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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF A THIRD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATION

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Abstract. This paper discusses oscillatory and asymptotic properties of solutions of a class of third-order nonlinear neutral differential equations. Some new sufficient conditions for a solution of the equation to be either oscillatory or to converges to zero are presented. The results obtained can easily be extended to more general neutral differential equations as well as to neutral dynamic equations on time scales. Two examples are provided to illustrate the results.

Keywords: third order, neutral differential equations, asymptotic behavior, nonoscillatory, oscillatory solution.

Mathematics Subject Classification: 34K10, 34K11, 34K15, 34C10.

1. INTRODUCTION

In this paper, we consider the third order neutral differential equation

$$\left((z''(t))^{\alpha}\right)' + q(t)x^{\alpha}(\delta(t)) = 0, \quad t \geq t_0 \geq 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and $\alpha > 0$ is the ratio of odd positive integers.

In the remainder of the paper we assume that:

(i) p, $q : [t_0, \infty) \to \mathbb{R}$ are real valued continuous functions with $q(t) > 0$, $p(t) \geq 1$, and $p(t) \not\equiv 1$ for large t;

(ii) τ, $\delta : [t_0, \infty) \to \mathbb{R}$ are real valued continuous functions such that $\tau(t) < t$, τ is strictly increasing on $[t_0, \infty)$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

The cases

$$\tau(t) \geq \delta(t) \tag{1.2}$$

and

$$\tau(t) \le \delta(t). \tag{1.3}$$

are both considered here.

By a *solution* we mean a function $x : [t_x, \infty) \to \mathbb{R}$ such that $z(t) \in C^2([t_x, \infty), \mathbb{R})$ and $(z''(t))^\alpha \in C^1([t_x, \infty), \mathbb{R})$, and which satisfies equation (1.1) on $[t_x, \infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$, for some $t_x \ge t_0$, and $\sup\{|x(t)| : t \ge T\} > 0$ for every $T \ge t_x$. Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$; otherwise it is called *nonoscillatory*.

Neutral differential equations have received a lot of attention as has the study of the oscillatory and asymptotic properties of their solutions. The problem of asymptotic and oscillatory behavior of solutions of neutral differential equations is of both theoretical and practical interest. One reason for this is that they arise, for example, in applications to electric networks containing lossless transmission lines. Such networks appear in high speed computers where lossless transmission lines are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to an elastic bar and in the solution of variational problems with time delays. We refer the reader to the book by Hale [7] for additional applications in science and technology.

The oscillatory and asymptotic behavior of (1.1) and it's special and more general forms such as

$$z'''(t) + q(t)x(\delta(t)) = 0, \tag{1.4}$$

$$\Big(a(t)\,(b(t)z'(t))'\Big)' + q(t)x(\delta(t)) = 0, \tag{1.5}$$

$$\big(a(t)\,(z''(t))^\alpha\big)' + q(t)x^\alpha(\delta(t)) = 0, \tag{1.6}$$

$$\big(a(t)\,(z''(t))^\alpha\big)' + \sum_{j=1}^{n} f_j(t, x(\tau_j(t))) = 0, \tag{1.7}$$

and if $n \ge 3$ is an odd integer,

$$z^{(n)}(t) + q(t)x^\alpha(\delta(t)) = 0 \tag{1.8}$$

and

$$z^{(n)}(t) + q_1(t)x(\delta(t)) + q_2(t)x(\sigma(t)) = 0 \tag{1.9}$$

have been studied by many authors utilizing different methods; see, for example, [2, 6, 11–13, 15–17] and the references therein.

In reviewing the literature, it becomes apparent that results on the oscillatory and asymptotic behavior of third order neutral differential equations are relatively scarce and most such results are concerned with the cases where $0 \le p(t) \le p_0 < 1$ and $-1 \le p(t) < 0$. We refer the reader to the papers [2, 6, 13–16] and the references cited therein as examples of recent results on this topic. For the classification of the oscillatory and nonoscillatory behavior of solutions of nonlinear neutral differential equations based on the values of the function $p(t)$, we refer the reader to the paper of Graef, Spikes, and Grammatikopoulos [5].

On the other hand, to the best of our knowledge, there are a few results regarding oscillatory and asymptotic behavior of third order neutral differential equations in case $p(t) > 1$; see [11, 12, 17]. In [11, 17], the authors considered equations (1.8) and (1.6), respectively, under the assumption that $\tau \circ \delta = \delta \circ \tau$. In [12], the authors studied equation (1.9) with $\tau \circ \delta = \delta \circ \tau$ and $\tau \circ \sigma = \sigma \circ \tau$. But, these conditions are very restrictive for applications; for example, letting $\tau(t) = t/2$ and $\delta(t) = t - 1$, we see that $\tau \circ \delta \neq \delta \circ \tau$. Therefore, our results also improve the results in [11, 12, 17] in the sense that we do not require that $\tau \circ \delta = \delta \circ \tau$.

The purpose of this paper then is to present some new oscillation and asymptotic results for (1.1) that are new even for the $\alpha = 1$ and constant delays such as $\tau(t) = t - a$ and $\delta(t) = t - b$ with a, $b > 0$. Furthermore, the results in this paper can easily be extended to the more general equations (1.5)–(1.9). It is our belief that the present paper will contribute significantly to the study of oscillatory and asymptotic behavior of solutions of third order neutral differential equations.

The type of results obtained here, namely that a solution either oscillates or converges to zero, is related to the Properties A and B introduced by Kiguradze (see, for example, [9]) for ordinary differential equations and studied by Koplatadze [10] for delay equations. For the oscillatory behavior of solutions and related properties of n-th order neutral delay equations, we refer the reader to the paper of Domoshnitskii [3].

2. MAIN RESULTS

We begin with the following lemmas that are essential in the proofs of our theorems. For simplicity in what follows, it will be convenient to set:

$$\eta'_+(t) := \max\{0, \eta'(t)\},$$

$$p^*(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right),$$

and

$$p_*(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))}\right),$$

where τ^{-1} is the inverse of τ and m is a function to be specified later.

Lemma 2.1 ([8]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda X Y^{\lambda-1} - X^\lambda \leq (\lambda - 1) Y^\lambda,$$

where equality holds if and only if $X = Y$.

Lemma 2.2 ([4, 9]). *If the function $f(t)$ satisfies $f^{(i)}(t) > 0, i = 1, 2, \dots, n$, and $f^{(n+1)}(t) < 0$, then*

$$\frac{f(t)}{t^n/n!} \geq \frac{f'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2.3. *Assume that conditions* (i)–(ii) *hold and let $x(t)$ be a positive solution of* (1.1). *Then for sufficiently large t, either*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $z'''(t) < 0$, *or*
(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $z'''(t) < 0$.

Proof. The proof is standard and so the details will be omitted. □

Remark 2.4. An analogous result holds if $x(t)$ is an eventually negative solution.

Lemma 2.5. *Assume that conditions* (i)–(ii) *hold, $p^*(t) > 0$, and let $x(t)$ be an eventually positive solution of* (1.1) *with $z(t)$ satisfying Case* (II). *If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\int_{u}^{\infty} q(s)\left(p^*(\delta(s))\right)^{\alpha} ds\right)^{1/\alpha} du dv = \infty, \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists $t_1 \in [t_{0,}\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \geq t_1$. From the definition of z, (see also (8.6) in [1]), we have

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}\left(z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))\right) \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{2.2}$$

Since $z(t)$ is decreasing and $\tau(t) < t$, we have

$$z\left(\tau^{-1}(t)\right) \geq z\left(\tau^{-1}(\tau^{-1}(t))\right)$$

and so from (2.2) we obtain

$$x(t) \geq p^*(t) z\left(\tau^{-1}(t)\right). \tag{2.3}$$

In view of (2.3), equation (1.1) can be written as

$$\left((z''(t))^{\alpha}\right)' + q(t)\left(p^*(\delta(t))\right)^{\alpha}\left(z\left(\tau^{-1}(\delta(t))\right)\right)^{\alpha} \leq 0 \quad \text{for } t \geq t_1. \tag{2.4}$$

Since $z(t)$ satisfies Case (II) of Lemma 2.3, there exists a constant κ such that

$$\lim_{t\to\infty} z(t) = \kappa < \infty,$$

where $\kappa \geq 0$. If $\kappa > 0$, then there exists $t_2 \geq t_1$ such that $\tau^{-1}(\delta(t)) > t_1$ and

$$z(t) \geq \kappa \text{ for } t \geq t_2. \tag{2.5}$$

Integrating (2.4) twice from t to ∞ gives

$$-z'(t) \geq \kappa \int_t^\infty \left(\int_u^\infty q(s) \left(p^*(\delta(s))\right)^\alpha ds \right)^{1/\alpha} du.$$

Integrating the last inequality from t_2 to t yields

$$z(t_2) \geq \kappa \int_{t_2}^t \int_v^\infty \left(\int_u^\infty q(s) \left(p^*(\delta(s))\right)^\alpha ds \right)^{1/\alpha} du dv,$$

which contradicts (2.1), and so we have $\kappa = 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) < z(t)$ on $[t_1, \infty)$, we get $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the lemma. □

Lemma 2.6. *Assume that conditions* (i)–(ii) *hold,* $p_*(t) > 0$*, and* $x(t)$ *is an eventually positive solution of* (1.1) *with* $z(t)$ *satisfying Case* (I)*. If there exist a function* $m \in C^1([t_0, \infty), (0, \infty))$ *such that*

$$2m(t) - tm'(t) \leq 0 \tag{2.6}$$

for large t*, then* $z(t)$ *satisfies the inequality*

$$\left((z''(t))^\alpha\right)' + q(t) \left(p_*(\delta(t))\right)^\alpha \left(z\left(\tau^{-1}(\delta(t))\right)\right)^\alpha \leq 0 \tag{2.7}$$

for large t.

Proof. Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, $z(t)$ satisfies Case (I), (2.6) holds, and $\tau^{-1}(\delta(t)) > t_0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.5, we again arrive at (2.2). Since $z(t)$ satisfies Case (I), in view of Lemma 2.2 with $n = 2$, we have

$$\frac{z(t)}{z'(t)} \geq \frac{t}{2} \quad \text{for} \quad t \geq t_1. \tag{2.8}$$

From (2.6) and (2.8), we see that

$$\begin{aligned} \left(\frac{z(t)}{m(t)}\right)' &= \frac{1}{m^2(t)} \left[z'(t)m(t) - z(t)m'(t)\right] \\ &\leq \frac{z(t)}{tm^2(t)} \left[2m(t) - tm'(t)\right] \leq 0, \end{aligned}$$

and so $z(t)/m(t)$ is nonincreasing. Using this and noting that $\tau(t) < t$ implies $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, we obtain

$$\frac{m(\tau^{-1}(\tau^{-1}(t)))z(\tau^{-1}(t))}{m(\tau^{-1}(t))} \geq z(\tau^{-1}(\tau^{-1}(t))). \tag{2.9}$$

Substituting (2.9) into (2.2) gives

$$x(t) \geq p_*(t) z\left(\tau^{-1}(t)\right). \tag{2.10}$$

In view of (1.1) and (2.10), we arrive at (2.7) and this completes the proof of the lemma. □

Remark 2.7. The assumption concerning the existence of an auxiliary function m holds, for example, for $m(t) = t^2$, $m(t) = t^3$, $m(t) = e^t$, $m(t) = t^\gamma e^{\epsilon t}$ with $\gamma \geq 2$ and $\epsilon \geq 0$, etc.

Our first main result is contained in the following theorem.

Theorem 2.8. *Assume that conditions* (i)–(ii), (1.2), *and* (2.1) *hold and there exists a function* $m \in C^1([t_0, \infty), (0, \infty))$ *such that* (2.6) *holds,* $p_*(t) > 0$, *and* $p^*(t) > 0$ *for sufficiently large* t. *If there exists a positive function* $\eta \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s) q(s) \left(p_*(\delta(s))\right)^\alpha \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \eta'_+(s) \left(\frac{8}{s^2}\right)^\alpha\right] ds = \infty, \tag{2.11}$$

then any solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, (2.6) holds, $p_*(t) > 0$, $p^*(t) > 0$, $\tau^{-1}(\delta(t)) > t_0$, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$.

Assume that Case (I) holds and define

$$w(t) = \eta(t) \frac{(z''(t))^\alpha}{z^\alpha(t)} \quad \text{for} \quad t \geq t_1. \tag{2.12}$$

Then $w(t) > 0$, and from (2.12) and (2.7), we see that, for $t \geq t_1$,

$$\begin{aligned} w'(t) &= \eta'(t) \frac{(z''(t))^\alpha}{z^\alpha(t)} + \eta(t) \left[\frac{\left((z''(t))^\alpha\right)'}{z^\alpha(t)} - \frac{(z''(t))^\alpha \left(z^\alpha(t)\right)'}{z^{2\alpha}(t)}\right] \\ &\leq \eta'_+(t) \frac{(z''(t))^\alpha}{z^\alpha(t)} - \eta(t) q(t) \left(p_*(\delta(t))\right)^\alpha \frac{\left(z\left(\tau^{-1}(\delta(t))\right)\right)^\alpha}{z^\alpha(t)} \\ &\quad - \alpha \eta(t) \frac{(z''(t))^\alpha z'(t)}{z^{\alpha+1}(t)}. \end{aligned} \tag{2.13}$$

From Lemma 2.2, we have

$$\frac{z'(t)}{z(t)} \leq \frac{2}{t} \quad \text{for } t \geq t_1, \tag{2.14}$$

so integrating (2.14) from $\tau^{-1}(\delta(t)) > t_1$ to t, we obtain

$$\frac{z(t)}{z\left(\tau^{-1}(\delta(t))\right)} \leq \left(\frac{t}{\tau^{-1}(\delta(t))}\right)^2. \tag{2.15}$$

Since $z(t)$ satisfies Case (I), we obtain

$$z'(t) = z'(t_1) + \int_{t_1}^{t} z''(s)ds \geq (t - t_1)\, z''(t) \geq \frac{t}{2} z''(t) \quad \text{for} \quad t \geq t_2 := 2t_1. \tag{2.16}$$

Integrating (2.16) from t_2 to t gives

$$z(t) \geq z(t_2) + \int_{t_2}^{t} \frac{s}{2} z''(s)ds \geq \frac{1}{4} z''(t)(t + t_2)(t - t_2) \geq \frac{t^2}{8} z''(t) \quad \text{for} \quad t \geq t_3 := 2t_2. \tag{2.17}$$

Using (2.15), (2.16), and (2.17) in (2.13), we obtain

$$w'(t) \leq \eta'_+(t) \left(\frac{8}{t^2}\right)^{\alpha} - \eta(t)q(t)\,(p_*(\delta(t)))^{\alpha} \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} - \frac{\alpha t w^{(\alpha+1)/\alpha}(t)}{2\eta^{1/\alpha}(t)} \tag{2.18}$$

for $t \geq t_3$, for some $t_3 \geq t_2$. Since $w(t) > 0$, (2.18) yields

$$w'(t) \leq \eta'_+(t) \left(\frac{8}{t^2}\right)^{\alpha} - \eta(t)q(t)\,(p_*(\delta(t)))^{\alpha} \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha}. \tag{2.19}$$

Integrating (2.19) from t_3 to t, we obtain

$$\int_{t_3}^{t} \left[\eta(s)q(s)\,(p_*(\delta(s)))^{\alpha} \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \eta'_+(s)\left(\frac{8}{s^2}\right)^{\alpha}\right] ds \leq w(t_3),$$

which contradicts (2.11).

This implies Case (II) holds, and so from Lemma 2.5, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. □

Our next theorem is in the same spirit as the previous one. Its proof makes use of Lemma 2.1.

Theorem 2.9. *Assume that conditions* (i)–(ii), (1.2), *and* (2.1) *hold and there exists a function* $m \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.6) *holds,* $p_*(t) > 0$, *and* $p^*(t) > 0$ *for sufficiently large* t. *If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s)q(s)\,(p_*(\delta(s)))^{\alpha} \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left(\eta'_+(s)\right)^{\alpha+1}}{\left(s\eta(s)\right)^{\alpha}}\right] ds = \infty, \tag{2.20}$$

then any solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, (2.6) holds, $p_*(t) > 0$, $p^*(t) > 0$, $\tau^{-1}(\delta(t)) > t_0$, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$.

Assume Case (I) holds. Proceeding as in the proof of Theorem 2.8, we again arrive at (2.13), (2.15), and (2.16). Using (2.12), (2.15), and (2.16) in (2.13), we obtain

$$w'(t) \leq \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} - \frac{\alpha t}{2\eta^{1/\alpha}(t)} w^{(\alpha+1)/\alpha}(t) \quad (2.21)$$

for $t \geq t_3$.

Applying Lemma 2.1 with

$$X = \frac{(\alpha t)^{1/\lambda}}{\left(2\eta^{1/\alpha}(t)\right)^{1/\lambda}} w(t), \quad \lambda = \frac{\alpha+1}{\alpha},$$

and

$$Y = \left[\frac{\alpha}{\alpha+1} \frac{\left(2\eta^{1/\alpha}(t)\right)^{1/\lambda}}{(\alpha t)^{1/\lambda}} \frac{\eta'_+(t)}{\eta(t)}\right]^\alpha,$$

we obtain

$$\frac{\eta'_+(t)}{\eta(t)} w(t) - \frac{\alpha t}{2\eta^{1/\alpha}(t)} w^{(\alpha+1)/\alpha}(t) \leq \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\left(\eta'_+(t)\right)^{\alpha+1}}{(t\eta(t))^\alpha},$$

and substituting this into (2.21) gives

$$w'(t) \leq -\eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} + \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\left(\eta'_+(t)\right)^{\alpha+1}}{(t\eta(t))^\alpha}.$$

Integrating from t_3 to t gives

$$\int_{t_3}^{t} \left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\left(\eta'_+(s)\right)^{\alpha+1}}{(s\eta(s))^\alpha}\right] ds \leq w(t_3),$$

which contradicts (2.20). Therefore Case (II) holds, and so $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.5. This proves the theorem. □

Our next result is for the situation where $\alpha \geq 1$.

Theorem 2.10. *Let $\alpha \geq 1$. In addition to conditions* (i)–(ii), (1.2), *and* (2.1), *assume that there exists a function $m \in C^1([t_0,\infty),(0,\infty))$ such that* (2.6) *holds, $p_*(t) > 0$, and $p^*(t) > 0$ for sufficiently large t. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \frac{\left(\eta'_+(s)\right)^2}{2^{4-3\alpha}\alpha s^{2\alpha-1}\eta(s)}\right] ds = \infty, \quad (2.22)$$

then any solution of (1.1) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1). We may assume, without loss of generality, that there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, (2.6) holds, $p_*(t) > 0$, $p^*(t) > 0$, $\tau^{-1}(\delta(t)) > t_0$, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$.

Assuming Case (I) holds and proceeding as in the proof of Theorem 2.9, we again obtain (2.21), which can be written as

$$w'(t) \leq \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} - \frac{\alpha t}{2\eta^{1/\alpha}(t)} w^2(t) w^{\frac{1}{\alpha}-1}(t). \tag{2.23}$$

In view of (2.12) and (2.17), for $t \geq t_3$ we have

$$w^{\frac{1}{\alpha}-1}(t) = \eta^{\frac{1}{\alpha}-1}(t) \left(\frac{(z''(t))^\alpha}{z^\alpha(t)}\right)^{\frac{1}{\alpha}-1} = \eta^{\frac{1}{\alpha}-1}(t) \left(\frac{z(t)}{z''(t)}\right)^{\alpha-1} \geq \eta^{\frac{1}{\alpha}-1}(t) \left(\frac{t^2}{8}\right)^{\alpha-1}. \tag{2.24}$$

Using this in (2.23), we obtain

$$w'(t) \leq \frac{\eta'_+(t)}{\eta(t)} w(t) - \eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} - \frac{\alpha t^{2\alpha-1}}{2^{3\alpha-2}\eta(t)} w^2(t). \tag{2.25}$$

Completing the square with respect to w, it follows from (2.25) that

$$w'(t) \leq -\eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha \left(\frac{\tau^{-1}(\delta(t))}{t}\right)^{2\alpha} + \frac{\left(\eta'_+(t)\right)^2}{2^{4-3\alpha}\alpha t^{2\alpha-1}\eta(t)}.$$

Integrating this inequality from t_3 to t gives

$$\int_{t_3}^{t} \left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha \left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha} - \frac{\left(\eta'_+(s)\right)^2}{2^{4-3\alpha}\alpha s^{2\alpha-1}\eta(s)}\right] ds \leq w(t_3),$$

which contradicts (2.22).

If Case (II) holds, then again from Lemma 2.5, we have $\lim_{t\to\infty} x(t) = 0$, to complete the proof of the theorem. □

Next, we present two results for the case where (1.3) holds.

Theorem 2.11. *In addition to conditions* (i)–(ii), (1.3), *and* (2.1), *assume that there exists a function* $m \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.6) *holds,* $p_*(t) > 0$, *and* $p^*(t) > 0$ *for sufficiently large* t. *If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha - \eta'_+(s)\left(\frac{8}{s^2}\right)^\alpha\right] ds = \infty, \tag{2.26}$$

then any solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1). We may assume, without loss of generality, that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, (2.6) holds, $p_*(t) > 0$, $p^*(t) > 0$, $\tau^{-1}(\delta(t)) > t_0$, and $z(t)$ satisfies either Case (I) or Case (II) for $t \geq t_1$.

Assume Case (I) holds and proceed as in the proof of Theorem 2.8 to arrive at (2.13), (2.16), and (2.17). In view of (1.3), we have

$$t \leq \tau^{-1}(\delta(t)),$$

and so we obtain

$$\frac{z\left(\tau^{-1}(\delta(t))\right)}{z(t)} \geq 1. \tag{2.27}$$

Using (2.16), (2.17) and (2.27) in (2.13), we arrive at

$$w'(t) \leq \eta'_+(t)\left(\frac{8}{t^2}\right)^\alpha - \eta(t)q(t)\left(p_*(\delta(t))\right)^\alpha - \frac{\alpha t w^{(\alpha+1)/\alpha}(t)}{2\eta^{1/\alpha}(t)}. \tag{2.28}$$

The remainder of the proof is similar to that of Theorem 2.8, and so the details are omitted. □

Theorem 2.12. *In addition to conditions* (i)–(ii), (1.3), *and* (2.1), *assume that there exists a function* $m \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.6) *holds,* $p_*(t) > 0$, *and* $p^*(t) > 0$ *for sufficiently large* t. *If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}}\frac{\left(\eta'_+(s)\right)^{\alpha+1}}{\left(s\eta(s)\right)^\alpha}\right]ds = \infty, \tag{2.29}$$

then any solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. The proof follows from (2.27) and Theorem 2.9. □

Our final theorem in this paper is again for the case $\alpha \geq 1$.

Theorem 2.13. *Let* $\alpha \geq 1$. *In addition to conditions* (i)–(ii), (1.3), *and* (2.1), *assume that there exists a function* $m \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.6) *holds,* $p_*(t) > 0$, *and* $p^*(t) > 0$ *for sufficiently large* t. *If there exists a positive function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)q(s)\left(p_*(\delta(s))\right)^\alpha - \frac{\left(\eta'_+(s)\right)^2}{2^{4-3\alpha}\alpha s^{2\alpha-1}\eta(s)}\right]ds = \infty, \tag{2.30}$$

then any solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. The proof follows from (2.27) and Theorem 2.10. □

We conclude this paper with two examples to illustrate our results.

Example 2.14. Consider third order neutral differential equation

$$\left[\left(\left(x(t)+\frac{5t+6}{t+1}x(t/2)\right)''\right)^{1/5}\right]'+\frac{\beta}{t^{7/5}}x^{1/5}(t/3)=0,\quad t\geq 1. \tag{2.31}$$

Here we have $\alpha = 1/5$, $\tau(t) = t/2$, $\delta(t) = t/3$, $q(t) = \beta t^{-7/5}$ and $p(t) = (5t+6)/(t+1)$. Since $p(t) = 5 + 1/(t+1)$, we have

$$5 \leq p(t) < 6,$$

and so $p^*(t) > 2/15$. With $m(t) = t^2$, we obtain $p_*(t) > 1/30$. Thus,

$$\begin{aligned}\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\int_{u}^{\infty}q(s)\left(p^*(\delta(s))\right)^{\alpha}ds\right)^{1/\alpha}dudv &> \int_{1}^{\infty}\int_{v}^{\infty}\left(\int_{u}^{\infty}\beta s^{-7/5}\left(2/15\right)^{1/5}ds\right)^{5}dudv\\ &=\beta^5\Big(2/15\Big)\Big(\frac{5}{2}\Big)^5\int_{1}^{\infty}\int_{v}^{\infty}\frac{1}{u^2}dudv\\ &=\frac{1}{3}\Big(\frac{5}{2}\Big)^4\beta^5\int_{1}^{\infty}\frac{1}{v}dv=\infty,\end{aligned}$$

so condition (2.1) holds.

With $\eta(t) = t$, condition (2.11) becomes

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)q(s)\left(p_*(\delta(s))\right)^{\alpha}\left(\frac{\tau^{-1}(\delta(s))}{s}\right)^{2\alpha}-(\eta'(s))_+\left(\frac{8}{s^2}\right)^{\alpha}\right]ds\\ &>\int_{1}^{t}\left[s\beta s^{-7/5}\left(\frac{1}{30}\right)^{1/5}\left(\frac{2}{3}\right)^{2/5}-\left(\frac{8}{s^2}\right)^{1/5}\right]ds\\ &=\int_{1}^{t}\left[\frac{\beta\sqrt[5]{2/135}}{s^{2/5}}-\left(\frac{8}{s^2}\right)^{1/5}\right]ds=\infty\end{aligned}$$

for $\beta > \sqrt[5]{540}$, i.e., condition (2.11) holds. Therefore, by Theorem 2.8, any solution of (2.31) is either oscillatory or converges to zero.

Example 2.15. Consider third order neutral differential equation

$$\left[\left(\left(x(t)+256x\Big(\frac{t}{4}\Big)\right)''\right)^{3}\right]'+tx^3(t-1)=0,\quad t\geq 2. \tag{2.32}$$

Here we have $\alpha = 3$, $\tau(t) = t/4$, $\delta(t) = t - 1$, $q(t) = t$ and $p(t) = 256$. It is clear that $p^*(t) = 255/4^8 > 0$. With $m(t) = t^3$, we get $p_*(t) = 3/4^5 > 0$. Since $\int_u^\infty s ds = \infty$ for $u \geq t_0 = 2$, we see that

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\int_{u}^{\infty} q(s)\left(p^*(\delta(s))\right)^{\alpha} ds\right)^{1/\alpha} du dv = \left(\frac{255}{4^8}\right)\int_{2}^{\infty}\int_{v}^{\infty}\left(\int_{u}^{\infty} s ds\right)^{3} du dv = \infty,$$

and so condition (2.1) holds. With $\eta(t) = t$ and $T = 2$, condition (2.29) becomes

$$\limsup_{t\to\infty}\int_{T}^{t}\left[\eta(s)q(s)\left(p_*(\delta(s))\right)^{\alpha} - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\eta'(s))_+^{\alpha+1}}{(s\eta(s))^{\alpha}}\right] ds$$
$$= \limsup_{t\to\infty}\int_{2}^{t}\left[s^2\left(3/4^5\right)^3 - \frac{8}{4^4}\frac{1}{s^6}\right] ds = \infty,$$

i.e., condition (2.29) holds. Therefore, by Theorem 2.12, a solution of (2.32) is either oscillatory or converges to zero.

Remark 2.16. The results of this paper can easily be extended to third order neutral dynamic equations of the form

$$\left[\left((x(t) + p(t)x(\tau(t)))^{\Delta\Delta}\right)^{\alpha}\right]^{\Delta} + q(t)x^{\alpha}(\delta(t)) = 0$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where $\alpha > 0$ is the ratio of odd positive integers, p, $q \in C_{rd}(\mathbb{T}, (0,\infty))$, $p(t) \geq 1$, $p(t) \not\equiv 1$ eventually, τ, δ, $\tau^{-1}(\tau^{-1})$, $\tau^{-1}(\delta) \in C_{rd}(\mathbb{T},\mathbb{T})$ with τ strictly increasing, $\tau(t) < t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$.

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