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On the mathematical condition-based maintenance modelling for continuously deteriorating systems

Keywords

condition-based maintenance, continuous deterioration, stochastic modelling, Gamma process, semi-regeneration

Abstract

This paper presents a modelling framework of condition-based maintenance policies for continuously deteriorating systems, based on semi-regenerative stochastic techniques.

1. Introduction & problem statement

This paper is devoted to the synthetic presentation of a stochastic modeling approach of condition-based maintenance policies for continuously deteriorating systems, which has been applied on several different condition-based maintenance policies, see e.g. [2,11].
The main motivation for this kind of work is the following. Nowadays, among maintenance practitioners, classical preventive maintenance policies [5,23,24] are more or less well established, but these static preventive maintenance approaches do not always meet the needs and requirements of high performance maintenance. In this context, condition-based (or predictive) maintenance strategies can help. Moreover, the development and dissemination of systems monitoring facilities and maintenance information management and the need for improved maintenance performance under budget and resources constraints foster the implementation of such dynamic maintenance strategies. However, to ensure a safe and efficient migration from static (but robust) preventive maintenance policies to dynamic condition-based maintenance policies, we need to develop practice-oriented maintenance performance model that can allow to assess and to optimize their performance [5,21,22].
For a gradually deteriorating system, the most advanced preventive maintenance strategies rely on the monitoring of a measurable system diagnostic parameter (``system state") and base the maintenance decisions on the level of deterioration of the system. Generally, such a condition-based preventive maintenance policy is more efficient than a preventive maintenance policy based only on the age of the system and on the knowledge of the statistical information on its lifetime [10,17,18]. However, as stressed in [9], the price for this higher efficiency is the requirement of a mathematical model for the stochastic deterioration process of the maintained system. There is an economic necessity to quantify and model the deterioration / maintenance process, since this model can be used by the maintenance decision-maker as a tool to optimize the maintenance decisions and to minimize the total maintenance cost of the system. Usually, the task of deriving such a mathematical model turns out to be more complex than just statistically describing the binary transition from a ``good state" to a ``failed state".

This paper precisely examines the problem of developing a mathematical maintenance cost model for assessing and optimizing a condition-based maintenance policy for continuously deteriorating system. This modelling task usually consists in several steps:

- deterioration and monitoring information modelling (modelling uncertainty, taking into account the lack of maintenance data, …);
- modelling the effect of maintenance;
- definition of maintenance decision process (pre-defined parametric structure vs. open-decision, control limit policy, inspection rule, …);
- building the performance model and optimizing the maintenance parameters.

2. Continuous deterioration modeling

For many real-world systems, the deterioration process due to wear and tear is intrinsically continuous, e.g. systems subject to erosion (hydraulic structures, dikes), corrosion (steel reinforcements of concrete structures, pipelines), consumption (tires, brakes), cumulative wear (cutting tools), … For these systems, the notion of "discrete states" often used in maintenance and reliability models [9] might be irrelevant whereas the level of deterioration (which can be measured in practice, eventually through a strongly correlated process) has generally a clearer and more physical signification for the maintenance decision-maker. Accordingly, the choice in our work on maintenance modelling is to develop a model to assess maintenance decisions based directly on an observed deterioration level, without involving more abstract quantities like one-step transition probabilities in a finite-state process (with artificial states constructed by discretization from the continuous deterioration process) in the modelling procedure, see e.g. [12].
The basic aim of deterioration modelling is to be able to predict in some sense the future evolution of the deterioration level and to interpret and give sense to a deterioration raw measurement (Is this deterioration acceptable? How much time left before failure? …). As presented by R. Nicolai in [16], deterioration can be represented using a "black-box" model (lifetime distribution), a "white-box" model (explicit model constructed from the physics of the deterioration phenomenon) or a "grey-box" model (kind of stress-strength model in which the system failure occurs when a measurable quantity representing time-dependent deterioration exceeds a threshold). In the approach presented in this paper, we consider the class of "grey-box" models.

The deterioration behavior of the unmaintained system is represented by a continuous-state univariate stochastic process $\{X(t), t \geq 0\}$ with initial level of deterioration $X_0 = 0$. The deterioration is strictly increasing which means that the system worsens with time due to ageing and accumulated wear or damage. The system failure occurs when a limit deterioration threshold L is crossed: beyond this level of deterioration, the system can no longer meet the user's requirements and is considered as failed, see *Figure 1*. This failure can be either an actual ``hard'' failure of an active system or a pending failure of a passive system or structure. The failure is not assumed to be self-announcing, i.e. it can be detected only by an inspection. After failure, the system remains unavailable until the next scheduled maintenance operation.

Assume that the system can be inspected or maintained only at periodic discrete maintenance times $t_k = k.\delta t$; thus, only the discrete-time stochastic process $\{X_k = X(t_k)\}$ can be observed. The elementary deterioration increments occurring between two successive maintenance times t_k and t_{k+1} are assumed to be positive, exchangeable and stationary. The positiveness of the increments corresponds to the increasing monotonicity, which is a behavior observed in physical deterioration process. The properties of exchangeable and stationary increments are very similar (even if weaker) to the properties of stationary and independent increments. The stationarity and independence entail the ``memoryless'' property that the future increment of deterioration depends neither on the current level of deterioration of the system nor on its age, but only on the period of time over which the system deteriorates, [4,13,16].

All these desirable deterioration properties lead to the class of Lévy processes [1] to develop a model of gradual deterioration. Any process belonging to the class of Lévy processes is the sum of a Wiener process and a jump process, [4,16]. Hence, Lévy processes can be used to model purely continuous deterioration processes as well as "jumps" deterioration processes. Forcing increasing monotonicity in the class of Lévy process leads to jumps processes which are clearly not the most adapted to model gradual continuous deterioration phenomena. One solution is then to consider the limit case of a jumps process with a countable infinite number of jumps on a finite interval of time, which in turn leads to the Gamma processes (because the pdf of the deterioration increments is Gamma and infinitely divisible, [7]). Gamma processes have been widely used in deterioration modelling for maintenance applications, see [14] for a complete review. In this paper, we will use stationary Gamma processes to model the deterioration phenomenon of the system to be maintained.

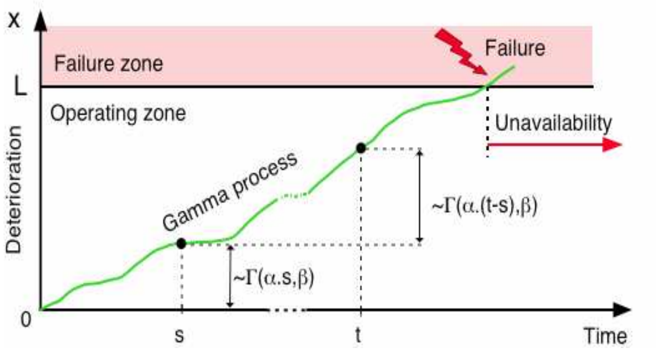

Figure 1. Deterioration and failure of the system

A stationary Gamma process satisfies the following properties:

- $X(0) = 0$;
- the deterioration increments are stationary and independents ;
- the random deterioration increment between times s and t, $X(t) - X(s)$, follows a Gamma law:

$$f_{\alpha(t-s),\beta}(x) = \frac{1}{\Gamma(\alpha(t-s))\beta^{\alpha(t-s)}} x^{\alpha(t-s)-1} e^{-x/\beta} \mathrm{I}_{\{x\geq 0\}}$$

with $\alpha(t)$ is a linear function of time.

3. Condition-based maintenance modelling

3.1. Maintenance problem statement

Once the stochastic deterioration model is available, our objective is to develop maintenance models for sequential condition-based maintenance policy for system monitored through inspections, i.e. whose state is not continuously observed. The main feature of this kind of policy is that the decision is sequentially and dynamically adapted to the level of system deterioration measured during an inspection. After an inspection at time t_k revealing the system deterioration level, we have to make a decision on two points, see *Figure 2*:

- What action A_k should be undertaken on the system to maintain it in a proper condition?
- When should be performed the next inspection on the system?

We want to consider mainly inspection/replacement policy, but also more complex policy involving different maintenance action such as partial repairs, with two main characteristics: (i) the inspection schedule is not periodic, and (ii) the nature and the time of the maintenance action are jointly optimized.

To solve the maintenance modelling problem and to develop the mathematical maintenance model, we propose a 3-steps approach: first, we choose a parametric structure for the maintenance decision rule. Secondly, we characterize the steady-state behaviour of the system maintained under the chosen parametric maintenance decision rule. Finally, thanks to the knowledge of the steady-state behaviour of the system, we evaluate the maintenance performance criterion of interest, so as to be able to optimize the maintenance policy.

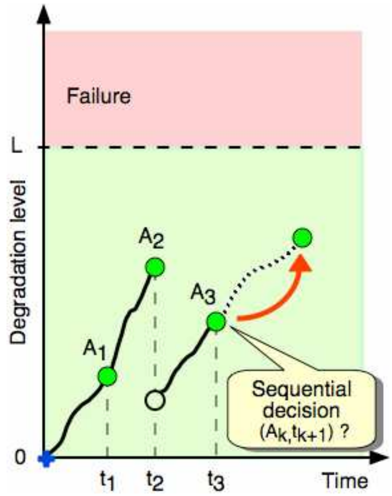

Figure 2. Principle of a sequential condition-based maintenance policy

3.2. Parametric structure-based decision rule

Developing a quite general condition-based maintenance model leads to the formulation of a decision problem at least in a two dimensions space. For a "simple" condition-based periodic inspection/replacement policy, we have to determine the replacement threshold and the inspection period. Obviously, the problem dimension increases with the complexity of the considered policy, e.g. to take into account several different maintenance operations (complete replacements or overhauls, minimal repairs, partial and imperfect repairs,…). In theory, the decision problems corresponding to these more complex policies can be formulated using the dynamic programming tools, and their performance can be evaluated using classical policy optimisation algorithms (Policy Iteration Algorithm). Several works have followed this approach [15]; however, we may see two main problems with it :

- from a theoretical point of view, for a general policy, it can be difficult and burdensome to formalise and solve the corresponding decision problem, even numerically ;
- from a practical point of view, the resulting structure of the optimal policy can be quite complex and hard to implement on a real system.

Our preferred approach is to impose a parametric structure to the maintenance policy, and the parameters of this structure constitute the maintenance decision variables. The advantage of this approach is to reduce the size of the problem space when searching for the optimal policy. Of course, there is a risk that the imposed structure does not correspond to the absolutely optimal policy. However, even if a proof is not available in the general case, several existing results [23,24] indicate that for systems with a stationary Markovian deterioration, the two following features are desirable:

- The replacement decision is based on “control limit decision rule”, viz., the system is replaced when the deterioration level exceeds a threshold. When several different maintenance actions are to be considered, a multi-level control limit decision rule can be adopted;
- The inspection schedule is constructed dynamically and sequentially to adapt to the observed deterioration level of the system: a more deteriorated system is inspected more frequently. After an inspection, the inter-inspection time (i.e. until the next inspection) is determined as a function of the system state (or deterioration level) using an “inspection scheduling function”. This “inspection scheduling function” can be either continuous (for continuous time problems) or step-wise (for discrete-time problems), see *Figures 3* and *Figure 4*.

The resulting policies belong to the class of stationary Markovian policies, deterministic of partly randomized (when e.g. the repair level can be random).

3.3. Policy performance assessment

The performance of maintenance policies are usually assessed through the availability of the maintained system and the overall maintenance costs balance. In this paper, we focus only on an asymptotic evaluation of these quantities, i.e. the expected cost per time unit C_∞ and the asymptotic unavailability $\overline{A}_\infty$ over an infinite horizon.

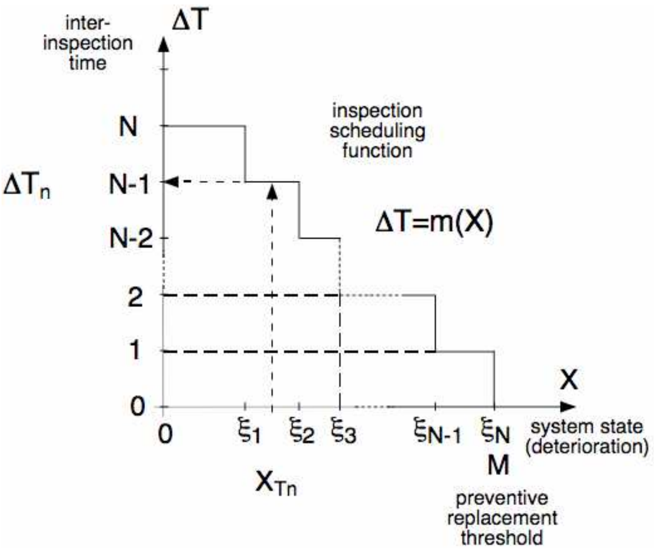

Figure 3. Step-wise inspection scheduling function

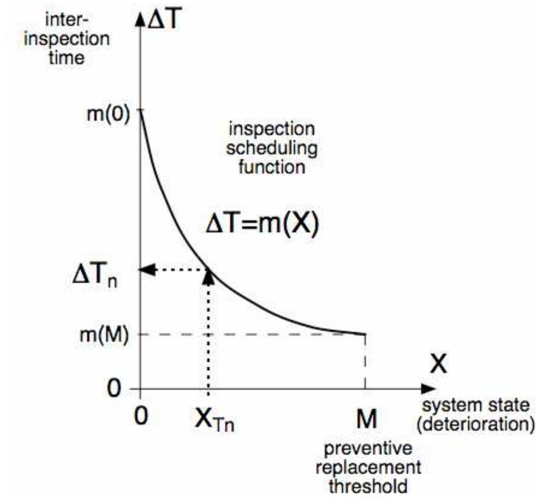

Figure 4. A continuous inspection scheduling function

Cost criterion: the cumulated maintenance cost at time t, denoted $C(t)$, includes both direct and indirect maintenance costs (maintenance crew costs, spare parts, production loss, performance degradation,…). The expected maintenance cost per time unit over an infinite time span is given by:

$$C_\infty = \lim_{t\to\infty} \frac{C(t)}{t} \quad \text{or} \quad C_\infty = \lim_{t\to\infty} \frac{E[C(t)]}{t}$$

Availability criterion: denoting $\overline{D}(t)$ the cumulated unavailability duration of the system up to time t, the asymptotic unavailability is given by:

$$\overline{A}_\infty = \lim_{t\to\infty} \frac{\overline{D}(t)}{t} \quad \text{or} \quad \overline{A}_\infty = \lim_{t\to\infty} \frac{E\left[\overline{D}(t)\right]}{t}$$

The main objective of a mathematical maintenance model is then to allow the evaluation of these quantities C_∞ and $\overline{A}_\infty$. The construction of such a model requires first the modelling of the stationary behaviour of the maintained system state.

3.4. Modelling the maintained system deterioration behaviour

Maintenance actions modify the deterioration behaviour and permit to control the deterioration process. Let assume that the action of the maintenance policy leads the maintained system to exhibit a stationary behaviour. The aim of the maintenance optimization is then to tune the maintenance parameters (thresholds, inter-inspection times, …) in order to keep the deterioration level of the system in an optimal zone where maintenance costs optimally balance the failure and deterioration cost, so that the profit from the system is maximum.

Regenerative properties of the maintained system state - A very classical approach to this problem is to derive an evaluation procedure for the asymptotic criteria taking advantage of the renewal properties of the stochastic process describing the maintained system state. Indeed, after an as-good-as new (preventive or corrective) replacement, the stochastic deterioration process $\{X(t), t \geq 0\}$starts again from the new state ($X(0)=0$); it evolves with the same probabilistic behaviour and is independent from the past, viz the history and past events of the system before the replacement. The stochastic process describing the maintained system is thus a regenerative process and the replacement times S_n are renewal points (and regenerative times) for this process, [1], [3], [19], see *Figure 5*. Applying classical renewal theorems, the expected cost per time unit on an infinite horizon can be computed as the ratio of the expected cost on a renewal cycle over the expected length of a cycle:

$$C_\infty = \lim_{t\to\infty} \frac{E[C(t)]}{t} = \frac{E[C(S)]}{E[S]} \tag{1}$$

where $_S$ is the time of the first replacement. Similarly, we have:

$$\overline{A}_\infty = \lim_{t\to\infty} \frac{E[\overline{D}(t)]}{t} = \frac{E[D(S)]}{E(S)}\,. \tag{2}$$

This classical approach has proven its efficiency and simplicity of use for classical "static" preventive maintenance policies, [23], [24]. However, for more complex, dynamic, condition-based maintenance policies, this approach based on renewal properties of the maintained system state turns out to be rather difficult to use.

When the maintenance policy includes several different maintenance operations (and not only as-good-as new replacements), a maintenance action does not necessarily ends up with a system renewal and a return to zero of the corresponding stochastic process. The time between two consecutive as-good-as new replacements can be very long and, consequently, the evolution of the system on a renewal cycle can be very difficult to apprehend and to characterize. It is thus difficult to compute the expectations in Equations (1) or (2).

When maintenance decisions are based on the deterioration level, and not only on the fact that the system is either running or failed, the knowledge of the lifetime law is not sufficient to solve the maintenance problem. It is also necessary to know the probability law of the deterioration level of the maintained system.

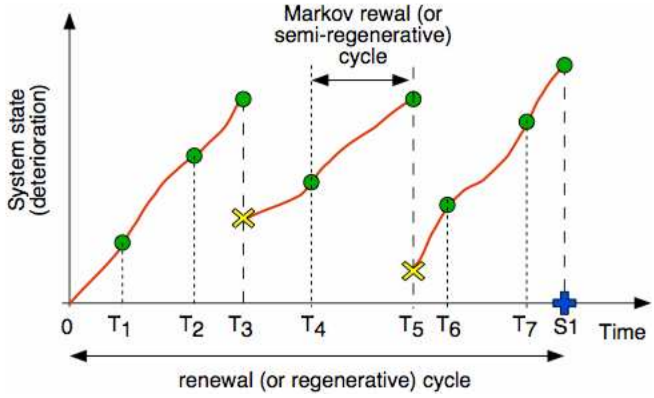

Figure 5. Schematic evolution of the maintained system state with inspection (bullets), partial repairs (x) and as-good-as new replacements (+). Note the difference between semi-regenerative (or Markov renewal) cycles and regenerative (or renewal) cycles.

In both cases, it is not easy to use the regenerative properties of the maintained system state to solve the problem. However, to overcome this difficulty, it is often convenient to use its semi-regenerative properties (if any).

Semi-regenerative properties of the maintained system state – The implementation of a Markovian stationary inspection & maintenance policy on a Markovian deteriorating system brings semi-regenerative (or Markov renewal) properties to the process describing the maintained deterioration state. Under these Markovian assumptions, the maintenance decisions depend only on the observed deterioration level, and the maintenance actions modify the deterioration level, but have no effect on the degradation process itself, viz the underlying deterioration mechanism (obviously, this is only a working assumption in the proposed Markovian modelling framework, and in general a maintenance action can very well modify the deterioration mechanism itself). Consequently, the future behaviour of the system after an inspection depends only if the deterioration level revealed by the inspection. In other words, conditionally to the deterioration level measured upon an inspection, and at steady-state, the deterioration process before and after the inspection are independent and follow the same probability law. The embedded discrete-time process describing the system deterioration level at the inspection times is a Markov chain with stationary law π, [1], [3]. The maintained deterioration process is also a semi-regenerative process and two successive inspection times T_k and T_{k+1} define thus a Markov renewal (or semi-regenerative) cycle, see *Figure 5*. These semi-regenerative properties can then be used with profit both to compute the asymptotic criteria of maintenance performance and to determine the stationary law of the maintained system state.

Evaluation of the asymptotic performance criteria – To compute these criteria, the semi-regenerative property offers a solution more efficient than the classical renewal theorem. Instead of considering a renewal cycle, the study can be conducted on a Markov renewal cycle between to inspections, which simplifies considerably the analysis. But, the price to pay for this simplification is that we have to consider such a cycle at steady-state, viz the expectations in Equations (1) and (2) have to be computed with respect to the stationary law of the maintained system state (which has to be evaluated). For example, we have:

$$C_\infty = \lim_{t\to\infty} \frac{E[C(t)]}{t} = \frac{E[C(S)]}{E[S]} = \frac{E_\pi[C(T)]}{E_\pi[T]} \qquad (3)$$

where T is the time between two inspections at steady-state.

Determination of the stationary law of the maintained system state – The semi-regenerative property allows one to shorten significantly the study horizon: a Markov renewal cycle (between two inspections) is enough and it becomes easier to analyze its behaviour. It is thus possible to determine the transition probability law of the process, and then solving the invariance equation to obtain the stationary law π.

In conclusion, for condition-based maintenance policies, once the parametric structure has been chosen for the maintenance decision rule, the evaluation of the performance criteria relies mainly on the probabilistic modeling of the deterioration of the maintained system at steady-state, which is completely characterized by its stationary law. The derivation of the transition probability law and the stationary law of the maintained system state is thus the enabling key, and of course, the main difficulty of this modeling work. When compared to more classical maintenance models based on lifetime law, this point constitutes the additional modeling task. Once the steady state model of the maintained system state is available, it is generally easy to compute the performance criteria as a function of the maintenance parameters, and then to find the optimal values of these parameters. The use of these semi-regenerative techniques can be seen as an extension to continuous time and deterioration level of the semi-Markovian techniques presented by Gertsbakh in [10] as one of the most powerful tools to tackle the discrete time and discrete state condition-based maintenance problems.

In the following, we illustrate this modeling approach on two examples of condition-based maintenance policies.

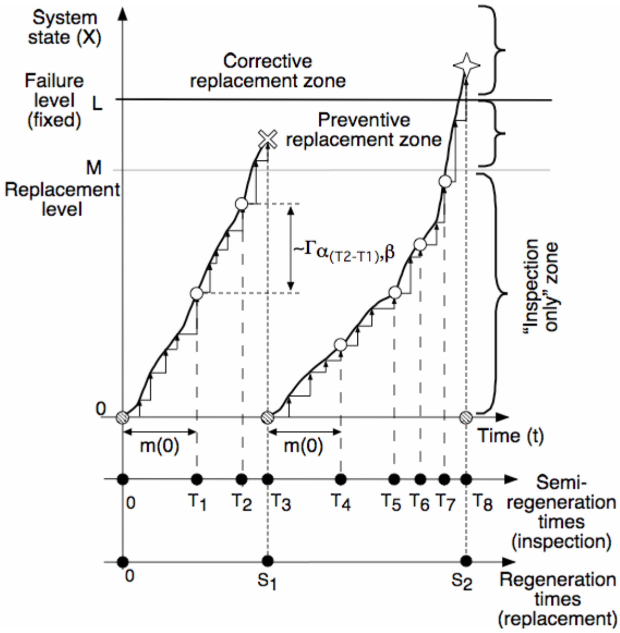

Figure 6. Schematic sample path of the maintained deterioration level

4. A continuous time condition-based inspection & replacement policy

This section presents the policy presented and studied in, e.g., [11].

4.1. Policy structure

Consider the following condition-based inspection and replacement policy. After the n^{th} inspection performed at time T_n, we have to decide whether the system should be replaced or not, and the time of the next inspection T_{n+1}. We adopt the decision rule (T_n^- refers to the time just before the maintenance action):

- If $X(T_n^-) \geq L$, the failed system is correctively replaced at a cost c_c and the unavailability duration since the previous inspection $d(t)$ generates a cost at a rate r_d. After replacement, the system is as good as new ($X(T_n) = 0$).
- If $M \leq X(T_n^-) < L$, the system is preventively replaced at a cost c_p (with $c_p \leq c_c$). After replacement, the system is as good as new ($X(T_n) = 0$).
- If $X(T_n^-) < M$, the system is left as it is ($X(T_n) = X(T_n^-)$).

- In any case, the time of the next inspection is computed as $T_{n+1} = T_n + \Delta T_n$ with $\Delta T_n = m(T_n)$, where $m(.)$ is the inspection scheduling function defined in Section 3.2, see also *Figure 4*.

All the maintenance actions are assumed to have a duration equal to zero, i.e. their duration is considered to be negligible and neglected in the model. With this decision structure, the two maintenance decision "variables" are the replacement threshold M and the function $m(.)$. *Figure 6* shows a sample path of the deterioration process of the maintained system under this policy.

4.2. Policy performance evaluation

The policy performance can be quantified by the total expected maintenance cost per time unit on an infinite time span due to both maintenance operations and unavailability of the maintained system, C_∞. The cumulated maintenance cost at time t is:

$$C(t) = c_i N_i(t) + c_p N_p(t) + c_c N_c(t) + r_d d(t)$$

where $N_i(t)$ is the number of inspections at time t and $N_p(t)$ (resp. $N_c(t)$) the number of preventive (resp. corrective) replacements at time t.
The choice of the policy parameters affects directly the number and the nature of the maintenance operations, and consequently the cumulated cost $C(t)$. The expected cost per time unit over an infinite time span C_∞ is given by:

$$C_\infty = c_i \lim_{t\to+\infty} \frac{E[N_i(t)]}{t} + c_p \lim_{t\to+\infty} \frac{E[N_p(t)]}{t} + c_c \lim_{t\to+\infty} \frac{E[N_c(t)]}{t} + r_d \lim_{t\to+\infty} \frac{E[d(t)]}{t}$$

At this point, all the difficulty is now to evaluate the four terms (limits and expectations) in the above expression. In order to simplify the analysis and as explained in Section 3.4., it can be noticed that the process $\{X(t), t \ge 0\}$ describing the maintained system state exhibits both regenerative and semi-regenerative properties:

- Each replacement returns the system to an as-good-as new state, and the deterioration process starts again with the same law. The process $\{X(t), t \ge 0\}$ is thus a regenerative process; the replacement times S_k are regenerative points and form a renewal process.
- After each inspection at time T_k, the future evolution of the maintained system state depends only on the observed deterioration level: conditionally to the deterioration level at T_k, the processes $\{X(t), t \ge 0\}$ and $\{X(t+T_k), t \ge 0\}$ have the same law. The process $\{X(t), t \ge 0\}$ is thus also a semi-regenerative process and the inspection times T_k are its semi-regenerative points. Moreover, the discrete-time stochastic process describing the system state at each inspection $\{Y(k) = X(T_k)\}_{k\ge 0}$ is a continuous state ($[0, M[$) Markov chain and the process $\{Y_k, T_k\}$ is a Markov renewal process. Since after an inspection, there is a non-zero probability for the system to return to the new state, the transition probability law of the process $\{Y(k)\}_{k\ge 0}$ is a convex combination of a pdf and a Dirac mass:

$$P(dy \mid x) = \bar{F}_{\alpha m(x),\beta}(M - x)\delta_0(dy) + f_{\alpha m(x),\beta}(y - x)\mathrm{I}_{\{x \le y > M\}} dy$$

where $f_{\alpha,\beta}$ and $\bar{F}_{\alpha,\beta}$ correspond respectively to the pdf and survival function associated to the Gamma law. Note that the two parts of $P(dy \mid x)$ correspond to two different evolution scenarios on a semi-renewal cycle: natural deterioration or return to 0 after replacement.

The regenerative property allows the use of classical renewal results, as described in Section 3.4.. However, the study of the maintained deterioration process remains difficult even if conducted on a renewal cycle and the computations of both $E[C(S)]$ and $E[S]$ are not easy because the processes $\{X(t), t \ge 0\}$ and $\{Y_k, T_k\}$ are "imbricated". In order to reduce further the analysis horizon, we can take advantage of the semi-regenerative property of the maintained deterioration process and use Equation (3) to compute the cost criterion. Now, all the difficulty comes from the evaluation of the stationary law π of the Markov chain $\{Y(k)\}_{k\ge 0}$ which can be obtained as the solution of the invariance equation:

$$\pi(.) = \int_{[0,M[} P(. \mid x)\pi(dx)$$

It can be shown [10] that the solution of this equation is also a convex combination of a pdf and a Dirac mass [1]:

$$\pi(dx) = a\delta_0(dx) + (1-a)b(x)dx$$

where

$$a = 1/(1 + \int_0^M B(x)dx),$$

$$b(y)=\frac{a}{1-a}B(y)$$

and $B(y)$ is solution of the renewal equation:

$$B(y)= f_{\alpha m(0),\beta}(y)+\int_0^y B(x) f_{\alpha m(x),\beta}(y-x)dx$$

Once the stationary law π is available, the cost criterion can be computed as (T is the time between two inspections at steady-state):

$$C_\infty = c_i \frac{E_\pi[N_i(T)]}{E_\pi[T]}+c_p \frac{E_\pi[N_p(T)]}{E_\pi[T]} + c_c \frac{E_\pi[N_c(T)]}{E_\pi[T]}+r_d \frac{E_\pi[d(T)]}{E_\pi[T]} \quad (4)$$

On a semi-regenerative cycle, one and only one inspection is performed; thus, we have $E_\pi[N_i(T)]=1$. The other quantities involved in Equation (4) can be computed as follows:

- Expected number of preventive replacements on a semi-regenerative cycle:

$$E_\pi[N_p(T)]= P_\pi(M \le X(T_1^-) < L) = \int_{[0,M[} (\bar{F}_{\alpha m(x),\beta}(M-x)-\bar{F}_{\alpha m(x),\beta}(L-x))\pi(dx)$$

- Expected number of corrective replacements on a semi-regenerative cycle:

$$E_\pi[N_c(T)]= P_\pi(X(T_1^-) \ge L) = \int_{[0,M[} \bar{F}_{\alpha m(x),\beta}(L-x)\pi(dx)$$

- Expected unavailability duration on a semi-regenerative cycle:

$$E_\pi[d(T)]= \int_{[0,M[}\left(E_x \int_0^T \mathrm{I}_{\{X(s)\ge L\}}ds\right)\pi(dx) = \int_{[0,M[}\left(\int_0^{m(x)} \bar{F}_{\alpha s,\beta}(L-x)ds\right)\pi(dx)$$

- Expected length of semi-regenerative cycle:

$$E_\pi[T]= \int_{[0,M[} m(x)\pi(dx)$$

4.3. Numerical example

For a numerical illustration of the use of this model to assess the performance of this policy, we consider a linear inspection scheduling function $m(x)=1+\max(n(x),0)$ where $n(x)$ is defined by $n(x)=a-(a/b)x$. In this case, the maintenance policy parameters to be optimized are a, b, and M. For the data set $\alpha=1$, $\beta=1$, $L=12$, $c_i=25$, $c_p=50$, $c_c=100$ and $r_d=250$, the optimal values (obtained by a classical optimization algorithm) are $a^*=5.5$, $b^*=5.5$ and $M^*=5.6$. *Figure 7* presents the iso-level cost curves in the plane (a,b).computed for $M=6$.

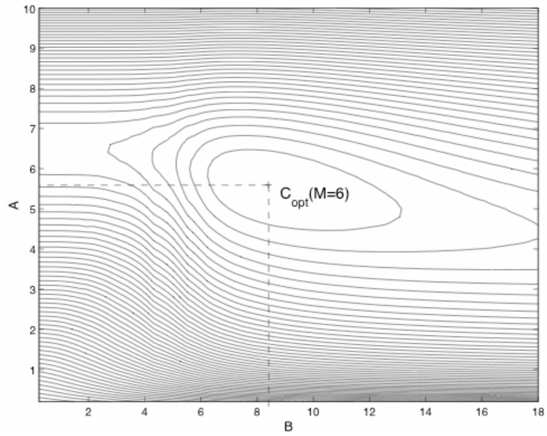

Figure 7. Iso-level maintenance cost curves for the condition-based inspection & replacement policy.

5. A discrete-time inspection, repair & replacement policy

In this section, we apply the proposed modelling approach to a more general policy in discrete time proposed in [2]. The considered maintenance policy includes partial repairs, together with inspections and as-good-as new replacements. The partial repairs do not renew completely the system, but they improve it by returning to a lower deterioration level. We also relax the assumption of instantaneous and immediate maintenance actions: during each maintenance action (even inspection) the system is stopped, and the maintenance action contribute in this sense to increase the system unavailability. Again, the model development proposed in this section illustrates the use of semi-regenerative techniques to solve the maintenance model.

5.1. Policy structure

The system deterioration is observed on a discrete-time grid $t_k = k\Delta t$, and the deterioration increment on a period Δt follows a Gamma law f. After an inspection at t_k revealing a deterioration level $X(t_k)=y$, we adopt a decision rule based on multi-level control limit rule involving $N+2$ thresholds. The first $N-1$ thresholds ξ_1 to ξ_{N-1} are used to schedule the inspections, see *Figure 3*. The thresholds ξ_N and ξ_{N+1} are respectively the repair and replacement thresholds. A threshold ζ is introduced to determine the depth repair and delimit the so-called "re-starting zone" in which the deterioration level must lie after a repair. We adopt the following decision rule, se *Figure 8*:

- *No action on the system and next inspection scheduling*: if $y \in [\xi_l, \xi_{l+1}[$ with $l = 0,...,N-1$, the system is left as it is and the next inspection is scheduled $(N-l)\Delta t$ later. This inspection incurs a cost c_i and lasts a random time τ_i.

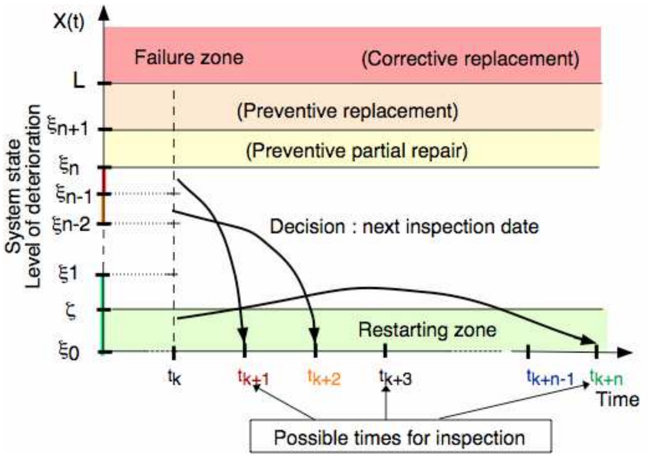

Figure 8. Parametric structure of the maintenance decision rule.

- *Repair:* if $y \in [\xi_N, \xi_{N+1}[$, the system undergoes a partial repair which improves its deterioration level by a random quantity Δy without renewing it completely. The deterioration level after repair is known and lies in the restarting zone, i.e. $y - \Delta y < \zeta$. If $(y - \Delta y) \in [\xi_l, \xi_{l+1}[$ with $l = 0,...,N-1$, the next inspection is scheduled $(N-l)\Delta t$ later. The repair cost $c_r(y, \Delta y)$ may depend on the efficiency (or depth) of the repair. The repair duration $\tau_r(y)$ may also depend on its efficiency.
- *Replacement:* if $y > \xi_{N+1}$, the system is undergoes and as-good-as new replacement, either preventively if $y \in [\xi_{N+1}, L]$, or correctively if $y \geq L$. The next inspection is scheduled $N\Delta t$ later. A preventive (resp. corrective) replacement incurs a cost c_p (resp. c_c) and stops the system for a random duration equal to τ_{rpc}.

Within this maintenance decision structure, the thresholds ξ_k allow the control of the balance between preventive and corrective actions, and the threshold ζ allows the control of the trade-off between the efficiency and cost of the repair actions.

Figure 9 presents a sample path of the maintained deterioration process under this policy.

5.2. Stochastic modelling of the maintained system state

For this policy again, the study, analysis and modelling of the behaviour of the maintained system state can be significantly simplified if we take advantage of the semi-regenerative properties of the corresponding process.

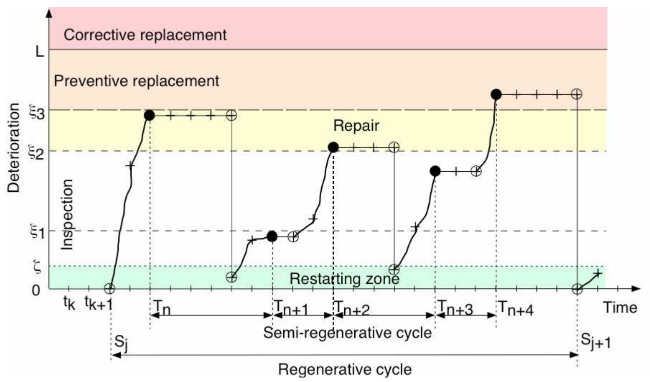

Figure 9. Sample path of the maintained system state for a 4-threshold policy $\xi_1, \xi_2, \xi_3, \zeta$ $(N = 2)$. ● marks the beginning of a maintenance operation; ❍ marks the end of an operation.

Because of the presence of perfect replacements, the maintained system state is a regenerative process. However, as shown in *Figure 9*, the renewal cycles (viz., between two replacements) can be very long and quite complex to describe. Consequently, this regenerative property does not help that much to solve the problem.

The system evolution after the beginning of any maintenance operation depends only on the level of deterioration y observed at its onset. Conditionally on y, the characteristics of the maintenance operation (length, cost, effect) and the future evolution of the system after this action do not depend on the past. The maintained deterioration process is thus a semi-regenerative process. The times T_n of the beginning of the maintenance operations are the semi-regeneration or Markov renewal points. The deterioration level at the beginning of the maintenance operations $\{Y_n = X(T_n), n \geq 0\}$ form the embedded Markov chain of the process; the Markov renewal points delimit independent cycle, conditionally on the Markov chain $\{Y_n, n \geq 0\}$.

In order to assess the performance of this maintenance policy, the horizon can be limited to a semi-regenerative cycle. A semi-regenerative cycle is completely characterized by the deterioration level y at the beginning of the maintenance action, and more precisely by the following quantities, all dependent on y: effect of the maintenance action on the deterioration level (Δy), maintenance action duration (τ), system deterioration level after maintenance ($y-\Delta y$) and time of the next inspection which marks the end of the cycle (the unknown system state at this time is denoted x). The evolution from the deterioration level y to the level x on the semi-regenerative cycle corresponds necessarily to one the three following scenarios.

- *Scenario 1 (inspection):* $y \in [\xi_l, \xi_{l+1}[$ with $l = 0,...,N-1$, an inspection is performed and the system is unavailable for a duration τ_i. The deterioration level is not altered by the inspection and the next inspection is scheduled $(N-l)\Delta t$ later; thus the deterioration increment $x-y$ until this next inspection follows the probability law $f^{(N-l)}$.
- *Scenario 2 (repair):* $y \in [\xi_N, \xi_{N+1}[$, a partial preventive repair is triggered which rejuvenates the system by a random quantity Δy, following a probability law $f_{\Delta y}$ defined on $[y-\zeta, y]$. The system is unavailable for a random duration $\tau_r(\Delta y)$. After repair, the degradation level $z = y - \Delta y$ lies in the restarting zone $[0, \zeta[$ and in an inspection zone $y \in [\xi_l, \xi_{l+1}[$, $l = 0,...,N-1$. An inspection is scheduled $(N-l)\Delta t$ later. The deterioration increment until this future inspection $x-(y-\Delta y)$ follows a probability law $f^{(N-l)}$.
- *Scenario 3 (replacement):* $y \geq \xi_{N+1}$, a replacement (preventive or corrective) is performed and the system is unavailable for a random duration $\tau_{rpc}(y)$. After replacement, the system is as-good-as new and the next inspection is scheduled $N\Delta t$ later. The deterioration increment until this next inspection x follows a probability law $f^{(N)}$.

The evolution of the maintained deterioration level at steady-state can be characterized by two quantities:

Stationary law of the maintained system state π which can be computed as the solution of the invariance equation

$$\pi(x) = \int_0^\infty \pi(y) F(x \mid y) dy$$

where the transition law $F(x|y)$ from a level y to a level x is constructed from the three above-mentioned scenarios as follows:

$$F(x \mid y) = \underbrace{\sum_{l=0}^{N-1} \mathrm{I}_{\{y \in [\xi_l, \xi_{l+1}[\}} f^{(N-l)}(x-y)}_{\text{scenario 1}}$$
$$+ \underbrace{\mathrm{I}_{\{y \in [\xi_N, \xi_{N+1}[\}} \sum_{l=0}^{N-1} \int_{y-\min(\xi_{l+1}, \zeta)}^{y-\xi_l} f_{\Delta y}(y_0) f^{(N-l)}(x-y+y_0) dy_0}_{\text{scenario 2}}$$
$$+ \underbrace{\mathrm{I}_{\{y \geq \xi_{N+1}\}} f^{(N)}(x)}_{\text{scenario 3}}$$

Probability $P^i(k,y)$ *of the length (in* Δt*) of a maintenance action initiated at the deterioration level* y*:* since the characteristics of a maintenance operation depends only on the deterioration level at its beginning, the probability $P^i(k,y)$ can be evaluated from the three scenarios as follows:

$$P^i(k,y) = \underbrace{\mathrm{I}_{\{y \in [0, \xi_N[\}} P_y(\tau_i \geq k)}_{\text{scenario 1}}$$
$$+ \underbrace{\mathrm{I}_{\{y \in [\xi_N, \xi_{N+1}[\}} \int_{y-\zeta}^{y} P_y(\tau_r(y_0) \geq k) f_{\Delta y}(y_0) dy_0}_{\text{scenario 2}}$$
$$+ \underbrace{\mathrm{I}_{\{y \geq \xi_{N+1}\}} P_y(\tau_{rpc}(y) \geq k)}_{\text{scenario 3}}$$

5.3. Policy performance evaluation

We can consider two performance criteria: the expected maintenance cost rate on an infinite time span and the asymptotic availability of the system.

Cost criterion: it is composed of the costs due to maintenance operations themselves $\Gamma_m(t)$, the cost of unavailability due to system failure $\Gamma_u(t)$ and the cost of inactivity of the system for maintenance $\Gamma_i(t)$:

$$C_\infty = \lim_{t \to \infty} \frac{E[\Gamma_m(t) + \Gamma_u(t) + \Gamma_i(t)]}{t}$$

Availability criterion: it is evaluated from the cumulated time spent in the operating phase $D_o(t)$ and the cumulated time spent in the failed state $D_u(t)$:

$$A_\infty = \lim_{t \to \infty} \frac{E[D_o(t) - D_u(t)]}{t}$$

Using the semi-regenerative properties and Equation (3), the criterion cost can be computed as (T is the time between two inspections at steady-state):

$$C_\infty = \frac{E_\pi[C(T)]}{E_\pi[T]} = \frac{E_\pi[\Gamma_m(T)] + E_\pi[\Gamma_i(T)] + E_\pi[\Gamma_u(T)]}{E_\pi[T]}$$

All the quantities involved in the expression of C_∞ can be evaluated numerically by integration from $\pi(x)$ and $P^i(k,y)$. Consider for example the cost associated to the maintenance operations, gathering costs of inspections, repairs and replacements:

$$E_\pi[\Gamma_m(T)] = E_\pi[C_{insp}(T)] + E_\pi[C_{repair}(T)] + E_\pi[C_{repl}(T)]$$

with

$$E_\pi[C_{insp}(T)] = c_i \int_0^{\xi_N} \pi(y)dy$$

$$E_\pi[C_{repair}(T)] = \int_{\xi_N}^{\xi_{N+1}} \left(\int_{y-\zeta}^{y} C_r(y, y_0) f_{\Delta y}(y_0) dy_0 \right) \pi(y) dy$$

$$E_\pi[C_{repl}(T)] = c_p \int_{\xi_{N+1}}^{L} \pi(y)dy + c_c \int_L^\infty \pi(y)dy$$

The availability criterion can be evaluated as:

$$A_\infty = \frac{E_\pi[D_o(T)] - E_\pi[D_u^+(T)]}{E_\pi[D_o(T)] + E_\pi[D_i(T)]}$$

Here again, all the quantities involved in the expression of A_∞ can be evaluated numerically by integration from $\pi(x)$ and $P^i(k,y)$.

Note however that a further modelling work is necessary to evaluate completely the criterions. For example, one has to determine the cost function $C_r(y, \Delta y)$ or the pdf $f_{\Delta y}$. Further information on the system, expert opinion or working assumptions have to be injected in the model at this point. After this criterion evaluation, the policy optimization requires algorithms adapted to the multi-criterion formulation of the problem, and to the number and nature of parameters to be optimized.

6. Possible extensions

The above-presented models can be extended in several directions. The deterioration has been assumed to be stationary; however, this is seldom the case in practice. The deterioration can undergo abrupt changes because, for example, of a varying environment or demand on the system. In this case, the condition-based maintenance policy has to be adapted:

An on-line monitoring procedure can be added to detect the changes in the deterioration process. When a change is detected, the maintenance thresholds are updated to match the actual deterioration mode. This approach offers a solution to adapt the maintenance decision to a varying deterioration process, but introduces new sources of uncertainties because of the false alarms generated by the monitoring/change detection procedure. The monitoring and maintenance parameters have to be jointly optimize in order to reach an overall optimal performance for the monitoring/maintenance policy, [8,20].

The changes in the deterioration process can be due to a stressful environment. If the effect of the environment on the deterioration process and if the environment can be monitored, better maintenance performances can be obtained if the environment monitoring information is introduced in the maintenance decision procedure [6].

However, in such cases, the maintenance modelling problem becomes more complex, and is not always solvable by analytical means, even with the semi-regenerative techniques and we may have to resort, even partially, to stochastic simulation techniques.

7. Conclusion

The two main lessons from this lecture could be:

1) to show the interest of condition-based maintenance policies based on “grey-box” deterioration models : such policies have allows the dynamic adaptation of the maintenance decisions to the temporal variability of the deterioration and the control of the system deterioration around a level where preventive and corrective maintenance costs reach an optimal balance;

2) to show the efficiency of semi-regenerative techniques which provide an efficient tool to solve maintenance modelling problems for “complex” maintenance policies.

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