSIS$ method proposed in the literature. Most of them cannot be recognized as complete ones as ideal solutions are often assumed to be real values (not as fuzzy ones) or are not achievable in the decision matrix [5, 6, 7, 8, 9, 10]. Often a defuzzification of fuzzy components of a decision matrix is applied [5, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. This causes a crucial informational loss and, as a consequence, often unacceptable results. The less critical simplifications were made in [21, 22, 23, 24, 25, 26, 27], but some limitations of the proposed in these papers methods remain and were analyzed in [28]. In this paper, a new direct fuzzy extension of the $TOPSIS$ method, free of the limitations of the known methods, was developed.

Nowadays, intuitionistic fuzzy sets introduced by Atanassov [29], which based on the reasons noted in [30] will be hereinafter abbreviated as $A-IFS$, are the most frequently used extension of fuzzy sets met in the scholarly literature. According to recent studies [31], the Scopus database gives 3048 document results if the keyword “intuitionistic fuzzy” (IF) is in the article title. Since $A-IFS$ are mainly used to solve $MCDM$ tasks, there are many papers in the literature devoted to the IF and interval-valued intuitionistic fuzzy ($IVIF$) extensions of the $TOPSIS$ method [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]. Of course, this list of papers can be considerably extended, but we do not intend to make here a comprehensive overview of them, because for us the most important thing that unites these publications is the use of Atanassov’s operational laws defined on intuitionistic fuzzy values IFVs and on interval-valued intuitionistic fuzzy values $IVIFV$s. Currently, the intuitionistic fuzzy extensions of the $TOPSIS$ method are based on the classical Atanassov’s arithmetical operations with IFVs (see [52, 53] and the comparison rule in [54]). The interval-valued intuitionistic fuzzy extensions of the $TOPSIS$ method are based on the direct interval extension of the classical operational laws defined on IFVs [36, 37, 55, 56].

Meanwhile, in our recent paper [57], it is shown that Atanassov’s operations with IFVs, including the aggregating operations and operation of comparison, possess six negative properties revealed to date in [57] that may lead (sometimes, but not always) to undesirable counter-intuitive results in the solution of $MCDM$ problems. In our opinion, this is not so surprising as some difficult to justify features of their definitions can be easily observed, e.g. usual multiplication is used in the definition of the sum operation and the power operation completely defines the multiplication by a scalar.

Having in mind this problem, in [57, 58], we demonstrated the existence of the strong correlation between the intuitionistic fuzzy sets theory and the Dempster-Shafer theory of evidence (DST). It was shown in [57] that operations with IFVs might be redefined in the framework of DST and substituted with the operations with belief intervals BI. Moreover, based on some relevant theorems it was proved in [57] that the introduced new set of operations with belief intervals representing IFVs is free of drawbacks of the classical Atanassov’s operations with IFVs. Besides, the semantic of DST allows us to introduce new operations that cannot be defined in the framework of canonical $A-IFS$ theory.

Looking ahead a bit, we note that in [59] we paid attention to some incompleteness of the classical definitions of interval-valued intuitionistic fuzzy sets $IVIFS$ and values $IVIFV$s introduced in [60, 61]. To solve this problem, we proposed new improved definitions of $IVIFS$ and $IVIFV$s in the framework of $A-IFS$ theory and DST and the associated set of new operations with interval extended belief intervals that represent $IVIFV$s. However, here we will limit ourselves only to considering the problems of IF extension of the $TOPSIS$ method.

For today we have found in the literature only one paper [62] devoted to our approach based on the treatment of IFVs in the framework of DST to the IF extension of the $TOPSIS$ method. The authors correctly transformed an initial intuitionistic-valued decision matrix into a matrix composed of BIs, but then they introduced the highly complicated and non-clear definitions for the distance between BIs as well as for ideal solutions, e.g. the positive ideal solutions were assumed to be BIs with maximal attainable belief values and minimal attainable plausibility values and so on. In our opinion, such definitions introduced in [62] without a substantive analysis of the problem cannot be reasonable treated in a natural way. So we can say that the paper [62] can be considered only as the first attempt to introduce the *IF* extension of the *TOPSIS* method.

When choosing an appropriate method to solve athe *MCDM* problem, the approaches, which allow us to use more information available or more correct operational laws are usually recognized to be better ones, especially when they are based on novel operations that cannot be introduced in the realm of known traditional methods.

In [28], we showed that the distances between rates of alternatives and ideal solutions may be considered (in a broad sense) as modified weighted sums of some local criteria. Weighted sums are of course a most popular, but not always the best approach to aggregate local criteria in many real-world tasks, as extremely low rates of some local criteria can be counterbalanced in the weighted sum with great rates of some rest local criteria that in some problem at hand may occur to be not so important for a decision maker. As a consequence, in some fields, e.g. in ecological modeling, the use of weighted sum aggregations is forbidden at all [63].

Therefore, in the current paper, we introduce in the framework of *IF* extended *TOPSIS* method, besides the weighted sum, some other approaches to the local criteria aggregation. When solving complex real-world problems, different aggregation modes may be applied and they usually generate different rankings of alternatives. So, the problem of aggregation of aggregation modes arises. Therefore, the method developed in [64] for the aggregation of used aggregating modes, based on the synthesis of Type 2 and Level 2 fuzzy sets will be used. It is worthy to note that we have successfully used this approach to generalize the fuzzy extension of the *TOPSIS* method [28]. The rest of the paper is set out as follows: In Section 2, we recall some definitions needed for our analysis. Section 3 presents our approach to the generalization of the *TOPSIS* method in the intuitionistic fuzzy setting in the framework of the evidence theory. A numerical example is presented in Section 4. Section 5 concludes the paper with some remarks.

2 Preliminaries

2.1 The basic definitions of the *TOPSIS* method

Let a *MCDM* problem comprises m alternatives $Al_1, Al_2,...,Al_m$ and n local criteria $LC_1, LC_2,...,LC_n$. All the rates assigned to alternatives according to local criteria are presented in the decision matrix $D[d_{ij}]_{m\times n}$, where d_{ij} is the rate of Al_i with respect to LC_j. Then let $CW = (cw_1, cw_2, ..., cw_n)$ be the vector of local criteria weights such that $\sum_{j=1}^{n} cw_j = 1$.

The conventional *TOPSIS* method involves five steps [1]:

1. The normalization of decision matrix

$$x_{ij} = \frac{d_{ij}}{\sqrt{\sum_{k=1}^{m} d_{kj}^2}},\ i = 1,...,m;\ j = 1,...,n. \quad (1)$$

   The weighting procedure:

$$r_{ij} = cw_j \times x_{ij},\ i = 1,...,m;\ j = 1,...,n. \quad (2)$$

2. Determination of the positive ideal and negative ideal solutions, respectively

$$\begin{aligned} S^+ &= \{r_1^+, r_2^+, ..., r_n^+\} = \\ &= \{(\max_i r_{ij} \,|\, j \in K_b)\ (\min_i r_{ij} \,|\, j \in K_c)\}, \end{aligned} \quad (3)$$

$$\begin{aligned} S^- &= \{r_1^-, r_2^-, ..., r_n^-\} = \\ &= \{(\min_i r_{ij} \,|\, j \in K_b)\ (\max_i r_{ij} \,|\, j \in K_c)\}, \end{aligned} \quad (4)$$

   where K_b and K_c are sets of benefit and cost criteria, respectively.

3. Calculating the distances of the existing alternatives from the positive ideal and negative ideal solutions: two Euclidean distances for each alternatives are, respectively, calculated as follows

$$\begin{aligned} Di_i^+ &= \sqrt{\sum_{j=1}^{n} (r_{ij} - r_j^+)^2},\ i = 1,...,m, \\ Di_i^- &= \sqrt{\sum_{j=1}^{n} (r_{ij} - r_j^-)^2},\ i = 1,...,m. \end{aligned} \quad (5)$$

4. Calculation of the relative closeness to the ideal alternatives

$$R_i = \frac{Di_i^-}{Di_i^+ + Di_i^-},\ i = 1,2,...,m,\ 0 \le R_i \le 1. \quad (6)$$

5. Ranking alternatives concerning to their relative closeness to the ideal alternatives: the greater is R_i, the more acceptable is the alternative Al_i.

2.2 Conventional operational laws defined on intuitionistic fuzzy values and their shortcomings

Atanassov [29] defined A-IFS as follows.

Definition 1. Let $Y = \{y_1, y_2, ..., y_n\}$ be a finite universal set. An intuitionistic fuzzy set B in Y is the following mathematical object: $B = \{< y_j, \mu_B(y_j), \nu_B(y_j) > | y_j \in Y\}$, where the functions $\mu_B : Y \to [0,1]$, $y_j \in Y \to \mu_B(y_j) \in [0,1]$ and $\nu_B : Y \to [0,1]$, $y_j \in Y \to \nu_B(y_j) \in [0,1]$ present the degree of membership and degree of non-membership of element $y_j \in Y$ to the set $B \subseteq Y$, respectively, and for any $y_j \in Y$, $0 \leq \mu_B(y_j) + \nu_B(y_j) \leq 1$. Atanassov [29], called $\pi_B(y_j) = 1 - \mu_B(y_j) - \nu_B(y_j)$ the intuitionistic index (or the hesitation degree) of the element y_j in the set B. Then for any $y_j \in Y$ we have $0 \leq \pi_B(y_j) \leq 1$. The operations of addition $\oplus$ and multiplication $\otimes$ on IFVs were introduced by Atanassov [65]. Let $X = \langle \mu_X, \nu_X \rangle$ and $Y = \langle \mu_Y, \nu_Y \rangle$ be IFVs. Then

$$X \oplus Y = \langle \mu_X + \mu_Y - \mu_X \mu_Y, \nu_X \nu_Y \rangle, \tag{7}$$

$$X \otimes Y = \langle \mu_X \mu_Y, \nu_X + \nu_Y - \nu_X \nu_Y \rangle. \tag{8}$$

Based on (7) and (8), in [66] for any integer n=1,2,.. the following operations were defined: $nX = X \oplus ... \oplus X = \langle 1 - (1 - \mu_X)^n, \nu_X^n \rangle$, $X^n = X \otimes ... \otimes X = \langle \mu_X^n, 1 - (1 - \nu_X)^n \rangle$.

It was shown later that the last operations are resulted in IFVs not only for integer n, but also for all real values $\lambda > 0$, i.e.

$$\lambda X = \left\langle 1 - (1 - \mu_X)^\lambda, \nu_X^\lambda \right\rangle, \tag{9}$$

$$X^\lambda = \left\langle \mu_X^\lambda, 1 - (1 - \nu_X)^\lambda \right\rangle. \tag{10}$$

The above operations possess conventional algebraic properties:

Theorem 1. [54]. Let $X = \langle \mu_X, \nu_X \rangle$ and $Y = \langle \mu_Y, \nu_Y \rangle$ be IFVs. Then

$$\begin{aligned} & X \oplus Y = Y \oplus X,\ X \otimes Y = Y \otimes X, \\ & \lambda(X \oplus Y) = \lambda X \oplus \lambda Y,\ (X \otimes Y)^\lambda = X^\lambda \otimes Y^\lambda, \\ & \lambda(X \oplus Y) = \lambda X \oplus \lambda Y, \\ & \lambda_1 X \oplus \lambda_2 X = (\lambda_1 + \lambda_2) X,\ \lambda_1, \lambda_2 > 0, \\ & X^{\lambda_1} \otimes X^{\lambda_2} = X^{\lambda_1 + \lambda_2},\ \lambda_1, \lambda_2 > 0. \end{aligned} \tag{11}$$

For the solution of $MCDM$ problems in the intuitionistic fuzzy setting based on the operations (7)-(10), we can aggregate local criteria as follows

For $IFA_1, ..., IFA_n$ being IFVs representing the rates of local criteria and $cw_1, ..., cw_n$, $\sum_{i=1}^{n} w_i = 1$, being their real-valued weights we obtain:

Intuitionistic Weighted Arithmetic Mean ($IWAM$)

$$IWAM = cw_1 IFA_1 \oplus cw_2 IFA_2 \oplus ... \oplus cw_n IFA_n = \left\langle 1 - \prod_{i=1}^{n} (1 - \mu_{IFA_i})^{cw_i}, \prod_{i=1}^{n} \nu_{IFA_i}^{cw_i} \right\rangle, \tag{12}$$

and the Intuitionistic Weighted Geometric Arithmetic operator ($IWWG$)

$$IFWG = IFA_1^{cw_1} \otimes IFA_2^{cw_2} ... \otimes IFA_n^{cw_n} = \left\langle \prod_{i=1}^{n} \mu_i^{cw_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{cw_i} \right\rangle. \tag{13}$$

It is easy to prove that the final rates of alternatives represented by (12),(13) are IFVs and therefore the problem of IFVs comparison arises. To solve it in [67] the so-called score function (or net membership) $S(x) = \mu(x) - \nu(x)$, where x is IFV was introduced. As a valuable complement to the score function, in [68] the so-called accuracy function $H(x) = \mu(x) + \nu(x)$ was proposed. It was noted in this paper that the link between functions S and H is similar to that between mean and variance in statistics. In [54], the functions S and H are applied to introduce the rule of comparison of IFVs x and y as follows

$$\begin{aligned} & S(x) = \mu_x - \nu_x,\ S(x) \in [-1, 1], \\ & H(x) = \mu_x + \nu_x, H(x) \in [0, 1]. \end{aligned} \tag{14}$$

$$\begin{aligned} & If\, (S(x) > S(y)), then\, y\, is\, smaller\, than\, x; \\ & If\, (S(x) = S(y)), then \\ & (1) If\, (H(x) = H(y)), then\, x = y; \\ & (2) If\, (H(x) < H(y))\, then\, x\, is\, smaller\, than\, y. \end{aligned} \tag{15}$$

Even at the first glance, the above operations on IFVs seem to be somewhat strange. For example, the presence of usual multiplication and especially subtraction operation in the definition of IFVs addition (7) looks curiously or at least as a brave heuristic. The usual addition and subtraction operations in the definition of IFVs multiplication (8) look no less strange as well as the multiplication of IFV by a scalar (9) completely defined by the usual power operation. Perhaps this is an incomplete list of strangeness concerned with the conventional operational laws defined on IFVs and more inconsistencies may be found with a deeper analysis. On the other hand, one can say that the noted above incomprehensibilities are not described in a strong mathematical form and do not matter since the operations with IFVs (7)-(10) provide IFVs as well and the important algebraic properties (11) are held. Therefore besides the above general, but not so strict qualitative consideration of the raised problem, in our paper [57], using strictly mathematical tools and argumentation we showed that the operations (7)-(10), (12) and (15) are burdened with a number of undesirable features which may provide unexpected non-acceptable results in the solution of *MCDM* problems and in collateral applications:

1. The operation of IFVs addition (7) is not an addition-invariant operation.

2. The operation (9) is not preserved under multiplication by a usual real value $\lambda > 0$.

3. The weighted sum aggregation operation (12) is not consistent with the aggregation operation on the ordinary fuzzy sets (Ordinary Weighted Arithmetic Mean *OWAM*).

4. The aggregating operation (12) is not monotone with respect to the ordering (15).
   The validity of the above statements supported by convincing numerical examples was proved in [57]. Later we have found some additional bad properties of the operations (8), (13) and (15).

5. The multiplication (8) is not always monotone with respect to the ordering (15). Consider the following example:
   Example 1. Let $x = \langle 0.1, 0.3\rangle$, $y = \langle 0.4, 0.5\rangle$, $z = \langle 0.2, 0.1\rangle$. Then $S(x)$=-0.2, $S(y)$=-0.1 and from (15) we get $y > x$.

   Meanwhile, $x \otimes z = \langle 0.02, 0.37\rangle$, $y \otimes z = \langle 0.08, 0.55\rangle$, $S(x \otimes z) = -0.35$, $S(y \otimes z) = -0.47$. So we have $S(x \otimes z) > S(y \otimes z)$ and $x \otimes z > y \otimes z$ opposite to $y > x$.

6. The aggregating operation (13) is not monotone with respect to the ordering (15). Consider the example:
   Let $x = \langle 0.4, 0.5\rangle$, $y = \langle 0.35, 0.448\rangle$, $z = \langle 0.5, 0.5\rangle$, cw_1=0.5, cw_2=0.5.
   Since $S(x)$=-0.1 and $S(y)$=-0.098 we have $y > x$.

   Meanwhile, since $IWGM(x,z)$=$\langle 0.4472, 0.5\rangle$, $IWGM(y,z)$=$\langle 0.41833, 0.474643\rangle$, $S(IWGM(x,z))$=-0.0528 and $S(IWGM(y,z))$=-0.0563 we obtain $IWGM(x,z) > IWGM(y,z)$ opposite to the $y > x$.

It is worthy to note that one of the important limitations of Atanassov's $A-IFS$ is that the aggregating operator $IWWG$ (13) can be defined only in the case of real-valued local criteria weights, although in applications such weights can be presented by IFVs. This limitation is easy to explain since the operation x^y for x and y being IFVs does not exist in the body of Atanassov's $A-IFS$.

Of course, we do not insist here that we have revealed all the existing negative properties of the conventional operations with IFVs. Nevertheless, what we have already found is enough for us to make the best of possible to solve the problem with operational laws defined on IFVs. To do this, so that to avoid the above-described shortcomings and limitations of classical $A-IFS$ following the paper [57], we will use here an approach based on the redefinition of Atanassov's IFS in the framework of DST.

2.3 Interpretation of A-IFS in the framework of DST

First we present a set of basic definitions of DST reduced to the necessary for our analysis extent. The DST was developed by Dempster [69, 70] and Shafer [71]. Its basics definitions may be presented as follows. Assume C is a subset of X. A subset C may also be treated as a question or proposition and X as a set of propositions or mutually exclusive hypotheses or answers [72]. A DST belief structure has the associated mapping m, called basic assignment function (or mass assignment function), from subsets of X into a unit interval, $m : 2^X \to [0,1]$ such that $m(\emptyset) = 0$, $\sum_{Z \subseteq X} m(Z) = 1$. The subsets of X with non-zero values are called focal elements.

The null set is never a focal element. The measures of belief and plausibility associated with DST

belief structure were introduced in [71] as follows.

The measure of belief is a mapping $Bel : 2^X \rightarrow [0,1]$ such that for any subset Y of X

$$Bel(Y) = \sum_{\emptyset \neq Z \subseteq Y} m(Z). \qquad (16)$$

The measure of plausibility associated with m is a mapping $Pl : 2^X \rightarrow [0,1]$ such that for any subset Y of X

$$Pl(Y) = \sum_{Z \cap Y \neq \emptyset} m(Z). \qquad (17)$$

It is seen that $Bel(Y) \leq Pl(Y)$.

An interval $[Bel(Y), Pl(Y)]$ is called the belief interval (*BI*). The belief interval is of dual nature: it can be treated as an usual regular interval containing a true probability or a true power of some statement (argument, proposition, hypothesis, *ets*) [72]. In [58], we showed that the triplet $\mu_Z(x)$, $\nu_Z(x)$, $\pi_Z(x)$ represents a correct basic assignment function in *DST* so that *IFV* $Z(x)=\langle \mu_Z(x), \nu_Z(x) \rangle$ may be redefined as follows
$Z(x) = BI_Z(x) = [Bel_Z(x), Pl_Z(x)]$
$= [\mu_Z(x), 1 - \nu_Z(x)]$
(see [57, 58] for more details and formal definitions). At first glance, this transformation seems to be a simple rewriting of *A-IFS* in terms of interval-valued fuzzy sets, but in [57, 58] we showed that the use of *DST* semantics allows us to boost the performance of *A-IFS* when using the operational laws defined on *IFV*s to solve *MCDM* tasks.

Also, it was shown in [57] that the operations with *IFV*s X and Y could be transformed to the corresponding operations on the belief intervals $BI(X)$ and $BI(Y)$.

Nevertheless, one methodologically meaningful characteristic of the proposed transformation was not noted in [57]: to represent all the basic concepts of *IFS* theory in the realm of *DST* only a small part of the mathematical tools of *DST* should be applied. Therefore we can say that the *IFS* theory may be qualified as an asymptotic approximation of the more general *DST*. This is not so unexpected, as it was proved that the possibility theory (and as a consequence, the fuzzy sets theory) and the probability theory are some asymptotic cases of *DST* which are valid in some predefined limiting conditions. Obviously, this does not mean that asymptotic theories are somewhat second rate. On the contrary, in specific applications they often allow the problem to be formulated more transparently and more simply than more general basic theories. On the other hand, when the limitations of the asymptotic theory don't allow us to solve a problem in the specific framework of this theory, we always have two approaches to solve the problem: using heuristics approaches or redefining the problem using a more general theory. It is easy to see that here we choose the second approach (the use of *DST*) which allows us to get rid of the restrictions and drawbacks of Atanassov's *IFS* theory.

In [57], the set of operations with *IFV*s defined as operations with belief interval was introduced.

These operations were developed in such a way that they provide as the results *BI*s, i.e. intervals which belong to [0,1]. In this context, the additional assumptions of a rather methodological nature were made to infer the operation of *BI*s addition as using the conventional rule of interval addition does not always provide a result in the form of *BI*. These assumptions were based on the dual nature of *BI* [72] as follows. Let Z be subsets of X. According to [72], in the context of *DST* a belief interval $BI(Z) = [Bel(Z), Pl(Z)]$ may be treated not only as a usual interval (in the sense of interval arithmetic), but also as an interval enclosing a true power of statement (argument, proposition, hypothesis, *ets*) that $x \in X$ belongs to the set $Z \subseteq X$. It is clear that the value of such a power belongs to the unit interval [0,1].

Consider the statement, e.g. "the weather is warm", with some degree of truth that belongs to the corresponding *BI*. If we repeat this statement once more or even make that many times, the truth value of that considered sentence does not change. It is easy to conclude that the above consideration makes sense only if the following property of the belief interval addition is held: $BI(Z) = BI(Z) + BI(Z) + ... + BI(Z)$. This is feasible if the operation of belief intervals addition $\oplus$ is defined as follows $BI(Z) \oplus BI(Z) = \left[\frac{Bel(Z)+Bel(Z)}{2}, \frac{Pl(Z)+Pl(Z)}{2}\right]$. So the sum of belief intervals is their average interval value. As a consequence, for n different statements S_i, i=1 to n represented by belief intervals $BI(A_i)$

their sum is defined as follows

$$BI(S_1)\oplus BI(S_2)\oplus ....\oplus BI(S_n) = \left[\frac{1}{n}\sum_{i=1}^{n} Bel(S_i), \frac{1}{n}\sum_{i=1}^{n} Pl(S_i)\right]. \quad (18)$$

The other operations with BIs were defined in [57] treating them as regular ones (in the sense of interval arithmetic) as follows

$$BI(Y)\otimes BI(Z) = [Bel(Y)Bel(Z), Pl(Y)Pl(Z)]. \quad (19)$$

$$\lambda BI(Y) = [\lambda Bel(Y), \lambda Pl(Y)], \quad (20)$$

where $\lambda \in [0,1]$ is a real value. This operation makes sense only for $\lambda < 1$ since in the opposite case it does not always produce a true belief interval. This limitation is not so important when we define operations with BIs to solve $MCDM$ problems, where the values of λ are less than 1 as they represent weights of local criteria.

$$BI(Y)^{\lambda} = [Bel(Y)^{\lambda}, Pl(Y)^{\lambda}], \quad (21)$$

where $\lambda \geq 0$.

$$BI(Y)^{BI(Z)} = [Bel(Y)^{Pl(Z)}, Pl(Y)^{Bel(Z)}]. \quad (22)$$

It is worth noting that this useful operation is absent in the framework of Atanassov's $A-IFS$. Using some relevant theorems it was proved in [57] that the introduced operational laws on BIs possess good algebraic properties (the same as (11)). From (18) and (20) we infer the Intuitionistic Weighted Arithmetic Mean operator:

$$IWAM_{DST} = \left[\frac{1}{n}\sum_{i=1}^{n} cw_i Bel_i, \frac{1}{n}\sum_{i=1}^{n} cw_i Pl_i\right]. \quad (23)$$

This is not an idempotent aggregating operator, but its multiplication by n gives us the idempotent operator

$$IWAM_{DST} = \left[\sum_{i=1}^{n} cw_i Bel_i, \sum_{i=1}^{n} cw_i Pl_i\right]. \quad (24)$$

It is seen that operators (23) and (24) provide the same orderings of competing alternatives. Based on expressions (19) and (21), the Intuitionistic Fuzzy Weighted Geometric operator $IFWG_{DST}$ is defined as follows

$$IFWG_{DST} = \left[\prod_{i=1}^{n} Bel_i^{cw_i}, \prod_{i=1}^{n} Pl_i^{cw_i}\right]. \quad (25)$$

From (19) and (22), the Intuitionistic Fuzzy Weighted Geometric operator $IFWGB_{DST}$ with weights presented by BIs is inferred as follows

$$IFWGB_{DST} = \left[\prod_{i=1}^{n} Bel_i^{Pl_i}, \prod_{i=1}^{n} Pl_i^{Bel_i}\right]. \quad (26)$$

We can see that the subtraction and division operations are not presented and not discussed in the framework of conventional Atanassov's IFS and $IVIFS$ theories. It may be so that the appropriate and useful heuristic mathematical expressions for such operations with acceptable properties were not found for simple reasons because they are not really needed. There are no these operations in the framework of our approach as well since the direct BI extensions of the real-valued interval subtraction and division operations do not always produce BIs as the results and the linguistic treatment of BIs used above to introduce the addition operation with BIs does not help to infer the operations of BIs subtraction and division with acceptable properties. So on today's stage of studies, we have to admit that the absence of subtraction and division operations is an inherent limitation of IFS theory and its generalization in the realm of DST. However it may be a relief to recognize that in general the IFS theory with its extensions is used to the solution of $MCDM$ problems and related accompanying problems when operations of IFVs subtraction and division are not needed at all.

The operators of BIs comparison were defined as follows

$$\begin{array}{l} if\ (Bel(A)+Pl(A)) > (Bel(B)+Pl(B)) \\ then\ BI(B) < BI(A), \\ if\ (Bel(A)+Pl(A)) = (Bel(B)+Pl(B)) \\ then\ BI(B) = BI(A). \end{array} \quad (27)$$

The introduction of such a simple operation of BIs comparison as (27), which actually is the simple comparison of BIs midpoints, is not so obvious and needs transparent and persuasive justification. In our paper [28], we provided the comparative analysis of known approaches to the real-valued interval comparison and showed that without loss of accuracy it can be simplified to the comparison of intervals midpoints. Generally this is not so surprising conclusion because even in the relatively early paper [73], it was emphasized that most of the known methods developed to compare intervals are "totally based on the midpoints of interval numbers".

Therefore here we present only a brief review of the results obtained in the studies presented in [28].

Since *BI*s may be treated as regular interval ones, the problem of *BI*s comparison is reduced to the problem of interval comparison. In the literature, we can find many different approaches to the interval comparison (see comprehensive reviews in [74] and [75]). Currently, the most popular methods for interval comparison are heuristic ones proposed in [76, 77, 78, 79]. These methods allow us to estimate the possibility for an interval to be greater/lesser than another one. It was proved in [80] that they are equivalent. The probability-based methods were analyzed in [81], where a fuzzy two-criteria method for the interval comparison was developed as well. Perhaps the most complex approach to interval comparison is the method proposed in [74], based on the seemingly somewhat unexpected but logically sound suggestion that since arithmetic operations with intervals give us intervals, the results of their comparison should also be intervals (more strictly *BI*s).

Therefore, taking into account the visible lack of consensus in the field of interval comparison,in [28] we presented the results of comparative analysis of the most sound methods for the interval comparison based on a wide set of persuasive numerical examples. The results seem to be somewhat unexpected since it was shown that the simplest method of intervals midpoints comparison produces more reasonable results than the more complicated approaches considered. It is also very important that the difference between midpoints of compared intervals occurred to be quantitatively almost the same as the known Hamming and Euclidean distances between intervals and therefore can serve as a measure of intervals inequality even when there is no their common area. It is easy to see that when comparing the midpoints of intervals, we must treat intervals with common midpoints as equal, although their widths may be different. To avoid this seemingly obvious flaw, a two-criteria approach to the interval comparison was developed in [81], but later we realized that actually there is no flaw since we are dealing with an inherent feature of interval analysis. A regular interval may be naturally interpreted as a support of some uniform probability distribution. Let X and Y be such distributions with common mean and different variances. Since using statistical methods it is impossible to prove that $X > Y$ or $X < Y$, the only logically justified option is that $X = Y$, i.e. the compared intervals are equal ones. The advantages of the approach to the interval comparison based on their midpoints comparison were presented in [28], where it is also shown that this simplest approach is more in line with common sense than other analyzed methods. It is important also that this method is not of pure heuristic nature, as it may be inferred based directly on the analysis of the interval subtraction operation.

Nevertheless, we recognize that generally the interval comparison is a context-dependent problem. So in the concrete real-world situations the use of other more complicated approaches, e.g. such that proposed in [81] or [74] may be justified.

However, in the current paper we will use the simplified but justified rule of comparison (27).

Using corresponding theorems, it was proved in [57] that introduced set of operations (18)-(27) with belief intervals representing IFVs is free of drawbacks and limitations of conventional $A-IFS$ described in Subsection 2.2.

3 The generalization of the $TOPSIS$ method in the intuitionistic fuzzy setting in the framework of evidence theory

3.1 A new interpretation of classical $TOPSIS$ method

A new interpretation of the $TOPSIS$ method have been proposed in our paper [28] in context of the fuzzy $TOPSIS$ method. Here we present it on the base of the classical definition of the $TOPSIS$ method (1)-(6).

Let r_{ij} be normalized, but not yet a weighted component of the decision matrix. Then using procedures (3) and (4) we obtain the positive r_j^+ and negative r_j^- ideal solutions attainable in the decision matrix.

Then it is easy to see that from expressions (3) and (4) we have:

$$\begin{aligned} r_j^+ &\geq r_{ij},\ i=1,2,...,m,\ j\in K_b,\\ r_{ij} &\geq r_j^+,\ i=1,2,...,m,\ j\in K_c,\\ r_{ij} &\geq r_j^-,\ i=1,2,...,m,\ j\in K_b,\\ r_j^- &\geq r_{ij},\ i=1,2,...,m,\ j\in K_c. \end{aligned} \quad (28)$$

Therefore, there is no need to use n-dimensional Euclidean or Hamming distances to obtain S_i^+ and S_i^-, $i=1,2,...,m$, as they can be calculated as follows

$$\begin{aligned} S_i^+ &= \sum_{j\in K_b} w_j(r_j^+ - r_{ij}) + \sum_{j\in K_c} w_j(r_{ij} - r_j^+),\\ S_i^- &= \sum_{j\in K_b} w_j(r_{ij} - r_j^-) + \sum_{j\in K_c} w_j(r_j^- - r_{ij}),\\ i &= 1,...,m. \end{aligned} \quad (29)$$

Then we can conclude that the non-negative differences

$$\begin{aligned} &r_j^+ - r_{ij},\ i=1,2,...,m,\ j\in K_b,\\ &r_{ij} - r_j^+,\ i=1,2,...,m,\ j\in K_c,\\ &r_{ij} - r_j^-,\ i=1,2,...,m,\ j\in K_b,\\ &r_j^- - r_{ij},\ i=1,2,...,m,\ j\in K_c. \end{aligned} \quad (30)$$

can be naturally treated as some modified values of local criteria based on the initial ones. So expressions (29) may be considered as the modified weighted sum aggregations of modified local criteria. This introduced a small modification of the canonical $TOPSIS$ method, i.e., the determination of ideal solutions before the weighting procedure in (29) does not violate the general idea of the $TOPSIS$ method, but as it will be shown below, is very fruitable and allows us to boost considerably the effectiveness of the method. It is important that aggregation (29) cannot be treated as unique nor as the best one within the framework of the $TOPSIS$ method in all cases.

An important property (which can be positive of negative depending on the problem at hand) of weighted sum aggregation is that the unacceptable small values of some local criteria may be compensated by great values of some rest ones in the final assessment (It was noted in Introduction that this aggregation mode is not used at all in some fields). Since this property of weighted sum often is undesirable, a decision-maker may prefer to use, e.g. the weighted geometric aggregation and a more cautious decision-maker may prefer the aggregation based on the "principle of maximal pessimism" proposed by Yager [82]. Generally, the choice of aggregation method is a context-dependent problem [83].

Let $\mu_1(LC_1),...,\mu_n(LC_n)$ be the values of local criteria $LC_1,...,LC_n$ and $cw_1,...,cw_n$ be their weights.

There are many aggregation modes proposed in the literature, but here we will use only the most popular ones, which are also often used as basic ones to build more complex aggregating operations

$$Ag_1 = \sum_{j=1}^{n} cw_j \mu_j(LC_j), \quad (31)$$

$$Ag_2 = min(\mu_1(LC_1)^{cw_1}, \mu_2(LC_2)^{cw_2}, ..., \mu_n(LC_n)^{cw_n}), \quad (32)$$

$$Ag_3 = \prod_{j=1}^{n} \mu_j(LC_j)^{cw_j}. \quad (33)$$

3.2 Implementation of aggregation modes Ag_1, Ag_2 and Ag_3 in intuitionistic fuzzy $TOPSIS$ method in the framework of evidence theory

Let us consider a $MCDM$ problem which is based on m alternatives Al_1, Al_2,...,Al_m and n local criteria LC_1, LC_2,...,LC_n. Each alternative is evaluated with respect to the n criteria. Let all ratings assigned to alternatives and are intuitionistic fuzzy values. Then the decision matrix may be presented as follows $D\left[\langle\mu_{ij},\nu_{ij}\rangle\right]_{m\times n}$, $i=1,2,...m$, $j=1,2,...,n$, where $\langle\mu_{ij},\nu_{ij}\rangle$ is the IF-valued rating of alternative Al_i with respect to the local criterion LC_j.

Let $W=\{\langle\mu_{cwj},\nu_{cwj}\rangle\}$, $j=1,2,...,n$, be the IF-valued vector of local criteria weights. Let us replace intuitionistic fuzzy values in the decision matrix and in the the vector of weights by corresponding belief intervals as follows
$BI_{ij}=[Bel_{ij},Pl_{ij}]$, $Bel_{ij}=\mu_{ij}$, $Pl_{ij}=1-\nu_{ij}$, $i=1,2,...m$, $j=1,2,...,n$,
$BI_{cwj}=[Bel_{cwj},Pl_{cwj}]$, $Bel_{cwj}=\mu_{cwj}$, $Pl_{cwj}=1-\nu_{cwj}$, $j=1,2,...,n$.
According to (18) a sum of BIs is always a BI belonging to the unit interval [0,1] as well. Therefore an additional normalization of the decision matrix is not needed. Then using the rule of belief interval comparison (27), the positive ideal and negative

ideal solutions can be obtained from the expressions

$$\begin{array}{l} BI^{+}=\left\{\left[Bel_{j}^{+},Pl_{j}^{+}\right]\right\}=\{\max_{i}\left\{[Bel_{ij},Pl_{ij}]\right\}|j\in K_{b}\ ,\\ \min_{i}\left\{[Bel_{ij},Pl_{ij}]\right\}|j\in K_{c}\ \},\\ BI^{-}=\left\{\left[Bel_{j}^{-},Pl_{j}^{-}\right]\right\}=\{\min_{i}\left\{[Bel_{ij},Pl_{ij}]\right\}|j\in K_{b}\ ,\\ \max_{i}\left\{[Bel_{ij},Pl_{ij}]\right\}|j\in K_{c}\ \}, \end{array} \tag{34}$$

where K_b is a set of benefit criteria and K_c is a set of cost criteria.

Of course, one can say that the natural restrictions $[0,0] \leqslant [Bel_{ij}, Pl_{ij}] \leqslant [1,1]$, $i = 1,2,...m$, $j = 1,2,...,n$, provide automatically the positive and negative ideal solutions as the intervals $[1,1]$ and $[0,0]$, respectively. In our opinion, such an approach seems to be not only significantly simpler than that based on the expressions (34), but may be more justified in some situations. However, here we will try to adhere to the canonical approach as much as possible. Let us consider the weighted sum type of aggregation Ag_1. Then the expressions for the calculation of the distances S_i^+ and S_i^-, $i = 1,2,...,m$, may be obtained as the belief interval extension of expressions (29) as follows

$$\begin{array}{l} S_{Ag_{1i}}^{+}=\sum\limits_{j\in K_b}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(\left[Bel_{j}^{+},Pl_{j}^{+}\right]-[Bel_{ij},Pl_{ij}]\right)\right)+\\ \sum\limits_{j\in K_c}\left([Bel_{cwj},Pl_{cwj}]\otimes\left([Bel_{ij},Pl_{ij}]-\left[Bel_{j}^{-},Pl_{j}^{-}\right]\right)\right),\\ S_{Ag_{1i}}^{-}=\sum\limits_{j\in K_b}\left([Bel_{cwj},Pl_{cwj}]\otimes\left([Bel_{ij},Pl_{ij}]-\left[Bel_{j}^{-},Pl_{j}^{-}\right]\right)\right)+\\ \sum\limits_{j\in K_c}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(\left[Bel_{j}^{-},Pl_{j}^{-}\right]-[Bel_{ij},Pl_{ij}]\right)\right),\ i=1,...,m. \end{array} \tag{35}$$

We can see that to use $S_{Ag_{1i}}^{+}$ and $S_{Ag_{1i}}^{-}$ presented in (35) for the subsequent analysis, the distance between belief intervals should be defined.

Since belief intervals are regular ones here, following to our papers [28, 84], we will use directly the operation of interval subtraction [85] to define the distance between intervals. This approach allows us to calculate also the possibility that an interval is greater/lesser that another one.

For intervals $X = [x^L, x^U]$ and $Y = [y^L, y^U]$, their subtraction results in the interval Z=$X-Y$=$[z^L, z^U]$; $z^L = x^L - y^U$, $z^U = x^U - y^L$. It is seen that for overlapping intervals X and Y, we always get a negative left bound of interval Z and a positive right bound.

Then, to obtain a measure of distance between intervals which additionally indicates which interval is greater/lesser, we will apply the expression

$$\Delta(X,Y) = \frac{1}{2}\left((x^L - y^U) + (x^U - y^L)\right). \tag{36}$$

It is seen that for intervals X and Y with a common midpoint, $\Delta(X,Y)$ is always equal to 0. The expression (36) may be presented in the form:

$$\Delta(X,Y) = \left(\frac{1}{2}(x^L + y^U) - \frac{1}{2}(x^U + y^L)\right). \tag{37}$$

We can see that the expression (37) is the distance between midpoints of compared intervals X and Y. It is shown in [28, 84] that the presented method may also be successfully used for the interval comparison and that the values of distances between intervals are nearly the same as Hamming and Euclidean distances when intervals have a common area and when they do not have any intersection.

Then based on the above results we can introduce the distances between belief intervals in (35). The midpoints of corresponding belief intervals may be calculated as follows

$$\begin{array}{l} p_j^{+}=\frac{Bel_j^{+}+Pl_j^{+}}{2},\ p_j^{-}=\frac{Bel_j^{-}+Pl_j^{-}}{2},\\ l_{ij}=\frac{Bel_{ij}+Pl_{ij}}{2}, \end{array} \tag{38}$$

$i = 1,2,...,m$, $j = 1,2,...,n$.

Then the distances from ideal solutions may be presented as follows

$$\begin{array}{l} p_j^{+}-l_{ij},\ i=1,2,...,m,\ j\in K_b,\\ l_{ij}-p_j^{+},\ i=1,2,...,m,\ j\in K_c,\\ l_{ij}-p_j^{-},\ i=1,2,...,m,\ j\in K_b,\\ p_j^{-}-l_{ij},\ i=1,2,...,m,\ j\in K_c. \end{array} \tag{39}$$

Therefore, the expressions (35) can be transformed as follows

$$\begin{array}{l} S_{Ag_{1i}}^{+}=\sum\limits_{j\in K_b}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(p_j^{+}-l_{ij}\right)\right)+\\ \sum\limits_{j\in K_c}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(l_{ij}-p_j^{+}\right)\right),\\ S_{Ag_{1i}}^{-}=\sum\limits_{j\in K_b}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(l_{ij}-p_j^{-}\right)\right)+\\ \sum\limits_{j\in K_c}\left([Bel_{cwj},Pl_{cwj}]\otimes\left(p_j^{-}-l_{ij}\right)\right), \end{array} \tag{40}$$

$i = 1,2,...,m$.

Then using the multiplication by scalar operation (20), we get from (40) the real-valued representations of S_i^+ and S_i^- as follows

$$\overline{S}^+_{Ag_{1i}} = \sum_{j\in K_b}\left(Bel_{cwj}\left(p_j^+ - l_{ij}\right) + Pl_{cwj}\left(p_j^+ - l_{ij}\right)\right) + \sum_{j\in K_c}\left(Bel_{cwj}\left(l_{ij} - p_j^+\right) + Pl_{wj}\left(l_{ij} - p_j^+\right)\right),$$
$$\overline{S}^-_{Ag_{1i}} = \sum_{j\in K_b}\left(Bel_{cwj}\left(l_{ij} - p_j^-\right) + Pl_{cwj}\left(l_{ij} - p_j^-\right)\right) + \sum_{j\in K_c}\left(Bel_{cwj}\left(p_j^- - l_{ij}\right) + Pl_{cwj}\left(p_j^- - l_{ij}\right)\right). \quad (41)$$
$i = 1,2,...,m,\ j = 1,2,...,n.$

To simplify the analysis and make it possible to compare the results obtained with those provided by other aggregation modes, the special normalization was applied

$$T^+_{Ag_{1i}} = \overline{S}^+_{Ag_{1i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{1i}}\right\}, \max\left\{\overline{S}^-_{Ag_{1i}}\right\}\right\},$$
$$T^-_{Ag_{1i}} = \overline{S}^-_{Ag_{1i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{1i}}\right\}, \max\left\{\overline{S}^-_{Ag_{1i}}\right\}\right\}, \quad (42)$$
$i = 1,2,...,m.$

The final ratings of alternatives ware calculated as follows

$$RD_{Ag_{1i}} = \frac{T^-_{Ag_{1i}}}{T^-_{Ag_{1i}} + T^+_{Ag_{1i}}},\ i = 1,2,...,m. \quad (43)$$

Let us consider the aggregation operator Ag_2 (32) introduced by Yager [82]. This aggregation approach is based on the assumption that the greater the minimal value of a local criterion is, the more valuable is the analyzing alternative. Using this method, first we obtain the aggregated separation of each alternative from the positive ideal solutions ($\overline{S}^+_{Ag_{2i}}, i = 1,...,m$) and from the negative ideal solutions ($\overline{S}^-_{Ag_{2i}}, i = 1,...,m$).

Nevertheless, in the considered case, we cannot use expression (32) directly as the existence of benefit and cost criteria should be taken into account. Hence we should adapt the aggregation Ag_2, also called “principle of maximal pessimism”, for the peculiarity of considering aggregation problem.

It is seen that with the increasing the distances $p_j^+ - l_{ij}$, $l_{ij} - p_j^+$ (see (39)), the value of the aggregated separation of alternatives from the positive ideal solution decreases. On the other hand, with rising the distances $l_{ij} - p_j^-$, and $p_j^- - l_{ij}$, the value of the aggregated separation of each alternative from the negative ideal solution is increasing.

Then based on the operator (26), which cannot be defined in the framework of Atanassov’s $A-IFS$, and having in mind that we are dealing with cost and profit criteria and omitting cumbersome intermediate conversions (like those used to present $\overline{S}^+_{Ag_{1i}}$ and $\overline{S}^-_{Ag_{1i}}$) we get

$$\overline{S}^+_{Ag_{2i}} = \max\left\{\max_{j\in K_b}\left\{\left(p_j^+ - l_{ij}\right)^{Bel_{cwj}} + \left(p_j^+ - l_{ij}\right)^{Pl_{cwj}}\right\}, \max_{j\in K_c}\left\{\left(l_{ij} - p_j^+\right)^{Bel_{cwj}} + \left(l_{ij} - p_j^+\right)^{Pl_{cwj}}\right\}\right\},$$
$$\overline{S}^-_{Ag_{2i}} = \min\left\{\min_{j\in K_b}\left\{\left(l_{ij} - p_j^-\right)^{Bel_{cwj}} + \left(l_{ij} - p_j^-\right)^{Pl_{cwj}}\right\}, \min_{j\in K_c}\left\{\left(p_j^- - l_{ij}\right)^{Bel_{cwj}} + \left(p_j^- - l_{ij}\right)^{Pl_{cwj}}\right\}\right\}, \quad (44)$$
$i = 1,2,...,m,\ j = 1,2,...,n.$

$$T^+_{Ag_{2i}} = \overline{S}^+_{Ag_{2i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{2i}}\right\}, \max\left\{\overline{S}^-_{Ag_{2i}}\right\}\right\},$$
$$T^-_{Ag_{2i}} = \overline{S}^-_{Ag_{2i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{2i}}\right\}, \max\left\{\overline{S}^-_{Ag_{2i}}\right\}\right\}, \quad (45)$$
$i = 1,2,...,m.$

$$RD_{Ag_{2i}} = \frac{T^-_{Ag_{2i}}}{T^-_{Ag_{2i}} + T^+_{Ag_{2i}}},\ i = 1,2,...,m. \quad (46)$$

Based on the nearly same reasoning, we infer the aggregated separation of each alternative, from the positive ideal solutions ($\overline{S}^+_{Ag_{3i}}, i = 1,...,m$) and from the negative ideal solutions ($\overline{S}^-_{Ag_{3i}}, i = 1,...,m$), for the aggregating mode Ag_3 (33) we get

$$\overline{S}^+_{Ag_{3i}} = \prod_{j\in K_b}\left(\left(p_j^+ - l_{ij}\right)^{Bel_{cwj}} + \left(p_j^+ - l_{ij}\right)^{Pl_{cwj}}\right) \times \prod_{j\in K_c}\left(\left(l_{ij} - p_j^+\right)^{Bel_{cwj}} + \left(l_{ij} - p_j^+\right)^{Pl_{cwj}}\right),$$
$$\overline{S}^-_{Ag_{3i}} = \prod_{j\in K_b}\left(\left(l_{ij} - p_j^-\right)^{Bel_{cwj}} + \left(l_{ij} - p_j^-\right)^{Pl_{cwj}}\right) \times \prod_{j\in K_c}\left(\left(p_j^- - l_{ij}\right)^{Bel_{cwj}} + \left(p_j^- - l_{ij}\right)^{Pl_{cwj}}\right), \quad (47)$$
$i = 1,2,...,m,\ j = 1,2,...,n.$

$$T^+_{Ag_{3i}} = \overline{S}^+_{Ag_{3i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{3i}}\right\}, \max\left\{\overline{S}^-_{Ag_{3i}}\right\}\right\},$$
$$T^-_{Ag_{3i}} = \overline{S}^-_{Ag_{3i}} / \max\left\{\max\left\{\overline{S}^+_{Ag_{3i}}\right\}, \max\left\{\overline{S}^-_{Ag_{3i}}\right\}\right\}, \quad (48)$$
$i = 1,2,...,m.$

$$RD_{Ag_{3i}} = \frac{T^-_{Ag_{3i}}}{T^-_{Ag_{3i}} + T^+_{Ag_{3i}}},\ i = 1,2,...,m. \quad (49)$$

No doubt, to solve complex real-world multiple criteria problems, all appropriate approaches to the aggregation of local criteria might be applied. It is clear the in practice different aggregating methods may result in competing ranks of alternatives. Because experts may have different preferences concerned with considering aggregating methods, the choice of compromise solution becomes an actual problem. Therefore, here we will use an approach that was developed to solve the multiple criteria problems with the use of method for aggregation of aggregating modes.

3.3 A method for aggregation of aggregating modes

There are not so many approaches to generalize aggregating operators proposed in the literature. Such approaches usually comprise some assumption of heuristic nature and are burdened by different shortcomings and limitations considered in our paper [28].

In the current paper, an approach to aggregation of aggregating modes developed in [64] and [86] will be applied. This approach is based on the synthesis of type-2 and level-2 fuzzy sets with the support being the set of compared alternatives. It was shown in [28] that this approach is generally free of the shortcomings and limitations of other known methods. An important merit of this approach is that for its implementation there is no need to use the *min*, *sum* and *multiplication* operations as they produce an infinite sequence of nested into each other aggregation problems. This approach was developed and presented in [64, 86, 87, 88, 89] in detail. Therefore, in the current paper, it will be presented briefly with the adaptation to the formulation of the considered problem. Let al_j, $j=1,\ldots,m$ be competing alternatives and Ag_i, $i=1,\ldots,n$ be acceptable aggregation modes. In applications, it is usually not so difficult to find relative importance (weights, reliability) of aggregating modes, e.g. in the form of verbal statements. Hence we can introduce the membership function $\mu(Ag_i)$, $i=1,\ldots,n$, representing the closeness of Ag_i to the best (ideal) approach to the aggregation. Then such an "ideal" approach Ag_{ideal} can be represented using its membership function defined on the set of considered aggregating modes:

$$Ag_{ideal}=\left\{\frac{\mu(Ag_i)}{Ag_i}\right\}, i=1,\ldots,n. \qquad (50)$$

Then the evaluations $Ag_i(al_j)$ of alternatives al_j, $j=1,\ldots,m$, can be obtained using Ag_i, $i=1,\ldots,n$. Hence the aggregating modes Ag_i can be formally predefined on the set of competing alternatives al_j. So Ag_i can be presented in the form of the fuzzy subset

$$Ag_i=\left\{\frac{Ag_i(al_j)}{al_j}\right\}, j=1,\ldots,m, \qquad (51)$$

where $Ag_i(al_j)$ may be considered as an extent (degree) to which an alternative al_j pertains to the set of "good" alternatives evaluated with the use of aggregation Ag_i. Substituting (51) into (50) with the use of level-2 fuzzy set definition introduced by Zadeh [90] we infer

$$Ag_{ideal}=\left\{\frac{\mu_{ideal}(al_j)}{al_j}\right\}, j=1,\ldots,m, \qquad (52)$$

where

$$\mu_{ideal}(al_j)=\max_i\left\{\mu(Ag_i)\cdot Ag_i(al_j)\right\}. \qquad (53)$$

With the use of the above definitions, the "ideal" alternative can be obtained as follows

$$al_{ideal}=arg\max_j \mu_{ideal}(al_j). \qquad (54)$$

Based on the numerical example, in the next Section we will demonstrate the ability of the developed method to produce a compromise in the above sense final ranking of alternatives.

4 Numerical examples

To exemplify the impact of the choice among aggregating modes Ag_1, Ag_2 and Ag_3 on the results provided by the developed extension of the *TOPSIS* method the example considered in [32] and bit modified for our purposes will be used. Suppose a company needs to select the most suitable supplier for one of more profitable branches of its business. After previous analysis five suppliers (alternatives al_1, al_2, al_3, al_4, al_5) were chosen for further consideration. Four local criteria were selected: LC_1 (Product quality), LC_2 (Relationship closeness), LC_3 (Delivery performance) and LC_4 (Price). Then assume that the components of the decision matrix and the local criteria are defined by experts as IFVs. The decision matrix used

$D\left[\langle\mu_{ij},\nu_{ij}\rangle\right]_{m\times n}$, $i=1,2,...m$, $j=1,2,...,n$, where $\langle\mu_{ij},\nu_{ij}\rangle$ is the IFV presenting the rate of alternative al_i in relation to the criterionLC_j is given in Table 1.

The intuitionistic fuzzy valued weights of local criteria are as follows $W=(<0.861,0.128>,<0.750,0.200>,<0.680,0.267>,<0.567,0.371>)$.

Then according to our approach, we replace IFVs in the decision matrix and in the definition of local criteria weights with corresponding belief intervals as follows

$BI_{ij}=[Bel_{ij},Pl_{ij}]$, $Bel_{ij}=\mu_{ij}$, $Pl_{ij}=1-\nu_{ij}$, $i=1,2,...m$, $j=1,2,...,n$,
$BI_{cwj}=[Bel_{cwj},Pl_{cwj}]$, $Bel_{cwj}=\mu_{cwj}$, $Pl_{cwj}=1-\nu_{cwj}$, $j=1,2,...,n$.
So the BI-valued local criteria weights take a form:

$$W=([0.861,0.872],[0.750,0.800],[0.680,0.733],[0.567,0.629]) \quad (55)$$

The obtained BI-valued decision matrix is presented in Table 2.

Then using the rule of belief interval comparison (27) and expressions (34) from the decision matrix presented in Table 2, we obtain the positive ideal and negative ideal solutions:

$$BI^{+}=([0.849,0.900],[0.780,0.882],[0.780,0.882],[0.526,0.626]),$$
$$BI^{-}=([0.562,0.663],[0.462,0.562],[0.644,0.744],[0.759,0.872]). \quad (56)$$

Let us consider the weighted sum type of aggregation Ag_1. Then using (41)-(43) from (55),(56) and Table 2 we get the final ratings of alternatives $RD_{Ag_{1i}}$ presented in Table 4.

Table 3. The final ratings of alternatives obtained using the aggregating mode Ag_1

$RD_{Ag_{11}}$	$RD_{Ag_{12}}$	$RD_{Ag_{13}}$	$RD_{Ag_{14}}$	$RD_{Ag_{15}}$
0.5284	0.3337	0.6965	0.3798	0.2139

From Table 3 we get the following ranking of alternatives:
$al_5<al_2<al_4<al_1<al_3$. Consider the aggregating mode Ag_2. Then using (44)-(46) from (55),(56) and Table 2 we obtain the final ratings of alternatives $RD_{Ag_{2i}}$ presented in Table 4.

Table 4. The final ratings of alternatives obtained using the aggregating mode Ag_2

$RD_{Ag_{21}}$	$RD_{Ag_{22}}$	$RD_{Ag_{23}}$	$RD_{Ag_{24}}$	$RD_{Ag_{25}}$
0.4832	0.1632	0.4093	0.3181	0.1644

From Table 4 we get the following ranking of alternatives:
$al_2<al_5<al_4<al_3<al_1$. Consider the aggregating mode Ag_3. Then using (47)-(49) from (55),(56) and Table 2 we obtain the final ratings of alternatives $RD_{Ag_{3i}}$ presented in Table 4.

Table 5. The final ratings of alternatives obtained using the aggregating mode Ag_3

$RD_{Ag_{31}}$	$RD_{Ag_{32}}$	$RD_{Ag_{33}}$	$RD_{Ag_{34}}$	$RD_{Ag_{35}}$
0.3180	0.8417	0.6419	0.6859	0.6426

From Table 4 we get the following ranking of alternatives:
$al_1<al_3<al_5<al_4<al_2$.

We can see that the final rankings of alternatives obtained using different aggregation modes are considerable different (according with our experience, this is not always the case). Therefore, since decision makers hesitate in choosing the best (ideal) method for aggregation, a compromise solution is needed.

To obtain such a solution we will use the method for aggregation of aggregating modes presented above in the Subsection 3.3.

In the considered example, the "ideal" aggregating mode may be presented as the following fuzzy subset:

$$Ag_{ideal}=\left\{\frac{\mu(Ag_1)}{Ag_1},\frac{\mu(Ag_2)}{Ag_2},\frac{\mu(Ag_3)}{Ag_3}\right\}, \quad (57)$$

where $\mu(Ag_1),\mu(Ag_2),\mu(Ag_3)$ are the weights or relative reliability of aggregating modes Ag_1,Ag_2 and Ag_3 ,respectively. On the other hand, each Ag_i can be formally defined on the set of competing alternatives al_j, $j=1,2,...,5$. Therefore, each Ag_i can be represented by the fuzzy subset

$$Ag_i=\left\{\frac{Ag_i(al_1)}{al_1},\frac{Ag_i(al_2)}{al_2},\frac{Ag_i(al_3)}{al_3},\frac{Ag_i(al_4)}{al_4},\frac{Ag_i(al_5)}{al_5}\right\}, \quad (58)$$

where $Ag_i(al_j)=RD_{Ag_{ij}}$, $i=1,2,3$, $j=1,2,...,5$. The values of $RD_{Ag_{ij}}$ are presented in Tables 3-5.

Table 1. IF-valued decision matrix

	LC_1	LC_2	LC_3	LC_4
al_1	$<0.728, 0.170>$	$<0.626, 0.272>$	$<0.780, 0.118>$	$<0.700.0.200>$
al_2	$<0.596, 0.302>$	$<0.605, 0.292>$	$<0.644, 0.256>$	$<0.578, 0.321>$
al_3	$<0.849, 0.100>$	$<0.780, 0.118>$	$<0.769, 0.170>$	$<0.769, 0.128>$
al_4	$<0.663, 0.236>$	$<0.538, 0.361>$	$<0.746, 0.151>$	$<0.644, 0.254>$
al_5	$<0.562, 0.337>$	$<0.462, 0.438>$	$<0.668, 0.231>$	$<0.526, 0.374>$

Table 2. Belief interval-valued decision matrix

	LC_1	LC_2	LC_3	LC_4
al_1	$[0.728, 0.830]$	$[0.626, 0.728]$	$[0.780, 0.882]$	$[0.700.0.800]$
al_2	$[0.596, 0.698]$	$[0.605, 0.708]$	$[0.644, 0.774]$	$[0.578, 0.679]$
al_3	$[0.849, 0.900]$	$[0.780, 0.882]$	$[0.769, 0.830]$	$[0.769, 0.872]$
al_4	$[0.663, 0.764]$	$[0.538, 0.639]$	$[0.746, 0.849]$	$[0.644, 0.746]$
al_5	$[0.562, 0.663]$	$[0.462, 0.562]$	$[0.668, 0.769]$	$[0.526, 0.626]$

Then according to the method presented in Subsection 3.3, we get

$$Ag_{ideal} = \left\{ \frac{\mu_{ideal}(al_1)}{al_1}, \frac{\mu_{ideal}(al_2)}{al_2}, \frac{\mu_{ideal}(al_3)}{al_3}, \frac{\mu_{ideal}(al_4)}{al_4}, \frac{\mu_{ideal}(al_5)}{al_5} \right\}, \tag{59}$$

where $\mu_{ideal}(al_j) = \max_i \left\{ \mu(Ag_i) \cdot RD_{Ag_{ij}} \right\}, i = 1,2,3,\ j = 1,2,...,5$. For $\mu(Ag_1) = 0.05$, $\mu(Ag_2) = 0.65$, $\mu(Ag_3) = 0.3$, we have obtained
$\mu_{ideal}(al_1)$=0.314,
$\mu_{ideal}(al_2)$=0.2525,
$\mu_{ideal}(al_3)$=0.1291, $\mu_{ideal}(al_4)$=0.2068,
$\mu_{ideal}(al_5)$=0.2823 and therefore
$\mu_{ideal}(al_3) < \mu_{ideal}(al_4) < \mu_{ideal}(al_2) < \mu_{ideal}(al_5) < \mu_{ideal}(al_1)$.

Therefore, we get the following ranking: $al_3 < al_4 < al_2 < al_5 < al_1$.

It is seen that this final ranking is substantially different from rankings we have obtained with the use of aggregating modes Ag_1, Ag_2 and Ag_3 solely. Hence, it may be treated as some consensus or compromise solution.

5 Conclusion

The technique for establishing order preference by similarity to the ideal solution ($TOPSIS$) probably is one of the most popular methods for the solution of multiple criteria decision making ($MCDM$) problems. The method was primarily developed for dealing with real-valued data. Nevertheless, in practice often it is hard to present precisely exact ratings of alternatives with respect to local criteria and as a result these ratings are often seen as interval, fuzzy or intuitionistic fuzzy values (IFVs). In this paper, we have developed a generalization of $TOPSIS$ method in the intuitionistic fuzzy setting. In our recent works, we showed that the operation on IFVs defined in the classical Atanassov's intuitionistic fuzzy sets theory ($A-IFS$) has some drawbacks and limitations which may produce controversial results in the solution of $MCDM$ problems or decrease the method's ability, e.g. the power operation, where both operands are IFVs is not defined in the framework of $A-IFS$. This is a very important limitation since it prevents introducing such an important aggregation operator as the weighted geometric mean, where the weights of local criteria are IFVs. Therefore, we introduced two additional aggregation operators that cannot be defined in the framework of conventional $A-IFS$ when weights of local criteria are IFVs.

Then we have used the redefinition of $A-IFS$ in the framework of the Dempster-Shafer theory of evidence (DST). This redefinition is free of drawbacks and limitations of operations defined in the conventional $A-IFS$. This claim was proved with the use of corresponding theorems in our previous works. In the current paper, we use this redefinition to generalize the conventional intuitionistic fuzzy $TOPSIS$ method. We showed that the distances of the alternatives from the ideal solutions might be treated (in some sense) as modified weighted sums of local criteria. The use of weighted sums is not the best approach to the aggregation of local criteria in many real-world applications, since the small values of some local criteria can be compensated by large values of other ones which in practice may be less important for the decision maker. It is worth noting that in some fields the weighted sums aggregation is not used at all. Obviously, if we deal with a complex real-world *MCDM* problem, all relevant aggregation modes should be used in the analysis. On the other hand, different aggregating modes generally provide different rankings of alternatives. Therefore, to obtain the compromise ranking, an appropriate method for aggregation of aggregation modes should be used.

Therefore here we use the method for aggregation of aggregation modes proposed in developed in our papers, which is based on the synthesis of type-2 and level-2 fuzzy sets defined on the support composed of compared alternatives. This method, in a great extent, is free of the drawbacks and limitations of other methods for aggregation of aggregating modes presented in the literature. It is important that the developed redefinition of the conventional intuitionistic fuzzy *TOPSIS* method in the framework of *DST*, besides many obvious advantages, seems to be more simple and easy to use than the traditional approach. The illustrative examples are presented to show the features of the proposed approach.

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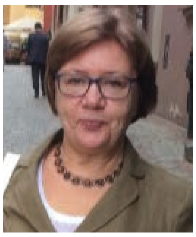

Ludmila Dymova received her Ph.D. in technical sciences from the Samara Technical University in Russia in 1986. In 2012, she obtained the title of Professor of technical sciences with specializations in Computer Science (Poland). The topics of her research activities in general may be defined as: "Development of modeling, identification, decision-making and optimization methods under conditions of objective and non-probabilistic types of uncertainty in economic, technological and ecological applications". In the years 1975-1999 she worked in Russia and Belarus, and since 2000 she has been working at the Częstochowa University of Technology. She is the author of about 150 publications and 2 books closely related to the subject of her interests.

Krzysztof Kaczmarek earned his Ph.D. degree in computer science at Czestochowa University of Technology, Poland, in 2009. Since 2009 he has been an Assistant Professor at the Institute of Computer and Information Sciences, Czestochowa University of Technology, Poland. Dr. Kaczmarek has authored about 20 publications. His research focuses on the synthesis of fuzzy logic, modern generalizations of fuzzy set theory and DST and methods of optimization of decision-making processes within automatic trading systems used in financial markets, such as Forex, NASDAQ.

Pavel Sevastjanov received his Ph.D. in technical sciences from the Samara Technical University in Russia in 1983. In 2003, he obtained the title of Professor of technical sciences with specializations in Computer Science, Control, Computer Technology. He recently focuses on the synthesis of fuzzy logic, modern generalizations of fuzzy set theory and DST with applications of MCDM and optimization for the solution of finance and medical diagnostic problems. From 1976 to 1999 he worked in Russia and Belarus, and since 2000 he has been working at the Częstochowa University of Technology. He is the author of about 260 publications and 4 books closely related to the subject of his interests.

Łukasz Sułkowski holds a professor degree in economics and a doctoral degree in humanities and specializes in issues related to university management. He is the head of the Department of Higher Education Institution Management at the Jagiellonian University as well as a professor at Clark University and at the Academy of Social Sciences. His research interests include organization and management, epistemology and methodology of social sciences and humanities, organizational culture and intercultural management as well as public management and family business management. He is the author of about 400 publications and 10 books closely related to the subject of his interests.

He is a member of the following international organizations and associations: American Academy of Management (AAofM - USA), International Family Entreprises Research Assotiation (IFERA), Reseau Pays du Groupe de Vysegrad (PGV) and the European Academy of Management (EURAM).

Krzysztof Przybyszewski is a professor at the University of Social Sciences in Łódź. His adventure with applied computer science began in the 1980s with a simulation of non-quantum collective processes (the subject of a Ph.D. dissertation). At present, he is involved in research and applications of various artificial intelligence technologies and soft computing methods in selected IT problems (in particular, in expert systems supporting the management of education quality in universities - the use of fuzzy numbers and sets). As a deputy dean at the University of Social Sciences, he is the designer and organizer of the on-Computer Science Faculty education program. He is the author of over 80 publications in the field of computer science and IT applications.