An Extension of Line Spring Model For Vibration Analysis of Thin Isotropic Plate Containing Multiple Part-Through Cracks: an Analytical Approach

N. K. JAIN
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, nkjmanit@rediffmail.com

Ankur GUPTA*
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, ankurgupta1729@gmail.com

R. SALHOTRA
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, salhotra_rahul@rediffmail.com

A. M. RAWANI
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, amrawani@rediffmail.com

Shashank SONI
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, shashanksoninitr@gmail.com

Ankur MISHRA
Department of Mechanical Engineering, National Institute of Technology, Raipur, (C.G.) INDIA 492010, ankurmishra2793@gmail.com

P. V. JOSHI
Department of Mechanical Engineering, Shri Shankaracharya Technical Campus, SSGI, Bhilai, (C.G.) INDIA 490001, psad@rediffmail.com

Abstract

In the present study, the effect of multiple part-through cracks on the vibration characteristics of thin isotropic rectangular plate is presented. The proposed analytical model is developed using Kirchhoff's classical plate theory and the crack terms are formulated using the simplified Line Spring Model (LSM). The application of Berger's formulation helps to transform the derived governing equation in the form of well known Duffing equation with cubic nonlinearities and then the solution for final governing equation is obtained using Galerkin's method for two different boundary conditions. The fundamental frequency of the plate as affected by the number of cracks, cracks length, cracks orientation, and plate aspect ratio for different boundary condition is presented. It is found that the results obtained for natural frequencies are maximally affected by number of cracks, crack length and orientations.

Keywords: multiple cracks, Line Spring Model, vibration, partial crack

**Corresponding author*

1. Introduction

Thin plates are widely used as structural components and find many broad applications in all fields of engineering such as mechanical, civil, aerospace, aviation and ship building industries. The presence of singularities in the form of cracks and holes is a quite common situation which adversely affects the vibration characteristics of such plate structures. Thus, the analysis for vibration characteristics of the structural members is very important for developing techniques to detect damages. An exhaustive literature review on vibration of plates has been given by Leissa [1] in his monograph and later the author presented a review on vibration of rectangular plates [2]. A variety of literatures are available onvibrations of plates due to its simplicity of solution. 'Theories and Applications of Plate Analysis' [3] and 'Thin Plates and Shells' [4] are the references covering statics and dynamics of plates. According to Szilard [3], the solution techniques are largely dependent on the edge conditions and shape of plates. Warburton [5] applied the Rayleigh's method and calculated natural frequencies and mode shapes of rectangular plates. The work of Dawe and Roufaeil [6] shows the application of the Rayleigh-Ritz method based on Mindlin plate theory to approximate the natural frequencies of lateral vibration of isotropic plates including the effects of shear deformation and rotary inertia. Kitipornchai et al. [7] applied the Rayleigh-Ritz method to study free vibrations of thick rectangular plates. Their approach was based on the Mindlin theory and they obtained Eigen value equations by minimizing the energy function. Domagalski [8] presented the analysis for nonlinear vibrations of beams with periodically varying material properties. They determined the linear natural frequencies and mode shapes for the vibrating beams. Huang et al. [9] applied the famous Ritz method to analyze the free vibrations of rectangular plates with internal cracks or slits and proposed a new set of admissible functions to retain the important and useful feature of the Ritz method providing the upper bounds on exact natural frequencies. Jędrysiak and Pazera [10] studied the free vibration problem of thin microstructured plates made of functionally graded material.Khadem and Rezaee [11] introduced a new functions named "modified comparison functions" and used for vibration analysis of a simply supported rectangular cracked plate assuming that the crack having an arbitrary length, depth and location is parallel to one side of the plate. Krawczuk et al. [12] developed a rectangular plate finite element with a through crack based on elasto-plastic fracture mechanics and the finite element method by considering that the crack is non-propagating and open. Stahl and Keer [13] presented Eigen value problems of cracked rectangular plates pertaining to vibration and buckling. They formulated the problem as dual series equations and reduced to homogeneous Fredholm integral equations of the second kind and compared numerical results for the natural frequencies and moment distributions with the work of other investigators. Liew et al. [14] reported an investigation on the vibrational behavior of cracked rectangular plates and carried out vibration analysis for plates with a crack (i) emanating from an edge or (ii) centrally located. They generated a governing Eigen value equation by complete coupling process and solved to obtain the vibration frequencies and compared these results, where possible with the work of other investigators Wu and Shih [15] studied the dynamic instability and nonlinear vibrations of simply supported plates containing edge cracks using von Karman theory. Their work shows the application of Galerkin's method and the harmonic balance method to obtain solution of the nonlinear model. They concluded that the increase in relative crack length decreases the natural frequency; also the vibration response is dependent on location of crack, plate aspect ratio and in-plane loading conditions. Xiao et al. [16] employed the Galerkin's method and the harmonic balance method to solve nonlinear equations of vibration. Their work shows the formulation of nonlinear equations based on Hamilton principle and Reissner theory for moderately thick cracked rectangular plate. The Line Spring Model (LSM) was proposed by Rice and Levy [17] for calculating the stress intensity at the crack tip of the plate. They represented the part through line crack as a continuously distributed spring with stretching and bending compliances. The three dimensional surface crack problem was reduced to two dimensional problem in which the constraining effects of the net ligament were incorporated in terms of moment and in-plane load on the crack surface. King [18] presented a simplified line-spring model along with detailed illustration of its application and accomplished the simplification by replacing the crack front with a crack of constant depth. Maruyama and Ichinomiya [19] used holographic interferometry and performed experiments on clamped rectangular plates to study the effect of slit length, location and inclination on natural frequencies and mode shapes of vibration. Huang and Leissa [20] introduced special displacement function in the Ritz method for free vibration analysis of rectangular plates with inclined side cracks. They investigated the effect of crack location and crack length on natural frequencies of vibrating plates based on the classical plate theory. Later, Huang et al. [21] presented a set of special functions for the vibration analysis of thick rectangular plate using Mindlin theory based Ritz formulation. Israr et al. [22] employed the LSM to obtain relations between nominal tensile and bending stresses at the crack location and at the far sides of the plate. They developed an analytical model for vibration analysis of cracked rectangular plate based on the classical plate theory. They employed Berger formulation for in-plane forces which rendered the non-linear analytical model. It can be concluded from their work that the natural frequencies go on decreasing as the crack length increases. Extending the work of Israr et al. [22], Ismail and Cartmell [23] developed an analytical model for vibration analysis of a cracked isotropic rectangular plate considering various angular orientation of a crack for three different boundary conditions i.e. all edges are simply supported (SSSS), two adjacent edges are simply supported and other two adjacent edges are clamped (CCSS) and third boundary condition, two edges are clamped and two other adjacent edges are free (CCFF). They extended the Berger formulation and included the effect of crack inclination into it by considering in-plane shear. The authors established relations for moment and in-plane force due to orientation of the crack and validated their findings with experimentation. They concluded that the vibration frequencies are dependent on the orientation of crack located at the centre of the plate. Recently, Joshi et al. [24] developed the analytical model for vibration analysis of isotropic plate containing two perpendicular partial cracks parallel to either edges of the plate considering the different location of partial cracks along the thickness of the plates. Further extending their research Joshi et al. [25, 26] presented results for vibration analysis of cracked orthotropic plate [25] and effect of thermal environment on vibration characteristics of cracked isotropic plate [26]. Gupta et al. [27] used the LSM and developed an analytical model for vibration analysis of cracked isotropic and FGM micro plate. They used classical plate theory in conjunction with modified couple stress theory and concluded that results for fundamental frequencies in case of micro plate are always higher for modified couple stress theory when compared to classical one. Further, they showed the effect of fibre orientation on vibration characteristics of cracked orthotropic micro-plate [28].

Review of literature shows the conventional and finite element formulations have been employed to study the linear and non-linear vibration characteristics of cracked plates. The development of analytical model for nonlinear vibrations of isotropic rectangular plate containing partial cracks and crack with various angular orientations have been studied by many researchers (Ref [22-24]). However, the literature lacks in analytical modelling for vibration analysis of isotropic plate containing multiple cracks of arbitrary orientation. Therefore, the present work addresses the effects of multiple cracks and their arbitrary orientation on vibration characteristics of the cracked plate by proposing a new analytical model.

The present work on multiple cracks references the analytical model developed by Israr et al. [22] for an isotropic plate containing a single crack which is extended by Joshi et al. [24] for two perpendicular cracks and Ismail et al. [23] for a variably oriented single crack. Further extending the recently developing field of vibration analysis of cracked plates, a new analytical model is proposed for a rectangular plate containing centrally located multiple concurrent cracks with different crack orientation. Equilibrium principle based on the Kirchhoff's Classical Plate Theory is used for deriving the equation of motion of the plate. The crack(s) considered in the present formulation are in the form of continuous line and the effect of rotary inertia is neglected. The crack terms are formulated by appropriate compliance coefficients and the relationship between nominal tensile and bending stress at the far sides of the plate and at the crack location is established using simplified Line Spring Model. The analytical model is developed by incorporating the additional twisting moment and in-plane forces due to presence of cracks. The in-plane deflections in x and y directions are assumed to be restricted by boundary conditions. Galerkin's method is employed to convert the equation of motion into Duffing equation from which the natural frequencies are evaluated for the cracked plate. The plate under consideration is shown in Fig. 1. Dimensions of the plate along x, y and z direction are L_1, L_2 and h respectively. 'e' is the depth of crack along the thickness of the plate. 2a, 2b, 2c and 2d are the crack lengths of the crack 'A', crack 'B', crack 'C' and crack 'D' respectively. Out of the four cracks, crack 'A' and crack 'B' are mutually perpendicular to each other and parallel to either edges of the plate whereas, crack 'C' and crack 'D' are at arbitrary orientation. The depth of the crack 'e' is assumed to be constant. Two boundary conditions, all edges simply supported (SSSS) and two edges clamped while two simply supported (CCSS) are considered for presenting the results. Results can also be obtained for other boundary conditions as well.

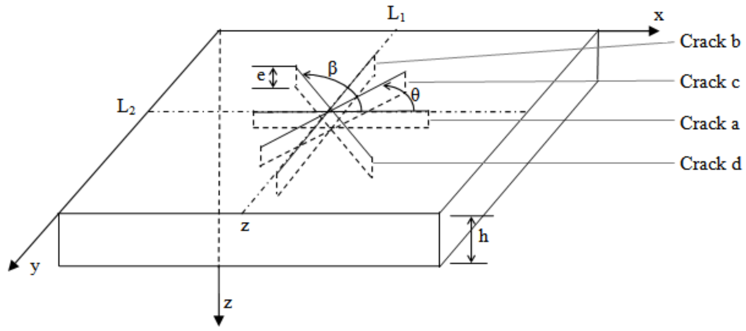

Figure 1. Isotropic plate containing multiple cracks

2. Governing equation

The governing equation of an intact rectangular plate based on the Classical plate theory considering the in-plane forces is rigorously treated in literature [3, 4].

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho h \frac{\partial^2 w}{\partial t^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + P_z \tag{1}$$

Where P_z is the lateral load per unit area, w is the transverse deflection in z direction, ρ is the density, h is the thickness of the plate and $N_x, N_y, N_{xy} = N_{yx}$ are the in-plane forces per unit length. D is flexural rigidity defined by $D = \frac{Eh^3}{12(1-\nu^2)}$, where E is the modulus of elasticity of isotropic material, and v is the Poisson's ratio. By using the force and moment equilibrium equations, the governing equation of motion of a rectangular plate with two mutually perpendicular cracks parallel to either edges of the plate and two variably oriented part-through cracks located at the centre of the plate is given by (Detailed derivation is given in Appendix A):

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho h \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_{\cancel{y}}}{\partial y^2} + \frac{\partial^2 M_{\cancel{x}}}{\partial x^2} + 2\frac{\partial^2 M_{\cancel{xy}}}{\partial x \partial y} + N_{\cancel{x}}\frac{\partial^2 w}{\partial x^2} + N_{\cancel{y}}\frac{\partial^2 w}{\partial y^2} + 2N_{\cancel{xy}}\frac{\partial^2 w}{\partial x \partial y} + P_z \tag{2}$$

Where $M_{\cancel{x}}$, $M_{\cancel{y}}$, $M_{\cancel{xy}}$ are bending moments per unit length and $N_{\cancel{x}}$, $N_{\cancel{y}}$, $N_{\cancel{xy}}$ are the membrane forces per unit length due to the presence of cracks.

2.1. *Crack terms*

The formulation of crack terms for the plate containing multiple part-through cracks is carried out using the application of Line Spring Model (LSM). Israr et al. [22] has initiated the use of Line Spring Model and applied the relationship of the stresses at the crack location and at far sides of the plate to formulate the crack terms. On extending the work of Ref. [22] and modifying the equation of motion to accommodate the effect of crack orientation Ismail and Cartmell [23] obtained new relations for the nominal tensile and bending stresses for the plate containing variably orientated single crack. Thus, in order to develop an analytical model for the plate containing multiple part-through cracks it is necessary to obtain new relationship of the stresses. Applying the Line Spring Model for the two variably oriented cracks, we find the out-plane transformation of stresses along the axis of plane "$x_C - y_C$" and "$x_D - y_D$" as depicted in Fig. 2.

Considering the uniaxial state of stresses and taking $m_x$and $\sigma_x$equal to zero for variably oriented cracks, a plane transformation of the stresses to the "$x_C - y_C$" plane can be expressed as:

$$\left.\begin{aligned} \sigma_{x_C} &= \frac{\sigma_y}{2}(1-\cos 2\theta) \\ \sigma_{y_C} &= \frac{\sigma_y}{2}(1+\cos 2\theta) \\ \tau_{xy_C} &= \frac{\sigma_y}{2}\sin 2\theta \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} m_{x_C} &= \frac{m_y}{2}(1-\cos 2\theta) \\ m_{y_C} &= \frac{m_y}{2}(1+\cos 2\theta) \\ m_{xy_C} &= \frac{m_y}{2}\sin 2\theta \end{aligned}\right\} \tag{4}$$

Similarly, by transformation of stresses along the "$x_D - y_D$", we get:

$$\left.\begin{aligned} \sigma_{x_D} &= \frac{\sigma_y}{2}-\frac{\sigma_y}{2}\cos 2\beta \\ \sigma_{x_D} &= \frac{\sigma_y}{2}-\frac{\sigma_y}{2}\cos 2\beta \\ \tau_{xy_D} &= \frac{\sigma_y}{2}\sin 2\beta \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} m_{x_D} &= \frac{m_y}{2}-\frac{m_y}{2}\cos 2\beta \\ m_{y_D} &= \frac{m_y}{2}+\frac{m_y}{2}\cos 2\beta \\ m_{xy_D} &= \frac{m_y}{2}\sin 2\beta \end{aligned}\right\} \tag{6}$$

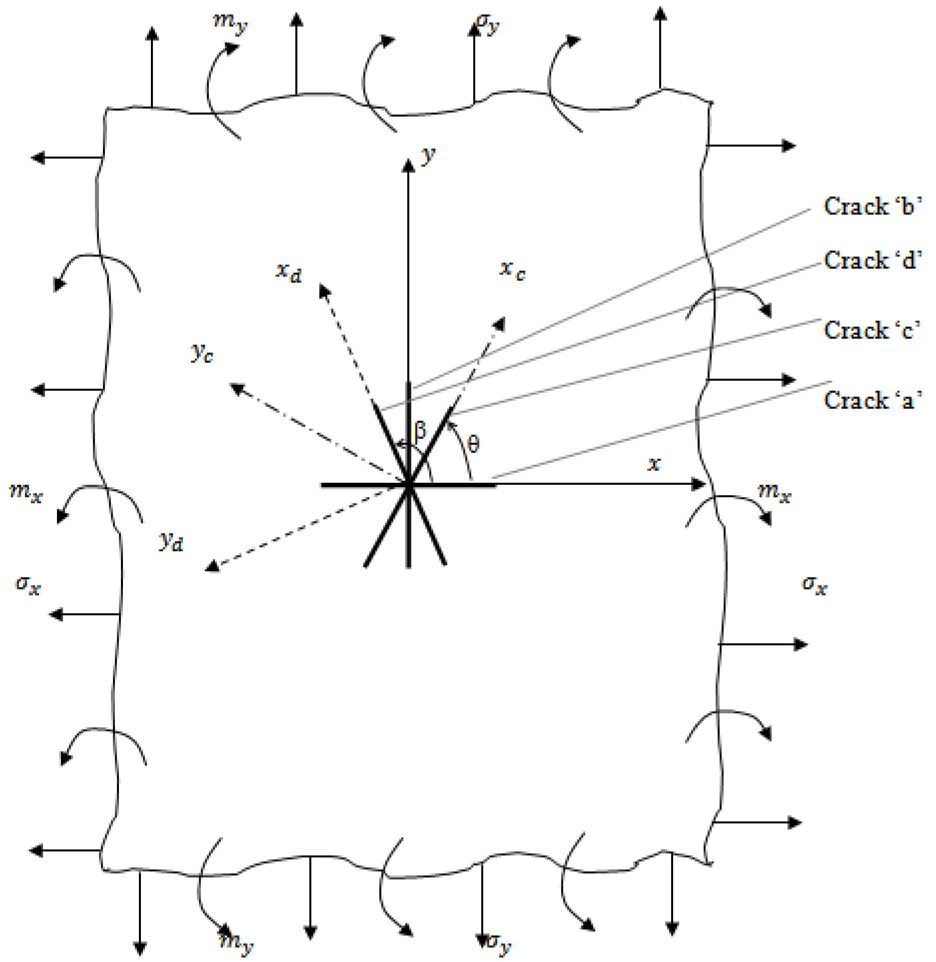

Figure 2. Tensile and bending stresses acting on the plate (Line Spring Model)

Considering the uniaxial loading, the relationship between the normal tensile and bending stress ($\sigma_{\cancel{mn}}$ and $m_{\cancel{mn}}$) at the crack location and the normal tensile and bending stress (σ_{mn} and m_{mn}) at the far sides of the plate with multiple cracks is given as

$$\sigma_{\cancel{mn}} = \frac{2a}{(6\alpha_{tb} + \alpha_{tt})(1-\nu^2)h + 2a}\sigma_{mn} \tag{7}$$

$$m_{\cancel{mn}} = \frac{2a}{3\left(\frac{\alpha_{tb}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2a}m_{mn} \tag{8}$$

In the tangential direction the relationship between the tangential tensile and bending stress ($\sigma_{\cancel{pq}}$ and $m_{\cancel{pq}}$) at the crack location and the tangential tensile and bending stress (σ_{pq} and m_{pq}) at the far sides of the plate is given as

$$\sigma_{\cancel{pq}} = \frac{2a}{(6c_{tb} + c_{tt})(1+\nu)h + 2a}\sigma_{pq} \tag{9}$$

$$m_{\cancel{pq}} = \frac{2a}{3\left(\frac{c_{tb}}{6} + c_{bb}\right)(1+\nu)h + 2a} m_{pq} \tag{10}$$

Where, α_{bb} and c_{bb} are the dimensionless bending or twisting compliances, α_{tt} and c_{tt} are the dimensionless stretching compliances and $\alpha_{tb} = \alpha_{bt}$ and $c_{tb} = c_{bt}$ are the dimensionless stretching-bending or stretching-twisting compliance coefficients at the crack centre, respectively. These compliance coefficients are rigorously used and can be found in literature (Ref. [22-24, 27]). The above relationship of tensile and bending stresses can be expressed in form of tensile and bending force effects. Therefore, Eq. (8) – (10) can be stated in terms of forces and moments for all the four cracks which are given as

$$N_{\cancel{x}} = \frac{2b}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2b} N_x \tag{11}$$

$$M_{\cancel{x}} = \frac{2b}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2b} M_x \tag{12}$$

$$\begin{aligned} N_{\cancel{y}} = {} & \frac{2a}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} N_y + \frac{c(1+\cos 2\theta)}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} N_y \\ & + \frac{d(1+\cos 2\beta)}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} N_y \end{aligned} \tag{13}$$

$$\begin{aligned} M_{\cancel{y}} = {} & \frac{2a}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2a} M_y \\ & + \frac{c(1+\cos 2\theta)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2c} M_y \\ & + \frac{d(1+\cos 2\beta)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2d} M_y \end{aligned} \tag{14}$$

$$N_{xy} = \frac{c\sin 2\theta}{(6C_{bt} + C_{tt})(1-\nu^2)h + 2c} N_y + \frac{d\sin 2\beta}{(6C_{bt} + C_{tt})(1-\nu^2)h + 2d} N_y \tag{15}$$

$$M_{\cancel{xy}} = \frac{c \sin 2\theta}{3\left(\frac{C_{bt}}{6} + C_{bb}\right)(3+\nu)(1-\nu)h + 2c} M_y + \frac{d \sin 2\beta}{3\left(\frac{C_{bt}}{6} + C_{bb}\right)(3+\nu)(1-\nu)h + 2d} M_y \tag{16}$$

These compliance coefficients are functions of crack depth and plate thickness. Since, the cracks cause reduction in overall stiffness of the plate, thus Eqs. (11) to (16) are employed with a negative sign, as discussed in literature [23, 24]. Expressing the terms $N_{\cancel{x}}$, $N_{\cancel{y}}$,$M_{\cancel{x}}$, $M_{\cancel{y}}$, $N_{\cancel{xy}}$, $M_{\cancel{xy}}$ from Eq. (11) to (16) and substituting M_x, M_y and M_{xy} in terms of deflection w, the governing equation of plate containing multiple cracks can be expressed as,

$$\begin{aligned}
D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) &= -\rho h \frac{\partial^2 w}{\partial t^2} + \frac{2aD\left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2a} \\
&+ \frac{cD(1+\cos 2\theta)\left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2c} \\
&+ \frac{dD(1+\cos 2\beta)\left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2d} \\
&+ \frac{2bD\left(\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2b} \\
&+ \frac{cD \sin 2\theta\left(\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2}\right)}{3\left(\frac{C_{bt}}{6} + C_{bb}\right)(3+\nu)(1-\nu)h + 2c}
\end{aligned} \tag{17}$$

$$+\frac{dD\sin 2\beta\,(\frac{\partial^4 w}{\partial y^4}+\ \nu\frac{\partial^4 w}{\partial x^2\partial y^2})}{3\left(\frac{C_{bt}}{6}+\ C_{bb}\right)(3+\nu)(1-\nu)h+2d}$$

$$-\frac{2a}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}N_y\frac{\partial^2 w}{\partial y^2}$$

$$-\frac{c(1+cos\,2\theta)}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}N_y\frac{\partial^2 w}{\partial y^2}$$

$$-\frac{d(1+\cos 2\beta)}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}N_y\frac{\partial^2 w}{\partial y^2}$$

$$-\frac{2b}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2b}N_x\frac{\partial^2 w}{\partial x^2}$$

$$-\frac{c\,sin\,2\theta}{(6C_{bt}+\ C_{tt})(1-\nu^2)h+2c}N_y\frac{\partial^2 w}{\partial x\partial y}$$

$$-\frac{d\sin 2\beta}{(6C_{bt}+\ C_{tt})(1-\nu^2)h+2d}N_y\frac{\partial^2 w}{\partial x\partial y}+P_z$$

2.2. *Solution of governing equation*

Galerkin's method is a weighted residual technique to obtain solution of integral and differential equations and has wide applicability to global phenomenon like vibrations. In this case, the lateral deflection is a function of in-plane coordinates and time which can be separated by the Galerkin's method to obtain approximate solution. Such application of Galerkin's method is evident in the recent literature for vibration analysis of cracked plates [22, 24, 27]. Hence on applying the Galerkin's method, the general solution for transverse deflection of the plate can be written as

$$w(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\,X_m Y_n \psi_{mn}(t) \tag{18}$$

Where, X_m and Y_n are characteristic or modal functions of cracked plate in x and y direction respectively. A_{mn} is arbitrary amplitude and $\psi_{mn}(t)$ is time dependent modal coordinate. Two boundary conditions SSSS and CCSS are considered in this work. The appropriate expressions for the characteristic or modal functions satisfying the stated boundary conditions are given below in Eq. (19) and (20).

1) SSSS: all edges simply supported

$$\left.\begin{aligned} X_m &= \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L_1}\right) \\ Y_n &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{L_2}\right) \end{aligned}\right\} \quad (19)$$

2) CCSS: Two adjacent edges clamped and the other two simply supported

$$\left.\begin{aligned} X_m &= \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{m\pi x}{2L_1}\right) \\ Y_m &= \sum_{m=1}^{\infty} \sin\left(\frac{m\pi y}{L_1}\right) \sin\left(\frac{m\pi y}{2L_1}\right) \end{aligned}\right\} \quad (20)$$

Using Berger's formulation it is possible to define N_x and N_y in terms of the lateral deflection. The formulation simplifies the analysis of deflection by neglecting the strain energy due to the second invariant of the middle surface strains. The in-plane forces N_x and N_y can be expressed in terms of middle surface strains as discussed in Ref. [24]. Hence on expressing the middle surface strain in terms of deflection, the in-plane forces can be written as

$$\begin{aligned} N_x = D \frac{6}{h^2 L_1 L_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{L_1} \int_0^{L_2} \Big\{ (\frac{\partial X_m}{\partial x})^2 {Y_n}^2 \\ + \nu (\frac{\partial Y_n}{\partial y})^2 {X_m}^2 \Big\} dx\, dy\, {A_{mn}}^2 \psi_{mn}(t)^2 \end{aligned} \quad (21)$$

$$\begin{aligned} N_y = D \frac{6}{h^2 L_1 L_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{L_1} \int_0^{L_2} \Big\{ (\frac{\partial Y_n}{\partial y})^2 {X_m}^2 \\ + \nu (\frac{\partial X_m}{\partial x})^2 {Y_n}^2 \Big\} dx\, dy\, {A_{mn}}^2 \psi_{mn}(t)^2 \end{aligned} \quad (22)$$

On substituting Eq. (21) and Eq. (22) and applying the definition of $w(x, y, t)$ to Eq. (17), multiplying both sides by X_m, Y_n and then integrating over the plate area, the governing equation becomes

$$
\begin{aligned}
&\frac{\rho h}{D}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\int_0^{L1}\int_0^{L2} X_m^2 Y_n^2\,dx\,dy\,\frac{\partial^2\psi_{mn}(t)}{\partial t^2}\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\,\psi_{mn}(t)\int_0^{L1}\int_0^{L2}\Bigg\{(X_m^{iv}Y_n+2X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)\\
&-\frac{2a(\nu X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)}{3\left(\frac{\alpha_{bt}}{6}+\ \alpha_{bb}\right)(3+\nu)(1-\nu)h+2a}-\frac{c(1+\cos 2\theta)(\nu X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)}{3\left(\frac{\alpha_{bt}}{6}+\ \alpha_{bb}\right)(3+\nu)(1-\nu)h+2c}\\
&-\frac{d(1+\cos 2\beta)(\nu X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)}{3\left(\frac{\alpha_{bt}}{6}+\ \alpha_{bb}\right)(3+\nu)(1-\nu)h+2d}-\frac{c\sin 2\theta(\nu X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)}{3\left(\frac{C_{bt}}{6}+\ C_{bb}\right)(3+\nu)(1-\nu)h+2c}\\
&-\frac{d\sin 2\beta\,(\nu X_m^{ii}Y_n^{ii}+Y_n^{iv}X_m)}{3\left(\frac{C_{bt}}{6}+\ C_{bb}\right)(3+\nu)(1-\nu)h+2d}\\
&-\frac{2b(\nu X_m^{ii}Y_n^{ii}+X_m^{iv}Y_n)}{3\left(\frac{\alpha_{bt}}{6}+\ \alpha_{bb}\right)(3+\nu)(1-\nu)h+2b}\Bigg\}X_mY_n\,dx\,dy\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} {A_{mn}}^3\,\psi_{mn}(t)^3\int_0^{L1}\int_0^{L2}\Bigg\{\frac{2aB_{2mn}Y_n^{ii}Y_n\,X_m^2}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}\\
&+\ \frac{2bB_{1mn}X_m^{ii}X_mY_n^2}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2b}+\frac{2aB_{2mn}Y_n^{ii}Y_n\,X_m^2}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}\\
&+\ \frac{c(1+\cos 2\theta)B_{2mn}Y_n^{ii}Y_n\,X_m^2}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}+\frac{d(1+\cos 2\beta)B_{2mn}Y_n^{ii}Y_n\,X_m^2}{(6\alpha_{bt}+\ \alpha_{tt})(1-\nu^2)h+2a}\\
&+\ \frac{c\,(\sin 2\theta)\,B_{2mn}X_m^iY_n^iX_mY_n}{(6C_{bt}+\ C_{tt})(1-\nu^2)h+2c}+\frac{d\,(\sin 2\beta)\,X_m^iY_n^iX_mY_n}{(6C_{bt}+\ C_{tt})(1-\nu^2)h+2d}\Bigg\}dx\,dy=0
\end{aligned}
\tag{23}
$$

where,

$$\left.\begin{aligned}
\mathrm{B}_{1\mathrm{mn}} &= \frac{6}{h^2 L_1 L_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{L_1}\int_0^{L_2}\left\{(\frac{\partial X_m}{\partial x})^2 {Y_n}^2 + \nu(\frac{\partial Y_n}{\partial y})^2 {X_m}^2\right\} dx\, dy \\
\mathrm{B}_{2\mathrm{mn}} &= \frac{6}{h^2 L_1 L_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^{L_1}\int_0^{L_2}\left\{\left(\frac{\partial Y_n}{\partial y}\right)^2 {X_m}^2 + \nu\left(\frac{\partial X_m}{\partial x}\right)^2 {Y_n}^2\right\} dx\, dy
\end{aligned}\right\} \tag{24}$$

The modal peak amplitude A_{mn} is normalized to unity. The lateral load P_z is neglected here for free vibrations. Eq. (23) may be expressed in the form of well-known Duffing equation as

$$M_{mn}\frac{\partial^2 \psi_{mn}(t)}{\partial t^2} + K_{mn}\psi_{mn}(t) + G_{mn}\psi_{mn}(t)^3 = 0 \tag{25}$$

where,

$$M_{mn} = \frac{\rho h}{D}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\int_0^{L1}\int_0^{L2} {X_m}^2 {Y_n}^2\, dx\, dy \tag{26}$$

$$\begin{aligned}
K_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\int_0^{L1}\int_0^{L2} \Bigg\{ & (X_m^{iv} Y_n + 2X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m) \\
& - \frac{2a(\nu X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2a} \\
& - \frac{c(1+\cos 2\theta)(\nu X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2c} \\
& - \frac{d(1+\cos 2\beta)(\nu X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2d} \\
& - \frac{c\sin 2\theta(\nu X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m)}{3\left(\frac{C_{bt}}{6} + C_{bb}\right)(3+\nu)(1-\nu)h + 2c} \\
& - \frac{d\sin 2\beta\,(\nu X_m^{ii} Y_n^{ii} + Y_n^{iv} X_m)}{3\left(\frac{C_{bt}}{6} + C_{bb}\right)(3+\nu)(1-\nu)h + 2d} \\
& - \frac{2b(\nu X_m^{ii} Y_n^{ii} + X_m^{iv} Y_n)}{3\left(\frac{\alpha_{bt}}{6} + \alpha_{bb}\right)(3+\nu)(1-\nu)h + 2b}\Bigg\} X_m\, Y_n\, dx\, dy
\end{aligned} \tag{27}$$

$$G_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}{}^{3} \int_{0}^{L1}\int_{0}^{L2} \left\{ \frac{2aB_{2mn}Y_n{}^{ii}Y_n\ X_m{}^{2}}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} + \frac{2bB_{1mn}X_m{}^{ii}X_m\ Y_n{}^{2}}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} + + \frac{c(1+\cos 2\theta)B_{2mn}Y_n^{ii}Y_n X_m^2}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} + \frac{d(1+\cos 2\beta)B_{2mn}Y_n^{ii}Y_n X_m^2}{(6\alpha_{bt} + \alpha_{tt})(1-\nu^2)h + 2a} + \frac{c\,(\sin 2\theta)\,B_{2mn}X_m^{i}Y_n^{i}X_mY_n}{(6C_{bt} + C_{tt})(1-\nu^2)h + 2c} + \frac{d\,(\sin 2\beta)\,X_m^{i}Y_n^{i}X_mY_n}{(6C_{bt} + C_{tt})(1-\nu^2)h + 2d} \right\} dx\,dy \tag{28}$$

The natural frequency ω can be calculated from Eq. (26) and (27) as

$$\omega_{mn} = \sqrt{\frac{K_{mn}}{M_{mn}}} \tag{29}$$

Eq. (26) shows the mass of the cracked plate which is same as the mass of intact plate. This is so because all the cracks are in the form of continuous line and due to presence of continuous line cracks the reduction in mass of the plate is negligible. Eq. (27) shows the stiffness of the cracked plate having the last six terms being the reduction in stiffness due to the presence of four cracks.

3. Results and discussion

This section presents new results for fundamental frequencies of cracked rectangular isotropic plate with two perpendicular cracks parallel to the either edges of the plate and two variably oriented cracks in the form of continuous line and located at the centre. The first mode natural frequencies are presented for different parameters such as crack length, crack orientation, plate aspect ratio and boundary conditions. Two different boundary conditions (SSSS and CCSS) and three plate aspect ratios (L_1/L_2) are considered for analysis. The material properties taken are $E = 7.03 \cdot 10^{10}$ N/m^2, density $\rho = 2660$ kg/m^3, Poisson's ratio $\nu = 0.33$, and plate thickness $h = 0.01$ m. The depth of cracks 'e' throughout this work is taken as 0.006 m.

The validation study of the proposed model is carried out by comparing the results of natural frequencies of an intact plate (Table 1 (a)) and with single crack (Table 1 (b)) with two perpendicular cracks (Table 2) and also for a plate with a single variably oriented crack (Table 3). Table 1 shows the first mode natural frequencies of an intact plate as well as plate with a single crack for three plate aspect ratios (L_1/L_2), two boundary conditions

(SSSS and CCSS) and two half crack lengths, a = 0.01 m, a = 0.025 m. The present results are compared in Table 1 by considering only one crack length 'a' (b = c = d = 0). The obtained results are in very close agreement when they are compared with the existing literature.

Table 1(a). First mode natural frequency (rad/sec) of intact plate for two different boundary conditions

BC*	Plate dimension		Intact plate		
	L_1	L_2	Present	Ref.[22][24]	% Difference
SSSS	1	1	310.31	310.30	0.0032
	0.5	1	775.90	775.90	0.0000
	1	0.5	775.91	775.90	0.0013
CCSS	1	1	445.73	445.70	0.0067
	0.5	1	1161.80	1161.80	0.0000
	1	0.5	1161.80	1161.80	0.0000

**Boundary Condition*

Table 1(b). First mode natural frequency (rad/sec) of cracked plate for two different boundary conditions

BC*	Plate Dimension		Cracked plate					
			a = 0.01			a = 0.025		
	L_1	L_2	Present	Ref. [24]	% Difference	Present	Ref. [24]	% Difference
SSSS	1	1	301.11	302.10	0.33	291.82	293.50	0.57
	0.5	1	769.44	770.10	0.09	763.17	764.30	0.15
	1	0.5	727.01	732.70	0.78	675.88	685.60	1.42
CCSS	1	1	430.79	432.50	0.40	415.74	418.50	0.66
	0.5	1	1153.30	1154.20	0.08	1145.10	1146.50	0.12
	1	0.5	1080.45	1089.90	0.87	994.51	1011.10	1.64

**Boundary Condition*

Similarly, Table 2 shows natural frequencies of a plate containing two perpendicular cracks parallel to the either edges of the plate. Results are compared by considering two crack lengths 'a' and 'b' in the proposed model (c = d = 0). It is seen from Table 2, the presence of two surface cracks in a plate further decreases the natural frequency as compared to single crack.

The results are also validated for a plate containing single crack of variable orientation (Ref. [23]). Considering two mutually perpendicular cracks '$a = b = 0$' and one arbitrarily oriented crack '$d = 0$' and presenting the result only for single arbitrarily oriented crack 'c', the proposed model reduced to the model proposed by Ismail and Cartmell [23]. Table 3 shows results for isotropic plate with a single arbitrarily oriented crack. It can be seen from the obtained results of Table 3 that the natural frequency of the cracked plate increases with increase in the crack orientation angle θ for both the boundary conditions. This similar trend was also obtained in the study of Ismail and Cartmell [23]. It is evident from the results that the natural frequency of the plate with same angular orientation of crack is higher in case of CCSS boundary condition as compared to SSSS boundary condition.

Table 2. First mode natural frequency (rad/sec) of plate containing two perpendicular cracks for two boundary conditions

Boundary condition	Plate Dimension		Cracked plate			
			$a = b = 0.01$		$a = b = 0.025$	
	L_1	L_2	Present	Ref. [24]	Present	Ref. [24]
SSSS	1	1	291.61	291.80	272.06	272.20
	0.5	1	720.22	720.70	661.34	661.80
	1	0.5	720.22	720.70	661.34	661.80
CCSS	1	1	415.39	408.90	383.48	384.70
	0.5	1	1071.35	1072.40	974.83	975.60
	1	0.5	1071.35	1072.40	974.83	975.60

New results for fundamental frequencies of an isotropic plate containing two mutually perpendicular cracks parallel to either edges of the plate and two arbitrarily oriented cracks located at the centre of the plate are presented for two boundary conditions i.e. all edges simply supported (SSSS) (Table 4) and two adjacent edges simply supported and other two clamped (CCSS) (Table 5). The results are presented for different plate aspect ratios, crack lengths and various combinations of crack orientations. The inclination angles of crack 'C' and 'D' are varied between 15° to 45° and 135° to 165° respectively. It is observed from results of above combination of inclination angles that when crack 'C' and 'D' goes on orienting towards crack 'A', the natural frequency of the plate decreases and when cracks orients away from the crack 'A', the natural frequency of the plate increases. The results show such a trend because the uniaxial state of stresses is considered for formulation of crack terms. The results are in similar fashion for both SSSS and CCSS boundary conditions. Similarly, on increasing the crack lengths, stiffness of the plate decreases which results in decrease of the natural frequency of the plate. The results shows that the plates are very sensitive to the number of cracks, crack length and crack orientation angle.

Table 3. First mode natural frequency (rad/sec) of plate with variably oriented single crack for SSSS and CCSS boundary condition

B.C.	Plate Dimension		Orientation Angle	Intact plate		Cracked plate			
						a = 0.003 (m)		a = 0.0075 (m)	
	L_1	L_2	θ (deg.)	Present	Ref. [23]	Present	Ref. [23]	Present	Ref. [23]
SSSS	0.3	0.3	0	1034.4	1034.0	1003.70	1007.20	972.74	978.60
			20			1007.34	1010.40	980.15	985.85
			40			1016.50	1018.60	998.68	1002.04
			60			1026.81	1027.70	1019.34	1020.72
			80			1033.49	1033.60	1032.62	1032.76
	0.15	0.15	0	4137.6	4138.0	4014.83	4028.20	3890.97	3914.40
			20			4029.39	4041.80	3920.63	3943.40
			40			4066.02	4074.20	3994.73	4008.16
			60			4107.27	4110.70	4077.36	4082.88
			80			4133.97	4134.40	4130.40	4131.04
CCSS	0.3	0.3	0	1485.6	1486.0	1435.99	1441.70	1385.80	1395.27
			20			1441.88	1446.80	1397.83	1406.06
			40			1456.67	1459.90	1427.86	1433.16
			60			1473.32	1474.60	1461.25	1463.41
			80			1484.08	1484.20	1482.64	1482.88
	0.15	0.15	0	5942.2	5942.0	5743.96	5766.80	5543.18	5581.08
			20			5767.51	5787.40	5591.33	5624.24
			40			5826.70	5839.50	5711.43	5732.64
			60			5893.28	5898.50	5845.01	5853.64
			80			5936.34	5936.90	5930.58	5931.52

Table 4. First mode natural frequency of plate with multiple cracks for SSSS boundary condition

Plate dimension		Crack orientation	Frequency (rad/sec)			
			Intact	Cracked		
L_1	L_2	θ, β (deg.)		$a = b = c = d = 0.01$	$a = b = c = d = 0.02$	$a = b = c = d = 0.05$
1	1	45,135	310.3	281.79	259.93	216.13
		30,150		276.75	250.55	196.09
		15,165		273.01	243.46	180.01
0.5	1	45,135	775.8	690.75	624.28	485.56
		30,150		675.54	595.29	418.03
		15,165		664.18	573.13	360.67
1	0.5	45,135	775.8	690.75	624.28	485.56
		30,150		675.543	595.29	418.03
		15,165		664.18	573.13	360.67
0.5	0.5	45,135	1241	1127.17	1039.74	864.55
		30,150		1107.01	1002.22	784.36
		15,165		1092.01	973.84	720.01

Table 5. First mode natural frequency of plate with multiple cracks for CCSS boundary condition

Plate dimension		Crack orientation	Frequency (rad/sec)			
			Intact	Cracked		
L_1	L_2	θ, β (deg.)		$a = b = c = d = 0.01$	$a = b = c = d = 0.02$	$a = b = c = d = 0.05$
1	1	45,135	445.66	399.4	363.48	289.58
		30,150		391.16	347.9	254.43
		15,165		385.01	336.04	225.24
0.5	1	45,135	1161.8	1023.15	913.56	678.8
		30,150		998.18	865.3	559.12
		15,165		979.5	828.18	451.84
1	0.5	45,135	1161.8	1023.15	913.56	678.8
		30,150		998.18	865.3	559.12
		15,165		979.5	828.18	451.84
0.5	0.5	45,135	1782.7	1597.6	1453.93	1158.35
		30,150		1564.64	1391.61	1017.72
		15,165		1540.05	1344.16	900.96

4. Conclusions

An analytical model for vibration analysis of thin isotropic rectangular plate with multiple cracks is presented. Cracks are in the form of continuous line and are of definite crack length and crack orientation. Effect of various parameters like crack length, plate aspect ratio, crack orientation on SSSS and CCSS boundary conditions on fundamental frequency of the isotropic plate is presented. It is concluded from the obtained results that the presence of cracks affects the fundamental frequencies, since the presence of crack reduces the stiffness of the plate. It is also found that the fundamental frequency decreases with the increase in number of cracks and their length whereas; the increase in crack orientation increases the fundamental frequencies of the plate. Also, the presence of cracks affects more on CCSS boundary condition as compared to SSSS boundary condition. The fundamental frequency of the plate containing multiple cracks is found to be maximum when $\theta = 45°$ and $\beta = 135°$ and goes on decreasing when θ orients from 45° to 15° and β orients from 135° to 165°. Although, the analytical model is preferred over FEM and experimental results owing to the fact that analytical models are fast and accurate but there are few limitations too. The present model gives comparatively correct results for the values of crack orientation between -45° to 45° and 135° to 225° than other orientations as the uniaxial state of stresses are considered for formulation of crack terms. Discontinuities in engineering structures are always a matter of concern owing to safe and continuous working of dynamic systems. The detection and effects of such flaws is a recent topic of interest for researchers and hence there is lot of scope in this growing area. Future scope of this study includes analytical and experimental analysis of fundamental frequencies for cylindrical, spherical and conical shells with some discontinuous line cracks on various boundary conditions.

Acknowledgments

This work is not funded by any organization.

References

1. A. W. Leissa, *Vibration of plate*s, 1969, DOI:10.1002/zamm.19710510331.
2. A. W. Leissa, *The Free Vibration Ofrectangular Plates*, J. Sound Vib., **31** (1973) 257 – 293, DOI: 10.1016/S0022-460X(73)80371-2.
3. R. Szilard, *Theories and Applications of Plate Analysis*, John Wiley & Sons, Inc., Hoboken, NJ, USA, 2004, DOI: 10.1002/9780470172872.
4. E. Ventsel, T. Krauthammer, *Thin Plates and Shells*, CRC Press, 2001, DOI: 10.1201/9780203908723.
5. G. B. Warburton, *The vibration of rectangular plates*, Arch. Proc. Inst. Mech. Eng. 1847-1982 (Vols 1-196). 168 (1954) 371 – 384, DOI: 10.1243/PIME_PROC_1954_168_040_02.
6. D. J. Dawe, L. Roufaeil, *rayleigh-ritz vibration plates analysis of mindlin*ii, **69** (1980) 345 – 359.

7. S. Kitipornchai, Y. Xiang, C. M. Wang, K. M. Liew, *Buckling of thick skew plates*, Int. J. Numer. Methods Eng., **36** (1993) 1299 – 1310, DOI: 10.1002/nme.1620360804.
8. Ł. Domagalski, *An Analytical-Numerical Approach to Analysis of Large Amplitude Vibrations of Slender Periodic Beams*, Vibrations in Physical Systems, **27** (2016) 99 – 106.
9. C. S. Huang, A. W. Leissa, C. W. Chan, *Vibrations of rectangular plates with internal cracks or slits*, Int. J. Mech. Sci., **53** (2011) 436 – 445, DOI: 10.1016/j.ijmecsci.2011.03.006.
10. J. Jędrysiak, E. Pazera, *Free Vibrations of Thin Microstructured Plates*, Vibrations in Physical Systems, **26** (2014) 93 – 98.
11. S. E. Khadem, M. Rezaee, *Introduction of modified comparison functions for vibration analysis of a rectangular cracked plate*, J. Sound Vib., **236** (2000) 245 – 258, DOI: 10.1006/jsvi.2000.2986.
12. M. Krawczuk, A. Żak, W. Ostachowicz, *Finite element model of plate with elasto-plastic through crack, Comput. Struct*., **79** (2001) 519 – 532, DOI: 10.1016/S0045-7949(00)00156-5.
13. B. Stahl, L. M. Keer, *Vibration and stability of cracked rectangular plates,* Int. J. Solids Struct., **8** (1972) 69 – 91, DOI: 10.1016/0020-7683(72)90052-2.
14. K. M. Liew, K. C. Hung, M. K. Lim, *A solution method for analysis of cracked plates under vibration*, Eng. Fract. Mech., **48** (1994) 393 – 404, DOI: 10.1016/0013-7944(94)90130-9.
15. G. Y. Wu, Y. S. Shih, *Dynamic instability of rectangular plate with an edge crack, Comput. Struc*t., **84** (2005) 1 – 10, DOI: 10.1016/j.compstruc.2005.09.003.
16. Y. G. Xiao, Y. M. Fu, X. D. Zha, *Bifurcation and chaos of rectangular moderately thick cracked plates on an elastic foundation subjected to periodic load*, Chaos, Solitons & Fractals, **35** (2008) 460 – 465, DOI: 10.1016/j.chaos.2006.04.074.
17. J. R. Rice, N. Levy, *The Part-Through Surface Crack in an Elastic Plate*, J. Appl. Mech., **39** (1972) 185, DOI: 10.1115/1.3422609.
18. R. B. King, *Elastic-plastic analysis of surface flaws using a simplified line-spring model*, Eng. Fract. Mech., **18** (1983) 217 – 231, DOI: 10.1016/0013-7944(83)90108-X.
19. K. Maruyama, O. Ichinomiya, *Experimental Study of Free Vibration of Clamped Rectangular Plates with Straight Narrow Slits*, JSME Int. Journal. Ser. 3, Vib. Control Eng. Eng. Ind., **32** (1989) 187 – 193.
20. C. S. Huang, A. W. Leissa, *Vibration analysis of rectangular plates with side cracks via the Ritz method*, J. Sound Vib., **323** (2009) 974 – 988, DOI: 10.1016/j.jsv.2009.01.018.
21. C. S. Huang, A. W. Leissa, R. S. Li, *Accurate vibration analysis of thick, cracked rectangular plates*, J. Sound Vib., **330** (2011) 2079 – 2093, DOI: 10.1016/j.jsv.2010.11.007.
22. A. Israr, M. P. Cartmell, E. Manoach, I. Trendafilova, W. Ostachowicz, M. Krawczuk, et al., *Analytical Modeling and Vibration Analysis of Partially Cracked*

Rectangular Plates With Different Boundary Conditions and Loading, J. Appl. Mech., **76** (2009) 11005, DOI: 10.1115/1.2998755.

23. R. Ismail, M. P.Cartmell, *An investigation into the vibration analysis of a plate with a surface crack of variable angular orientation*, J. Sound Vib., **331** (2012) 2929 – 2948, DOI: 10.1016/j.jsv.2012.02.011.
24. P. V Joshi, N. K. Jain, G. D. Ramtekkar, *Analytical modeling and vibration analysis of internally cracked rectangular plates*, J. Sound Vib., **333** (2014) 5851 – 5864, DOI: 10.1016/j.jsv.2014.06.028.
25. P. V Joshi, N. K. Jain, G. D. Ramtekkar, *Analytical modelling for vibration analysis of partially cracked orthotropic rectangular plates*, Eur. J. Mech. A/Solids, **50** (2015) 100 – 111, DOI:10.1016/j.euromechsol.2014.11.007.
26. P. V Joshi, N. K. Jain, G. D. Ramtekkar, *Effect of thermal environment on free vibration of cracked rectangular plate: An analytical approach*, Thin-Walled Struct., **91** (2015) 38 – 49, DOI: 10.1016/j.tws.2015.02.004.
27. A. Gupta, N. K. Jain, R. Salhotra, P. V. Joshi, *Effect of microstructure on vibration characteristics of partially cracked rectangular plates based on a modified couple stress theory*, Int. J. Mech. Sci., **100** (2015) 269 – 282, DOI: 10.1016/j.ijmecsci.2015.07.004.
28. A. Gupta, N. K. Jain, R. Salhotra, A. M. Rawani, P. V. Joshi, *Effect of fibre orientation on non-linear vibration of partially cracked thin rectangular orthotropic micro plate: An analytical approach*, Int. J. Mech. Sci., **105** (2016) 378 – 397, DOI: 10.1016/j.ijmecsci.2015.11.020.

APPENDIX I

Consider the plate element with multiple cracks located at the centre of the plate as shown in Fig. A1. The moments and in-plane forces are considered according to the Kirchoff's classical plate theory [3]. Taking moments along the *x* and *y* directions and summation of the forces along the z direction, we get

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{\cancel{x}}}{\partial x^2} + \frac{\partial^2 M_{\cancel{y}}}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - 2\frac{\partial^2 M_{\cancel{xy}}}{\partial x \partial y} = \rho h \frac{\partial^2 w}{\partial t^2} - P_z \tag{A1}$$

Where, M_x ,M_y are the bending moments per unit length along the *x* and *y*-directions and M_{xy} is the twisting moment. $M_{\cancel{x}}$,$M_{\cancel{y}}$ are the bending moments per unit length due to multiple cracks and $M_{\cancel{xy}}$ is the twisting moment which occurs when the crack is inclined to either edges [23].

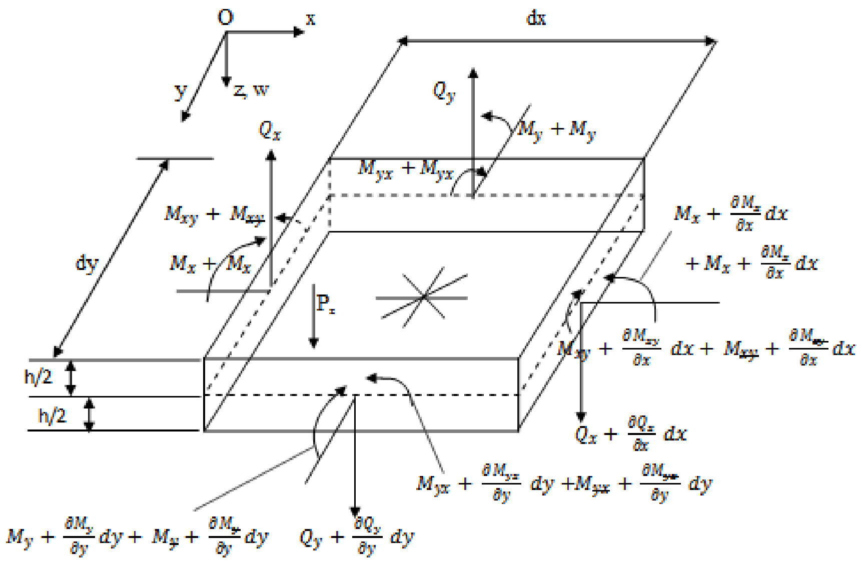

Figure A1. Plate element showing moments and transverse forces acting on mid plane

M_x ,M_y and M_{xy} are related with the transverse displacement as

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{A2}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{A3}$$

$$M_{xy} = -M_{yx} = -D(1-\nu)\frac{\partial^2 w}{\partial_x \partial_y} \tag{A4}$$

Where D is the flexural rigidity. E is the modulus of elasticity and *v* is Poisson's ratio. On Substituting Eq. (A2) – (A4) into Eq. (A1) the following equation becomes

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -\rho h\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_{\cancel{y}}}{\partial y^2} + \frac{\partial^2 M_{\cancel{x}}}{\partial x^2} - 2\frac{\partial^2 M_{\cancel{xy}}}{\partial x \partial y} + P_z \tag{A5}$$

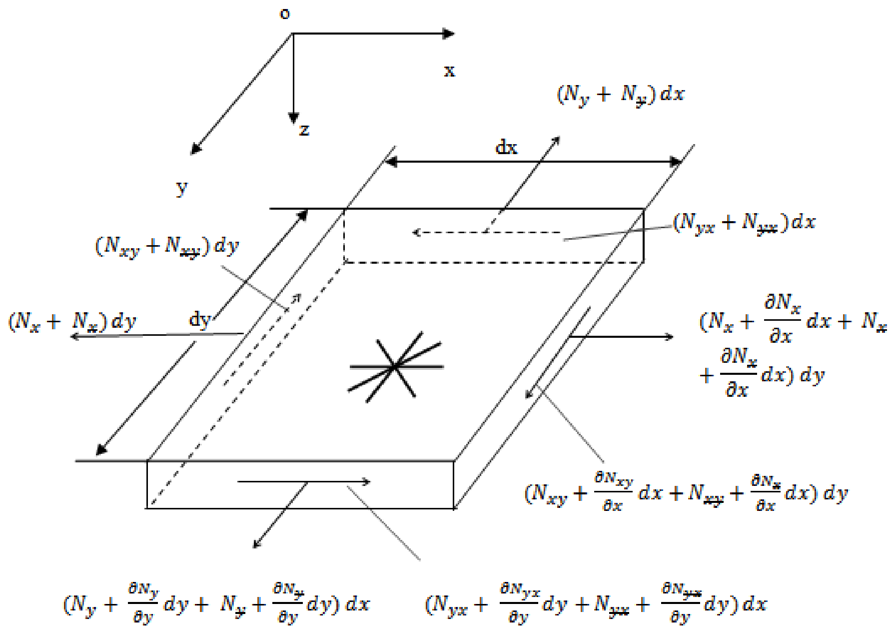

Figure A2. In-plane forces in a cracked plate element

The equilibrium of in-plane forces per unit length for the cracked plate element is shown in Fig. A2. $N_{\cancel{y}}$ is the in-plane force due to presence of cracks along *x*-axis or inclination of cracks towards x-axis. Similarly $N_{\cancel{x}}$ and $N_{\cancel{xy}}$ is due to presence of cracks along *y*-axis and inclined cracks respectively.
In the absence of body forces, the summation of membrane forces along *x* axis leads to,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\cancel{x}}}{\partial x} + \frac{\partial N_{yx}}{\partial y} + \frac{\partial N_{\cancel{yx}}}{\partial y} = 0 \tag{A6}$$

The summation of all forces along *y* axis leads to:

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{\cancel{y}}}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{\cancel{xy}}}{\partial x} = 0 \tag{A7}$$

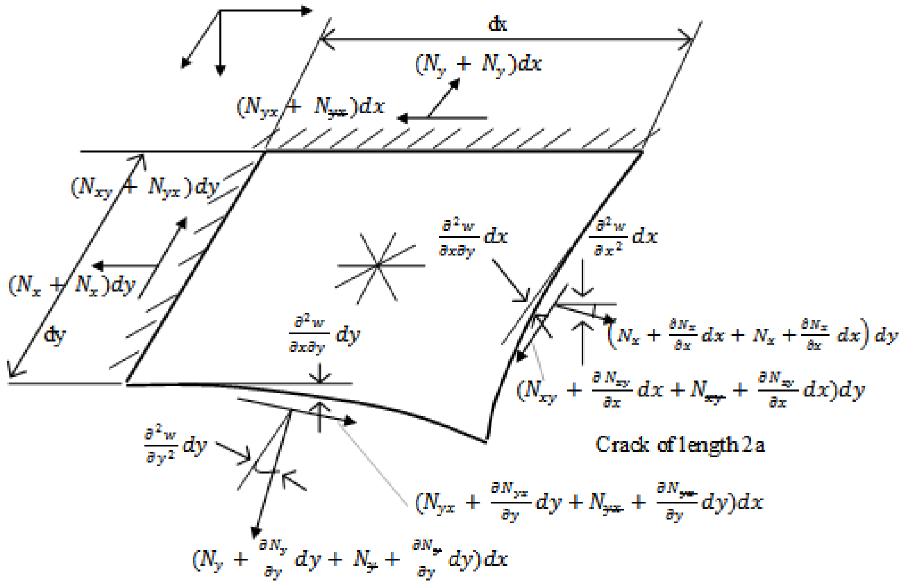

Figure A3. Equilibrium of in-plane forces along z axis

From Fig. A3, the projection of in-plane forces along *z* axis leads to,

$$\sum F_z = (N_x + \frac{\partial N_x}{\partial x}dx + N_{\cancel{x}} + \frac{\partial N_{\cancel{x}}}{\partial x}dx)dy\frac{\partial^2 w}{\partial x^2}dx + (N_y + \frac{\partial N_y}{\partial y}dy + N_{\cancel{y}} + \frac{\partial N_{\cancel{y}}}{\partial y}dy)dx\frac{\partial^2 w}{\partial y^2}dy + (N_{xy} + \frac{\partial N_{xy}}{\partial x}dx + N_{\cancel{xy}} + \frac{\partial N_{\cancel{xy}}}{\partial x}dx)dy\frac{\partial^2 w}{\partial x \partial y}dx + (N_{yx} + \frac{\partial N_{yx}}{\partial y}dy + N_{\cancel{yx}} + \frac{\partial N_{\cancel{yx}}}{\partial y}dy)dx\frac{\partial^2 w}{\partial x \partial y}dy \tag{A8}$$

Simplifying this equation by neglecting the higher order terms leads to,

$$\sum F_z = N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + N_{\cancel{x}}\frac{\partial^2 w}{\partial x^2} + N_{\cancel{y}}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + 2N_{\cancel{xy}}\frac{\partial^2 w}{\partial x \partial y} \tag{A9}$$

Due to the crack of length 2a, there is discontinuity across *y* direction and hence N_y and N_{xy} has to be neglected for equilibrium. Similarly along the crack length 2b, N_x andN_{yx} has to be neglected due to the discontinuity across *x* axis. Ismail and Cartmell [23] considered various angular orientations for a single crack and developed a relation between $N_{\cancel{xy}}$ and N_y. Their work shows that the in-plane shear $N_{\cancel{xy}}$ appears only when the crack is inclined to the edge of the plate. In this work, the two perpendicular cracks are parallel to the edges of the plate and two other cracks are of arbitrary angular orientation, thus neglecting the terms N_x, N_y and N_{xy}, Eq. (A9) can be written as,

$$\sum F_z = N_{\cancel{x}} \frac{\partial^2 w}{\partial x^2} + N_{\cancel{y}} \frac{\partial^2 w}{\partial y^2} + 2N_{\cancel{xy}} \frac{\partial^2 w}{\partial x \partial y} \tag{A10}$$

Adding the membrane forces given by Eq. (A10) to the equation of motion Eq. (A5) leads to the governing equation of cracked plate with simultaneous bending and stretching.

$$\begin{aligned} &D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) \\ &= -\rho h \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M_{\cancel{y}}}{\partial y^2} + \frac{\partial^2 M_{\cancel{x}}}{\partial x^2} + 2\frac{\partial^2 M_{\cancel{xy}}}{\partial x \partial y} + N_{\cancel{x}} \frac{\partial^2 w}{\partial x^2} \\ &+ N_{\cancel{y}} \frac{\partial^2 w}{\partial y^2} + 2N_{\cancel{xy}} \frac{\partial^2 w}{\partial x \partial y} + P_z \end{aligned} \tag{A11}$$

Eq. (A11) is the final equation of motion for rectangular plate containing multiple cracks.