Journal of Applied Mathematics and Computational Mechanics 2016, 15(1), 149-159
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2016.1.15 e-ISSN 2353-0588

VIBRATION ANALYSIS OF COMPOSITE CIRCULAR AND ANNULAR MEMBRANES

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Abstract. In this paper a solution to the free vibration problem of composite circular and annular membranes is presented. The vibrations of membranes whose material densities and/or thicknesses varied step-wise with the radial co-ordinate are considered. This approach is applied to approximate the solution to the vibration problem of a membrane with continuously varying density and/or thickness with the radial co-ordinate. The obtained analytical solutions are used in numerical investigations into the effect of parameters characterizing the composite membranes on their eigenfrequencies.

Keywords: *composite membrane, free vibration, Green's function*

1. Introduction

The vibration analysis of membranes is of particular interest in the design of various acoustical devices. The solution to the vibration problem of a non-homogenous membrane in a closed form can be derived only for the cases of some functions describing the change in the material density and thickness of this membrane. Free vibration problems of circular and annular membranes when the density varies with the radial co-ordinate are the subject of papers [1-5]. The solution to the vibration problem of a membrane comprising two concentric annular membranes has been derived in an exact form by Laura et al. in paper [1]. Gottlieb [2] gives the explicit values of the radial spectrum of an annular membrane with a stepped density which contains inverse fourth power logarithmic terms in the density function. The exact solutions to both the axisymmetric and antisymmetric modes of non-homogenous circular and annular membranes with polynomial variation of the density are given by Jabareen and Eisenberger in paper [3]. The eigenfrequencies for variable density membranes are obtained by a power series expansion and for multiple-connected regions by the dynamic stiffness method.

Approximate methods have been applied to solve the vibration problems of non-homogeneous membranes in numerous papers (for example in references [4-9] various approximate methods are used). An application of the boundary point collocation method to determine the eigenfrequencies of membranes with varying mass density is presented by Cap in paper [4]. Gutierrez et al. [5] present numerical results for two lower free vibration frequencies of circular and annular membranes whose densities varied linearly, quadratically or cubically with the radial coordinate. The frequencies were calculated by using the differential quadrature method, the finite element technique, an optimized and/or improved Rayleigh quotient method and a lower bound based on the Stodola-Vianello method. The multi-symplectic methods for free vibration of the membrane are proposed by Wei-Peng et al. in paper [6]. Numerical results presented in the paper verify the efficiency of the methods. The free vibration frequencies of an annular membrane for axisymmetric modes by the discrete singular convolution method, based on the regularized Shannon's delta kernel, were determined by Civalek and Gürses in reference [7]. The numerical technique for problems of free vibrations of non-homogenous membranes is presented by Reutskiy in paper [8]. The method is based on the mathematical modeling of the response of a system to external excitation over a range of frequencies. Buchanan [9] has studied vibrational properties and has demonstrated the accuracy of finite element formulation for circular and annular membranes with density variation assumed as a linear function of the radius. The free vibration of an annular membrane consisting of three concentric annular membranes was considered in the paper [10]. The solution of the problem has been derived by using the properties of Green's functions.

In this paper, a solution to the free vibration problem of composite circular and annular membranes whose densities and/or thicknesses varied step-wise with the radial co-ordinate is derived. This approach has been applied for numerical computation of eigenfrequencies of membranes with a continuous change in density and thickness in the radial direction.

2. Formulation and solution to the free vibration problem of a composite membrane

Consider the vibration problem of an annular composite membrane consisting of m annular homogenous membranes as shown in Figure 1. The thickness and the material density of the composite membrane change step-wise at circles with radii $r_1, r_2, \ldots, r_{m-1}$. The j-th annular membrane includes an elastic support distributed along the circle $r = \bar{r}$, where $\bar{r} \in [r_{j-1}, r_j)$.

Free vibrations of the homogenous annular membranes with an elastic support are governed by the following differential equations:

$$s\nabla^2 u_j - \rho_j h_j \frac{\partial^2 u_j}{\partial t^2} = 0, \quad j = 1, 2, \ldots, m \tag{1}$$

where u_j is the displacement of the j-th annular membrane, s is the tension per unit length, ρ_j is the mass per unit area, h_j is the thickness of the j-th annular membrane, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacian operator, r and θ are polar coordinates and t is time. The radial variable r for the j-th annular membrane is in the interval: $r_{j-1} \le r \le r_j$, $j = 1,2,...,m$, where $r_0 = a$, $r_m = b$.

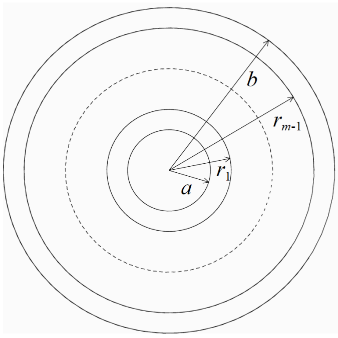

Fig. 1. The sketch of the composite annular membrane under study

Functions u_j $(j = 1,2,...,m)$ satisfy the continuity conditions

$$u_j\left(r_j,\theta,t\right) = u_{j+1}\left(r_j,\theta,t\right),\ j = 1,2,...,m-1 \tag{2}$$

$$\left.\frac{\partial u_j\left(r,\theta,t\right)}{\partial r}\right|_{r=r_j} = \left.\frac{\partial u_{j+1}\left(r,\theta,t\right)}{\partial r}\right|_{r=r_j},\ j = 1,2,...,m-1 \tag{3}$$

and the boundary conditions

$$u_1\left(r_0,\theta,t\right) = 0,\ u_m\left(r_m,\theta,t\right) = 0 \tag{4}$$

Considering the free vibration of the membrane we assume functions $u_j\left(r,\theta,t\right)$ in the form

$$u_j\left(r,\theta,t\right) = U_{jn}\left(r\right)\cos\omega_n t\cos n\theta,\ j = 1,2,...,m,\ n = 0,1,2,... \tag{5}$$

where ω_n is the natural frequency of the composite membrane. Taking equation (6) into account in differential equation (1), in continuity conditions (3)-(4) and in boundary conditions (5), we obtain a differential equation, continuity conditions and boundary conditions for functions $U_{jn}\left(r\right)$:

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+\lambda_{jn}^2\right]U_{jn}(r)=0 \quad \text{for} \quad r_{j-1}\le r\le r_j,\ j=1,2,...,m \tag{6}$$

$$U_{jn}(r_j)=U_{j+1n}(r_j), \quad j=1,2,...,m-1 \tag{7}$$

$$\left.\frac{dU_{jn}(r)}{dr}\right|_{r=r_j}=\left.\frac{dU_{j+1n}(r)}{dr}\right|_{r=r_j}, \quad j=1,2,...,m-1 \tag{8}$$

$$U_{1n}(r_0)=0, \quad U_{mn}(r_m)=0 \tag{9}$$

where $\lambda_{jn}=\omega_n\sqrt{\rho_j h_j/s}$.

The general solution of the differential equation (7) is given by

$$U_{jn}(r)=C_{1j}J_0(\lambda_{jn}r)+C_{2j}Y_0(\lambda_{jn}r) \quad \text{for} \quad r\in[r_{j-1},r_j],\ j=1,2,...,m \tag{10}$$

where C_{1j} and C_{2j} are arbitrary constants. Substituting the functions (10) into conditions (7)-(9), we obtain a set of $2m$ equations which can be written in a matrix form

$$\mathrm{A}\,\mathrm{C}=0 \tag{11}$$

where $\mathbf{A}=\left[a_{pq}\right]_{0\le p,q\le 2m}$, and $\mathbf{C}=\left[C_{11}\ \ C_{21}\ \ ...\ \ C_{1m}\ \ C_{2m}\right]^T$. For $r_0>0$ (an annular membrane) the non-zero elements of the matrix **A** are (index n is omitted)

$a_{11}=J_0(\lambda_1 r_0)$, $a_{12}=Y_0(\lambda_1 r_0)$, $a_{2j,2j-1}=J_0(\lambda_j r_j)$, $a_{2j,2j}=Y_0(\lambda_j r_j)$,
$a_{2j+1,2j+1}=-J_0(\lambda_{j+1}r_j)$, $a_{2j,2j+2}=-Y_0(\lambda_{j+1}r_j)$, $a_{2j+1,2j-1}=J_{-1}(\lambda_j r_j)-J_1(\lambda_j r_j)$,
$a_{2j+1,2j}=Y_{-1}(\lambda_j r_j)-Y_1(\lambda_j r_j)$, $a_{2j+1,2j+1}=-\dfrac{\lambda_{j+1}}{\lambda_j}\left[J_{-1}(\lambda_{j+1}r_j)-J_1(\lambda_{j+1}r_j)\right]$,
$a_{2j+1,2j+2}=-\dfrac{\lambda_{j+1}}{\lambda_j}\left[Y_{-1}(\lambda_{j+1}r_j)-Y_1(\lambda_{j+1}r_j)\right]$
for $j=1,...,m-1$ and
$a_{2m,2m-1}=J_0(\lambda_m r_m)$, $a_{2m,2m}=Y_0(\lambda_m r_m)$.

For a nontrivial solution of the equation (11) the determinant of the matrix **A** is set equal to zero yielding the frequency equation of the composite membrane

$$\det(\mathrm{A}(\omega))=0 \tag{12}$$

The equation (12) is then solved numerically with respect to ω by using an approximate method. Note that for a fixed n we obtain a sequence ω_{kn} , $k = 0,1,...$, of the roots of the equation (12).

The frequency equation for a circular membrane ($r_0 = 0$) is obtained similarly. In this case, the first row in the matrix **A** for the annular membrane should be changed by assuming: $a_{11} = 0$, $a_{12} = 1$. The remainder elements of the matrix **A** for the circular membrane are the same as for annular membrane.

For the computed values $\lambda_{jk} = \omega_k \sqrt{\rho_j h_j / s}$, $k = 1,2,\ldots$, the corresponding eigenfunctions (the mode shapes of vibration) are given by equation (10) where the coefficients C_{1j} , C_{2j} are determined by solving equation (11), in which $C_{2m} = 1$ should be assumed. The eigenfunctions satisfied the following orthogonality conditions

$$\sum_{j=1}^{m} \mu_j^2 \int_{r_{j-1}}^{r_j} rU_{jk'n}(r)U_{jkn}(r)\,dr = \begin{cases} 0 & k = k' \\ N_{kn} & k \neq k' \end{cases} \tag{13}$$

where $\mu_j = \sqrt{\rho_j h_j / s}$ and

$$N_{kn} = \sum_{j=1}^{m} \mu_j^2 \int_{r_{j-1}}^{r_j} rU_{jkn}^2(r)\,dr \tag{14}$$

3. Forced vibration of a composite membrane

The differential equation to vibration of an annular membrane forced by an outer force is

$$\nabla^2 u_j - \mu_j^2 \frac{\partial^2 u_j}{\partial t^2} = \frac{1}{s} f_j(r,\theta,t), \quad j = 1,2,...,m \tag{15}$$

The functions u_j $(j = 1,2,...,m)$ satisfy the continuity conditions (2)-(3) and the boundary conditions (4). Moreover, the initial conditions are

$$u_j(r,\theta,0) = p_j(r,\theta) \tag{16}$$

$$\left.\frac{\partial u_j(r,\theta,t)}{\partial r}\right|_{t=0} = q_j(r,\theta) \tag{17}$$

We seek a solution of the problem in the form of a series

$$u_j(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}U_{jkn}(r)\Gamma_{kn}(t)\cos n\theta \tag{18}$$

where $U_{jkn}(r)$ were derived in Section 2 (equation (10)) and functions $\Gamma_{kn}(t)$ will be now determined.

Substituting the function $u_j(r,\theta,t)$ in the form (18) into equation (15) and using the orthogonality condition (13), we obtain an equation with unknown functions $\Gamma_{kn}(t)$ in the form

$$\frac{d^2}{dt^2}\Gamma_{kn}(t)+\omega_{kn}^2\Gamma_{kn}(t)=F_{kn}(t) \tag{19}$$

where $F_{kn}(t)=\dfrac{-1}{sN_{kn}\kappa_n}\sum_{j=1}^{m}\int_0^{2\pi}\int_{r_{j-1}}^{r_j} rU_{jkn}f(r,\theta,t)\cos n\theta drd\theta$ and $\kappa_0=2\pi$, $\kappa_n=\pi$ for $n=1,2,\ldots$. This equation is complemented by initial conditions which follow from (16) and (17). Using (18) and (13) one obtains the initial conditions in the form

$$\Gamma_{kn}(0)=\frac{1}{\kappa_n N_{kn}}\sum_{j=1}^{m}\mu_j^2\int_0^{2\pi}\int_{r_{j-1}}^{r_j} rU_{jkn}(r)p_j(r,\theta)\cos n\theta\, drd\theta \tag{20}$$

$$\left.\frac{d\Gamma_{kn}}{dt}\right|_{t=0}=\frac{1}{\kappa_n N_{kn}}\sum_{j=1}^{m}\mu_j^2\int_0^{2\pi}\int_{r_{j-1}}^{r_j} rU_{jkn}(r)q_j(r,\theta)\cos n\theta drd\theta \tag{21}$$

The solution of the equation (19) with conditions (20), (21) is as follows:

$$\begin{aligned}\Gamma_{kn}(t)=&\frac{\cos\omega_{kn}t}{\kappa_n N_{kn}}\sum_{j=1}^{m}\mu_j^2\int_0^{2\pi}\int_{r_{j-1}}^{r_j} rU_{jkn}(r)p_j(r,\theta)\cos n\theta\, drd\theta\\&+\frac{\sin\omega_{kn}t}{\kappa_n N_{kn}\omega_{kn}}\sum_{j=1}^{m}\mu_j^2\int_0^{2\pi}\int_{r_{j-1}}^{r_j} rU_{jkn}(r)q_j(r,\theta)\cos n\theta\, drd\theta\\&-\frac{1}{s\kappa_n N_{kn}\omega_{kn}}\sum_{j=1}^{m}\int_0^{2\pi}\int_{r_{j-1}}^{r_j}\int_0^{t} rU_{jkn}(r)f_j(r,\theta,\tau)\sin\omega_k(t-\tau)\cos n\theta d\tau drd\theta\end{aligned} \tag{22}$$

Finally, the solution of the forced vibration problem of the composite membrane can be written in the form

$$u_j(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{j'=1}^{m}\mu_{j'}^{2}\int_{0}^{2\pi}\int_{r_{j'-1}}^{r_{j'}}\sum_{k=1}^{\infty}\frac{r'p_{j'}(r',\theta')}{\kappa_n N_{kn}}U_{j'kn}(r')U_{jkn}(r)\cos n\theta\cos n\theta'\cos\omega_{kn}t\,dr'd\theta'$$

$$+\sum_{n=0}^{\infty}\sum_{j'=1}^{m}\mu_{j'}^{2}\int_{0}^{2\pi}\int_{r_{j'-1}}^{r_{j'}}\sum_{k=1}^{\infty}\frac{r'q_{j'}(r',\theta')}{\kappa_n N_{kn}\omega_{kn}}U_{j'kn}(r')U_{jkn}(r)\cos n\theta\cos n\theta'\sin\omega_{kn}t\,dr'd\theta'$$

$$-\sum_{n=0}^{\infty}\sum_{j'=1}^{m}\int_{0}^{2\pi}\int_{r_{j'-1}}^{r_{j'}}\int_{0}^{t}\sum_{k=1}^{\infty}\frac{r'f_{j'}(r',\theta',\tau)}{s\kappa_n N_{kn}\omega_{kn}}U_{j'kn}(r')U_{jkn}(r)\cos n\theta\sin\omega_{kn}(t-\tau)\cos n\theta' d\tau dr'd\theta' \quad (23)$$

4. Numerical examples

The numerical computations presented here concern the frequency analysis of composite annular and circular membranes for various values of parameters which characterize their non-uniformity. The calculations of the non-dimensional free vibration frequencies were performed using frequency equation (12). The roots of this equation were determined by the application of the false position method [11].

In the first example, the eigenfrequencies of circular and annular membrane were computed with the density function given by the formula: $\rho(r)=\rho_0(1+\alpha r)$. The results of calculations: $\Omega_{nk}=\omega_{kn}b\sqrt{\rho_m h_m/s}$ ($n=0,1,2,3$; $k=1,2,3,4,5$) for various values of α and various numbers of annular membranes m are shown in Table 1 - for the annular membrane and in Table 2 - for the circular membrane. The eigenfrequencies for these membranes were determined earlier by using the power series method in paper [3] by Jabareen and Eisenberger. The results for $m=10,15,20$, obtained by using the presented method, are compared with the results given in reference [3]. The calculated free vibration frequencies Ω_{nk} are consistent for both annular (Table 1) and circular (Table 2) membranes for $m=20$ and those presented in reference [3]. The differences do not exceed $5\cdot10^{-4}$ for $k=1$ and $3\cdot10^{-2}$ for $k=5$. Moreover, it can be shown that the differences decrease as the number of annular membranes m is increased.

The frequencies of the circular composite membrane of radius b, consisting of three parts (a circular inner membrane and two annular membranes) as functions of the ratio r_1/b, are presented in Figure 2 for $n=0$. The material density of the inner annular membrane is much greater than the density of the other parts of the membrane, i.e. $\rho_1=\rho_3$, and $\rho_1 << \rho_2$. The calculations were performed for: $\sigma=\rho_2/\rho_1=2;\,5;\,10;\,15;\,20;\,30$ and $(r_2-r_1)/b=0.01$. Figure 2 shows that both the material density (or membrane thickness) and the location of the inner annular membrane cause significant changes in the eigenfrequencies of the composite membrane.

Table 1

Values of Ω_{nk} for an annular membrane, $a/b = 0.2$; $\rho(r) = \rho_0(1+\alpha r)$

	$\alpha = 1.0$				$\alpha = 2.0$			
	m			Ref [3]	m			Ref [3]
	10	15	20		10	15	20	
Ω_{01}	3.0156	3.0156	3.0156	3.0156	2.5691	2.5691	2.5691	2.5691
Ω_{02}	6.1639	6.1640	6.1640	6.1640	5.2653	5.2654	5.2654	5.2654
Ω_{03}	9.2537	9.2937	9.2937	9.2937	7.9442	7.9442	7.9443	7.9443
Ω_{04}	12.4161	12.4163	12.4163	12.4163	10.6161	10.6163	10.6164	10.6164
Ω_{05}	15.5350	15.5354	15.5354	15.5354	13.2845	13.2851	13.2852	13.2852
Ω_{11}	3.3349	3.3349	3.3349	3.3349	2.8360	2.8360	2.8360	2.8360
Ω_{12}	6.3801	6.3801	6.3802	6.3802	5.4503	5.4504	5.4504	5.4504
Ω_{13}	9.4532	9.4532	9.4532	9.4532	8.0825	8.0826	8.0826	8.0826
Ω_{14}	12.5414	12.5415	12.5416	12.5416	10.7255	10.7258	10.7258	10.7259
Ω_{15}	15.6377	15.6381	15.6381	15.6381	13.3746	13.3753	13.3753	13.3754
Ω_{21}	4.0734	4.0734	4.0734	4.0734	3.4492	3.4492	3.4492	3.4492
Ω_{22}	6.9718	6.9718	6.9718	6.9718	5.9526	5.9526	5.9526	5.9526
Ω_{23}	9.9155	9.9156	9.9156	9.9156	8.4816	8.4817	8.4817	8.4817
Ω_{24}	12.9120	12.9122	12.9122	12.9123	11.0488	11.0491	11.0492	11.0492
Ω_{25}	15.9440	15.9444	15.9445	15.9445	13.6434	13.6441	13.6442	13.6443
Ω_{31}	4.9359	4.9359	4.9359	4.9359	4.1603	4.1603	4.1603	4.1603
Ω_{32}	7.7955	7.7956	7.7956	7.7956	6.6424	6.6424	6.6425	6.6425
Ω_{33}	10.6269	10.6271	10.6271	10.6271	9.0889	9.0891	9.0892	9.0892
Ω_{34}	13.5092	13.5095	13.5095	13.5096	11.5663	11.5668	11.5669	11.5669
Ω_{35}	16.4474	16.4480	16.4481	16.4481	14.0838	14.0848	14.0850	14.0850

Table 2

Values of Ω_{nk} for a circular membrane, $\rho(r) = \rho_0(1+\alpha r)$

	$\alpha = 1.0$				$\alpha = 2.0$			
	m			Ref [3]	*m*			Ref [3]
	10	15	20		10	15	20	
Ω_{01}	2.0108	2.0108	2.0108	2.0108	1.7598	1.7598	1.7598	1.7598
Ω_{02}	4.5548	4.5549	4.5549	4.5549	3.9802	3.9802	3.9802	3.9802
Ω_{03}	7.1194	7.1195	7.1196	7.1196	6.2147	6.2149	6.2150	6.2150
Ω_{04}	9.6897	9.6900	9.6901	9.6901	8.4537	8.4542	8.4543	8.4543
Ω_{05}	12.2622	12.2629	12.2631	12.2631	10.6943	10.6956	10.6957	10.6958
Ω_{11}	3.0678	3.0678	3.0678	3.0678	2.6273	2.6273	2.6273	2.6273
Ω_{12}	5.6990	5.6991	5.6991	5.6991	4.9267	4.9268	4.9268	4.9268
Ω_{13}	8.3004	8.3006	8.3006	8.3006	7.1990	7.1993	7.1993	7.1994
Ω_{14}	10.8914	10.8919	10.8920	10.8920	9.4608	9.4616	9.4617	9.4618
Ω_{15}	13.4773	13.4785	13.4786	13.4787	11.7170	11.7188	11.7191	11.7192
Ω_{21}	4.0223	4.0223	4.0224	4.0224	3.4110	3.4110	3.4110	3.4110
Ω_{22}	6.7453	6.7454	6.7454	6.7454	5.7872	5.7874	5.7874	5.7874
Ω_{23}	9.3940	9.3943	9.3944	9.3944	8.1011	8.1015	8.1016	8.1016
Ω_{24}	12.0150	12.0157	12.0158	12.0159	10.3905	10.3916	10.3918	10.3919
Ω_{25}	14.6217	14.6234	14.6236	14.6236	12.6665	12.6691	12.6694	12.6696
Ω_{31}	4.9283	4.9283	4.9284	4.9284	4.1549	4.1549	4.1550	4.1550
Ω_{32}	7.7381	7.7383	7.7383	7.7383	6.6029	6.6031	6.6032	6.6032
Ω_{33}	10.4357	10.4362	10.4362	10.4363	8.9581	8.9587	8.9588	8.9588
Ω_{34}	13.0894	13.0904	13.0906	13.0906	11.2758	11.2774	11.2776	11.2776
Ω_{35}	15.7197	15.7220	15.7223	15.7224	13.5726	13.5763	13.5767	13.5768

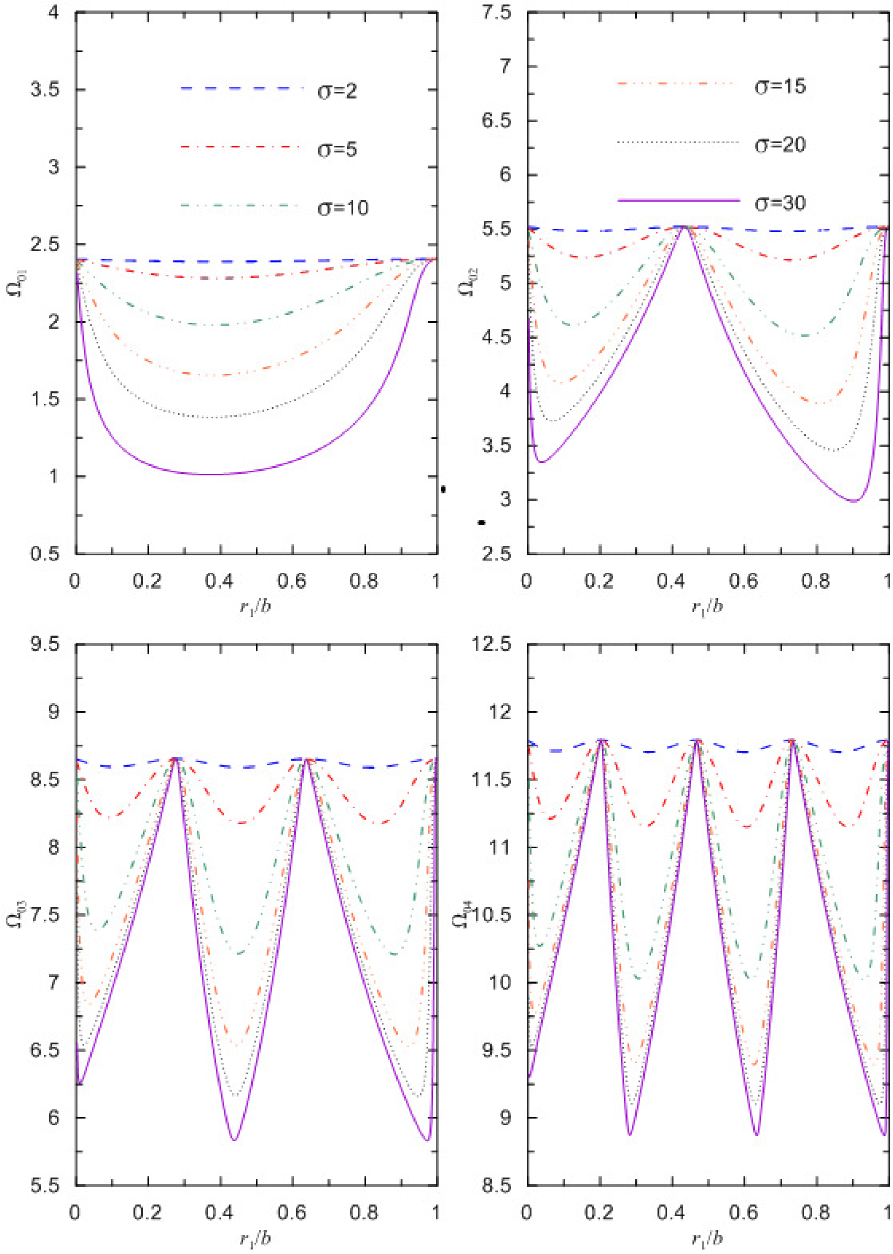

Fig. 2. Four free vibration frequencies Ω_{nk} of the circular composite membrane consisting of three segments for $n = 0$, as functions of the ratio r_1/b for various σ and $(r_2 - r_1)/b = 0.01$

5. Conclusions

In this paper, a solution to the free vibration problem of annular and circular composite membranes has been derived. Numerical analysis has shown the effect of parameters characterizing the composite annular or circular membranes on their eigenfrequencies. The frequency analysis was performed for a circular membrane with an inner highlighted annular membrane whose material density (or thickness) is much greater than the material density (or thickness) of the remaining part of the membrane. The numerical results show that the location of the highlighted annular membrane has a significant effect on eigenfrequencies of the composite membrane. It follows that the application of this annular membrane can be used to introduce a specific change in the dynamical characteristic of the composite membrane. Numerical examples show that the presented method can be used to determine approximate natural frequencies of composite membranes with continuously varying densities or thicknesses.

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