Design of regular positive and stable descriptor systems via state-feedbacks for descriptor continuous-time linear systems with singular pencils

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A new method is proposed of design of regular positive and asymptotically stable descriptor systems by the use of state-feedbacks for descriptor continuous-time linear systems with singular pencils. The method is based on the reduction of the descriptor system by elementary row and column operations to special form. A procedure for the design of the state-feedbacks gain matrix is presented and illustrated by a numerical example

Key words: design, positivity, descriptor, linear, system, singular pencil

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [6, 9, 10]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4, 5, 8]. Standard descriptor control systems have been addressed in [7, 11]. Positive descriptor linear systems have been analyzed in [1-3, 15]. A method based on shuffle algorithm for checking of the positivity of descriptor linear systems with regular pencil has been proposed in [13] and for descriptor systems with singular pencils in [14].

In this paper a new method of design of regular positive and asymptotically stable descriptor systems by the use of state-feedbacks for descriptor continuous-time linear systems with singular pencils will be proposed.

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The paper is organized as follows. In sec. 2 definitions and theorems concerning positive linear systems are recalled and the problem formulation is given. A method for solving of the problem is presented in sec. 3. Concluding remarks are given in sec. 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ - the set of $n \times m$ matrices with nonnegative entries, and $\Re_+^n = \Re^{n \times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), M_{nS} – the set of $n \times n$ asymptotically stable Metzler matrices, I_n – the $n \times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the descriptor continuous-time linear system with singular pencil

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t)
\end{aligned}
\tag{1}
$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

It is assumed that rank $B = m$ and

$$
\text{rank } E = \text{rank } [Es - A] = r < n \quad \text{and} \quad \text{(the pencil is singular)}
\tag{2}
$$

and the matrix E has only r nonzero columns.

If rank $E <$ rank $[Es - A]$ then by the use of the shuffle algorithm we obtain rank $E' =$ rank $[Es - A] = r$ [14].

Let U_{ad} be a set of all given admissible inputs $u(t) \in \Re^m$ of the system (1). A set of all initial conditions $x_0 \in \Re^n$ for which the equation (1) has a solution $x(t)$ for $u(t) \in U_{ad}$ is called the set of consistent initial conditions and is denoted by X_c^0. The set X_c^0 depends on the matrices E, A, B but also on $u(t) \in U_{ad}$ [11].

The following elementary row (column) operations will be used [11, 14]:

1. Multiplication of the ith row (column) by a real number c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).

2. Addition to the ith row (column) of the jth row (column) multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$ ($R[i + j \times c]$).

3. Interchange of the ith and jth rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

A method for checking the positivity of the descriptor linear system will be proposed. The method is based on elementary row and column operations.

Consider the standard continuous-time linear system

$$
\dot{x}(t) = Ax(t) + Bu(t)
\tag{3}
$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$ are the state and input vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$. The system (3) is called (internally) positive if $x(t) \in \mathfrak{R}^n_+$, $t \geqslant 0$ for every nonnegative initial condition $x_0 \in \mathfrak{R}^n_+$ and all inputs $u(t) \in \mathfrak{R}^m_+$, $t \geqslant 0$.

Theorem 1 *The standard continuous-time linear system* (3) *is positive if and only if*

$$A \in M_n, \quad B \in \mathfrak{R}^{n \times m}_+, \quad C \in \mathfrak{R}^{p \times n}_+. \tag{4}$$

The problem under considerations can be stated as follows:

Given the descriptor system (1) with singular pencil (2), find s state-feedback gain matrix $K \in \mathfrak{R}^{m \times n}$ of

$$u(t) - Kx(t) \tag{5}$$

such that the closed-loop system

$$E\dot{x}(t) = (A + BK)x(t) + Bu(t), \tag{6a}$$

$$y(t) = Cx(t) \tag{6b}$$

is positive, asymptotically stable and with regular pencil, i.e.

$$\det[Es - (A + BK)] \neq 0 \text{ for some } s \in C \text{ (the field of complex numbers).} \tag{7}$$

3. Problem solution

If the matrix E has only r nonzero columns then by permutation of the columns it can be reduced to the form

$$E_1 P = [E_1 \ 0], \quad E_1 \in \mathfrak{R}^{n \times r} \quad \text{and} \quad \text{rank } E_1 = r \tag{8}$$

and $P \in \mathfrak{R}^{n \times n}$ is a monomial matrix of the permutation of the columns and $P^{-1} \in \mathfrak{R}^{n \times n}_+$ [10, 11].

Next applying to the matrix E_1 suitable elementary row operations it is possible to reduced it to the form

$$QE_1 = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \tag{9}$$

where $Q \in \mathfrak{R}^{n \times n}$ is the matrix of elementary row operations.

Defining the new state vector

$$\bar{x}(t) = P^{-1}x(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \quad \bar{x}_1(t) \in \mathfrak{R}^r, \quad \bar{x}_2(t) \in \mathfrak{R}^{n-r} \tag{10}$$

and premultiplying the equation (6a) by the matrix Q we obtain

$$QEPP^{-1}\dot{x}(t) = QAPP^{-1}x(t) + QBu(t),$$
$$y(t) = CPP^{-1}\bar{x}(t) = \bar{C}\bar{x}(t) \tag{11a}$$

and

$$\dot{\bar{x}}_1(t) = \bar{A}_{11}\bar{x}_1(t) + \bar{A}_{12}\bar{x}_2(t) + \bar{B}_1 u(t) \tag{11b}$$

$$0 = \bar{A}_{21}\bar{x}_1(t) + \bar{A}_{22}\bar{x}_2(t) + \bar{B}_2 u(t) \tag{11c}$$

where

$$
\begin{aligned}
\bar{A} = QAP &= \left[\begin{array}{cc} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{array}\right], \\
\bar{A}_{11} \in \Re^{r \times r}, \quad & \bar{A}_{12} \in \Re^{r \times (n-r)}, \quad \bar{A}_{21} \in \Re^{(n-r) \times r}, \quad \bar{A}_{22} \in \Re^{(n-r) \times (n-r)}, \\
\bar{B} = QB &= \left[\begin{array}{c} \bar{B}_1 \\ \bar{B}_2 \end{array}\right], \\
\bar{B}_1 \in \Re^{r \times m}, \quad & \bar{B}_2 \in \Re^{(n-r) \times m}, \quad \bar{C} = CP.
\end{aligned}
\tag{11d}
$$

From (10) it follows that

$$x(t) \in \Re_+^n \quad \text{and} \quad y(t) \in \Re_+^p \quad \text{if and only if} \quad \bar{x}(t) \in \Re_+^n \quad \text{and} \quad y(t) \in \Re_+^p, \quad t \geqslant 0 \tag{12}$$

since $P \in \Re^{n \times n}$ and $P^{-1} \in \Re_+^{n \times n}$.

Therefore, by this transformation the positivity of the descriptor systems is preserved since the positivity is invariant under the elementary row operations [13, 14].

Using (5) and (10) we obtain

$$u(t) = KPP^{-1}x(t) = \bar{K}\bar{x}(t) = \bar{K}_1\bar{x}_1(t) + \bar{K}_2\bar{x}_2(t) \tag{13a}$$

where

$$\bar{K} = [\ \bar{K}_1 \quad \bar{K}_2\ ], \quad \bar{K}_1 \in \Re^{m \times r}, \quad \bar{K}_2 \in \Re^{m \times (n-r)}. \tag{13b}$$

Taking into account that

$$Q[A + BK]P = \bar{A} + \bar{B}\bar{K} = \left[\begin{array}{cc} \bar{A}_{11} + \bar{B}_1\bar{K}_1 & \bar{A}_{12} + \bar{B}_1\bar{K}_2 \\ \bar{A}_{21} + \bar{B}_2\bar{K}_1 & \bar{A}_{22} + \bar{B}_2\bar{K}_2 \end{array}\right] \tag{14}$$

from (11) we obtain

$$\dot{\bar{x}}_1(t) = (\bar{A}_{11} + \bar{B}_1\bar{K}_1)\bar{x}_1(t) + (\bar{A}_{12} + \bar{B}_1\bar{K}_2)\bar{x}_2(t) + \bar{B}_1 u(t) \tag{15a}$$

$$0 = (\bar{A}_{21} + \bar{B}_2\bar{K}_1)\bar{x}_1(t) + (\bar{A}_{22} + \bar{B}_2\bar{K}_2)\bar{x}_2(t) + \bar{B}_2 u(t). \tag{15b}$$

If $\bar{B}_1 \in \Re^{r \times m}$, $\bar{B}_2 \in \Re^{(n-r) \times m}$ and there exist matrices $\bar{K}_1 \in \Re^{m \times r}$, $\bar{K}_2 \in \Re^{m \times (n-r)}$ such that

$$\bar{A}_{21} + \bar{B}_2 \bar{K}_1 \in \Re_+^{(n-r) \times r} \quad \text{and} \quad \bar{A}_{22} + \bar{B}_2 \bar{K}_2 \in M_{(n-r)S} \tag{16}$$

then from (15b) we have

$$\bar{x}_2(t) = -(\bar{A}_{22} + \bar{B}_2 \bar{K}_2)^{-1}[(\bar{A}_{21} + \bar{B}_2 \bar{K}_1)\bar{x}_1(t) + \bar{B}_2 u(t)] \in \Re_+^{n-r}, \quad t \geqslant 0 \tag{17}$$

for $\bar{x}_1(t) \in \Re_+^r$, $u(t) \in \Re_+^m$, $t \geqslant 0$ since $-(\bar{A}_{22} + \bar{B}_2 \bar{K}_2)^{-1} \in \Re_+^{(n-r) \times (n-r)}$ [10].
Substitution of (17) into (15a) yields

$$\dot{\bar{x}}_1(t) = \bar{A}'_1 \bar{x}_1(t) + \bar{B}'_1 u(t) \tag{18a}$$

where

$$\begin{aligned}
\bar{A}'_1 &= \bar{A}_{11} + \bar{B}_1 \bar{K}_1 - (\bar{A}_{12} + \bar{B}_1 \bar{K}_2)(\bar{A}_{22} + \bar{B}_2 \bar{K}_2)^{-1}(\bar{A}_{21} + \bar{B}_2 \bar{K}_1), \\
\bar{B}'_1 &= \bar{B}_1 - (\bar{A}_{12} + \bar{B}_1 \bar{K}_2)(\bar{A}_{22} + \bar{B}_2 \bar{K}_2)^{-1}\bar{B}_2.
\end{aligned} \tag{18b}$$

If there exist matrices $\bar{K}_1$ and $\bar{K}_2$ such that $\bar{A}'_1 \in M_{rS}$ and $\bar{B}'_1 \in \Re_+^{r \times m}$ then $\bar{x}_1(t) \in \Re_+^r$ and $\bar{x}_2(t) \in \Re_+^{(n-r)}$ for $t \geqslant 0$ and the closed-loop system is positive and asymptotically stable. The closed-loop system is regular since the matrix $\bar{A} + \bar{B}\bar{K}$ is nonsingular.

Let $\bar{A}_d \in M_{nS}$ be a desired closed-loop system matrix. Then the solution K of the equation

$$\bar{B}\bar{K} = \bar{A}_d - \bar{A} \tag{19}$$

is given by

$$\bar{K} = (\bar{B}^T \bar{B})^{-1} \bar{B}^T (\bar{A}_d - \bar{A}) \tag{20}$$

if and only if

$$(I_n - \bar{B}(\bar{B}^T \bar{B})^{-1} \bar{B}^T)(\bar{A}_d - \bar{A}) = 0. \tag{21}$$

By assumption $\operatorname{rank} B = m$ and $\operatorname{rank} \bar{B} = m$ since $\bar{B} = QB$ and the matrix Q is nonsingular. The matrix $\bar{B}^T \bar{B} \in \Re^{m \times m}$ is nonsingular and the matrix (20) is well defined. The condition (21) can be obtained by substitution of (20) into (19).

Remark 1 The characteristic polynomials of the closed-loop system are related by the equality

$$\det[\bar{E}s - (\bar{A} + \bar{B}\bar{K})] = \det[Es - (A + BK)] \det Q \det P. \tag{22}$$

The equality (22) follows from (14) since

$$\begin{aligned}
\det[\bar{E}s - (\bar{A} + \bar{B}\bar{K})] &= \det[QEPs - Q(A + BK)P] = \det[Q(Es - (A + BK))P] \\
&= \det[Es - (A + BK)] \det Q \det P.
\end{aligned} \tag{23}$$

Remark 2 The descriptor closed-loop system (6a) is asymptotically stable if and only if the closed-loop system (11) is asymptotically stable.

Therefore, the following theorem has been proved.

Theorem 2 *Let the descriptor system* (1) *satisfy the assumption* (2) *and the matrix E have only r* = rank *E nonzero columns. Then there exists a state-feedback gain matrix K of* (5) *such that the closed-loop system* (6) *is positive, asymptotically stable with regular pencil satisfying* (7) *if there exist matrices* $\bar{K}_1$ *and* $\bar{K}_2$ *such that* (16) *holds,* $\bar{A}_1' \in M_{rS}$, $\bar{B}_1' \in \mathfrak{R}_+^{r \times m}$ *and the condition* (21) *is met.*

From Theorem 2 and the above considerations we have the following procedure for computation of the desired gain matrix K.

Procedure 1

Step 1 Reduce the matrix E to the form (9) and find the matrices P and Q.

Step 2 Knowing P and Q compute the matrices $\bar{A}$ and $\bar{B}$ given by (11d).

Step 3 Choose a desired matrix $\bar{A}_d \in M_{nS}$ and check the condition (21). If the condition is met then compute the gain matrix $\bar{K}$ given by (20) and $K = \bar{K}P^{-1}$.

Example 1 Consider the descriptor system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}. \tag{24}$$

Compute the state-feedback gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \tag{25}$$

such that closed-loop system is positive, asymmetrically stable and its pencil is regular.

It is easy to see that the pencil of the system with (24) is singular. Using Procedure 1 we obtain the following:

Step 1. To reduce the matrix

$$E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{26}$$

to the desired form (9) we perform the following elementary operations: $R[2,3]$ and $L[3+1 \times (-2)]$, $L[1+3 \times 1]$, $L[3(-0.5)]$, $L[2,3]$. The matrices P and Q have the

form

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -0.5 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

since

$$QEP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{28}$$

Step 2. Using (11d), (24) and (27) we compute

$$\bar{A} = QAP = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -0.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B} = QB = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -0.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\tag{29}$$

Step 3. The desired matrix $\bar{A}_d \in M_{3S}$ is chosen in the form

$$\bar{A}_d = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}. \tag{30}$$

In this case the condition (21) is met since

$$(I_n - \bar{B}(\bar{B}^T \bar{B})^{-1} \bar{B}^T)(\bar{A}_d - \bar{A})$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & -0.5 \\ 1 & 2 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\tag{31}$$

Using (20) we compute the desired state-feedback gain matrix

$$\bar{K} = (\bar{B}^T \bar{B})^{-1} \bar{B}^T (\bar{A}_d - \bar{A})$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & -0.5 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -0.5 \\ 1 & 2 & -1 \end{bmatrix} \tag{32}$$

and

$$K = \bar{K}P^{-1} = \begin{bmatrix} 0 & -3 & -0.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & -3 \\ 1 & -1 & 2 \end{bmatrix}. \quad (33)$$

4. Concluding remarks

A new method of design of positive asymptotically stable descriptor systems with regular pencil by the use of state-feedbacks for descriptor continuous-time linear systems with singular pencils has been proposed. The method is based on the reduction by the use of elementary row and column operations of the matrices E, A, B to the special form so that $PEQ = \text{blockdiag}[I_r, 0]$ where Q (P) is the matrix of elementary row (column) operations. Sufficient conditions for existence of a solution to the problem has been established (Theorem 2). A procedure for the design of the state-feedbacks gain matrix K has been proposed and illustrated by a numerical example. The proposed method can be extended to fractional descriptor linear systems.

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