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NEW OSCILLATION CONDITIONS FOR FIRST-ORDER LINEAR RETARDED DIFFERENCE EQUATIONS WITH NON-MONOTONE ARGUMENTS

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Abstract. In this paper, we study the oscillatory behavior of the solutions of a first-order difference equation with non-monotone retarded argument and nonnegative coefficients, based on an iterative procedure. We establish some oscillation criteria, involving lim sup, which achieve a marked improvement on several known conditions in the literature. Two examples, numerically solved in MAPLE software, are also given to illustrate the applicability and strength of the obtained conditions.

Keywords: oscillation, difference equations, non-monotone argument.

Mathematics Subject Classification: 39A10, 39A21.

1. INTRODUCTION

Consider the difference equation

$$\Delta x(m) + b(m)x(\eta(m)) = 0, \quad m \in \mathbb{N}_0, \tag{1.1}$$

where $\Delta x(m) = x(m+1) - x(m)$, $\mathbb{N}_0$ is the set of all nonnegative integers and the sequences $(b(m))_{m\geq 0}$, $(\eta(m))_{m\geq 0}$ are nonnegative real numbers and integers, respectively with

$$\eta(m) \leq m-1 \quad \text{for all} \quad m \in \mathbb{N}_0 \quad \text{and} \quad \lim_{m\to\infty} \eta(m) = \infty.$$

A sequence of real numbers $(x(m))_{m\geq -\varrho}$ which satisfies Eq. (1.1) for all $m \geq 0$, is called a solution of Eq. (1.1) where $\varrho = -\min_{s\geq 0} \eta(s)$. As customary, a solution $(x(m))_{m\geq -\varrho}$ of Eq. (1.1) is called *oscillatory*, if the terms of the sequence $(x(m))_{m\geq -\varrho}$ are neither eventually positive nor eventually negative; otherwise it is called *nonoscillatory*. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Throughout this paper, we assume that the exists a nondecreasing sequence $(\gamma(m))_{m\geq m_0}$ such that $\eta(m)\leq\gamma(m)\leq m-1$ for $m\geq m_0$, $m_0\in\mathbb{N}_0$. We are going to use the following notation:

$$a^*=\liminf_{m\to\infty}\sum_{s=\gamma(m)}^{m-1}b(s),\quad a=\liminf_{m\to\infty}\sum_{s=\eta(m)}^{m-1}b(s),\quad a\leq\frac{1}{\mathrm{e}},$$

$$\theta(m)=\max_{0\leq s\leq m}\eta(s), \tag{1.2}$$

$$\omega(d):=\begin{cases}0, & \text{if}\quad d>\frac{1}{\mathrm{e}},\\ \dfrac{1-d-\sqrt{1-2d-d^2}}{2}, & \text{if}\quad d\in\left[0,\,\frac{1}{\mathrm{e}}\right],\end{cases}$$

and

$$\sum_{s=u}^{u-1}A(s)=0\quad\text{and}\quad\prod_{s=u}^{u-1}A(s)=1.$$

Also, let $\lambda(\varkappa)$ be the smaller positive root of $\lambda=\mathrm{e}^{\varkappa\lambda}$.

In recent few decades, a great effort has been performed to investigate the oscillatory behaviour of difference equations with deviating arguments, see [1, 3, 5–14, 16–25]. For example, the oscillation problem for the autonomous retarded difference equation

$$\Delta x(m)+b_1x(m-\varsigma)=0,\quad m\in\mathbb{N}_0, \tag{1.3}$$

where b_1 is a nonnegative real number and ς is a positive integer, has been completely solved by Ladas et al [20]. Indeed, Eq. (1.3) is oscillatory if and only if

$$b_1>\frac{\varsigma^\varsigma}{(\varsigma+1)^{\varsigma+1}},$$

while the results for the non-autonomous form of (1.3), i.e.,

$$\Delta x(m)+b(m)x(m-\varsigma)=0,\quad m\in\mathbb{N}_0, \tag{1.4}$$

are incomplete. Many sufficient oscillatory criteria have been obtained for this equation. For example, Erbe and Zhang [18] proved that, if

$$\liminf_{m\to\infty}b(m)>\frac{\varsigma^\varsigma}{(1+\varsigma)^{\varsigma+1}}, \tag{1.5}$$

or

$$\limsup_{m\to\infty}\sum_{s=m-\varsigma}^{m}b(s)>1, \tag{1.6}$$

then all solutions of Eq. (1.4) are oscillatory, while Ladas *et al.* [20] improved (1.5) to

$$\liminf_{m\to\infty}\left(\frac{1}{\varsigma}\sum_{s=m-\varsigma}^{m-1}b(s)\right)>\frac{\varsigma^\varsigma}{(1+\varsigma)^{\varsigma+1}}.$$

Chatzarakis *et al.* [13, 14] showed that all solutions of Eq. (1.1) with nondecreasing retarded argument are oscillatory, if

$$\limsup_{m\to\infty} \sum_{s=\eta(m)}^{m} b(s) > 1, \tag{1.7}$$

or

$$\liminf_{m\to\infty} \sum_{s=\eta(m)}^{m-1} b(s) > \frac{1}{\mathrm{e}}.$$

In the following we highlight some recent lim sup criteria for Eq. (1.1).

Chatzarakis *et al.* [15] improved (1.7) to

$$\limsup_{m\to\infty} \sum_{s=\theta(m)}^{m} b(s) = 1 - \frac{1}{2}\left(1 - a - \sqrt{1-2a}\right), \quad a > 0. \tag{1.8}$$

Braverman and Karpuz [5] and Stavroulakis [24] established the sufficient conditions

$$\limsup_{m\to\infty} \left(\sum_{s=\theta(m)}^{m} b(s) \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1-b(s_1)} \right) > 1 \tag{1.9}$$

and

$$\limsup_{m\to\infty} \left(\sum_{s=\theta(m)}^{m} b(s) \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1-b(s_1)} \right) > 1 - \omega(a), \quad a > 0, \tag{1.10}$$

respectively.

Braverman *et al.* [6] defined the sequence $\{\phi_{l+1}(\kappa, v)\}_{l\geq 0}$ recursively as follows:

$$\phi_1(\kappa, v) = \prod_{s=v}^{\kappa-1} \left(1 - b(s)\right), \quad \phi_{l+1}(\kappa, v) = \prod_{s=v}^{\kappa-1} \left(1 - b(s)\phi_l^{-1}\left(s, \eta(s)\right)\right) \tag{1.11}$$

and showed that Eq. (1.1) is oscillatory if

$$\limsup_{m\to\infty} \left(\sum_{s=\theta(m)}^{m} b(s)\phi_l^{-1}\left(\theta(m), \eta(s)\right) \right) > 1 \tag{1.12}$$

or

$$\limsup_{m\to\infty} \left(\sum_{s=\theta(m)}^{m} b(s)\phi_l^{-1}\left(\theta(m), \eta(s)\right) \right) > 1 - \omega(a), \quad a > 0, \tag{1.13}$$

for some $l \in \mathbb{N}$.

Very recently Chatzarakis *et al.* [3, 7–12, 17] established many iterative oscillation tests for Eq. (1.1). For example, Chatzarakis and Jadlovska [10], and Attia and Chatzarakis [3] established the conditions (1.14) and (1.15), respectively

$$\limsup_{m\to\infty} \sum_{s=\theta(m)}^{m} b(s)\mathrm{e}^{\sum_{s_1=\eta(s)}^{\theta(m)-1} b(s_1) \prod_{s_2=\eta(s_1)}^{s_1-1} \frac{1}{1-\Upsilon_l(s_2)}} > 1 \tag{1.14}$$

for some $l \in \mathbb{N}$, where

$$\Upsilon_l(m) = b(m)\left[1 + \sum_{s=\eta(m)}^{m-1} b(s) e^{\sum_{s_1=\eta(s)}^{m-1} b(s_1) \prod_{s_2=\eta(s_1)}^{s_1-1} \frac{1}{1-\Upsilon_{l-1}(s_2)}}\right],$$

with

$$\Upsilon_0(m) = b(m)\left[1 + \sum_{s=\eta(m)}^{m-1} b(s) e^{\lambda(a) \sum_{s_1=\eta(s)}^{m-1} b(s_1)}\right], \quad a > 0,$$

and

$$\limsup_{m\to\infty} \left(\frac{\vartheta_l(m)}{1 - \sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\theta(m)}^{m} b(s) \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1 - b(s_1)\Psi_l(s_1, \eta(s_1))} \right) > 1 \tag{1.15}$$

for some $l \in \mathbb{N}$, where

$$\vartheta_l(m) = \sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1) \prod_{s_2=\eta(s_1)}^{\theta(m)-1} \frac{1}{1 - b(s_2)\Psi_l(s_2, \eta(s_2))},$$

$$\Psi_l(\kappa, v) = 1 + \sum_{s=v}^{\kappa-1} \frac{b(s)}{\left(1 - \frac{1}{\kappa-\eta(s)} \sum_{s_1=\eta(s)}^{\kappa-1} b(s_1) \prod_{s_2=\eta(s_1)}^{s_1-1} \frac{1}{1-b(s_2)\Psi_{l-1}(s_2,\eta(s_2))}\right)^{\kappa-\eta(s)}},$$

$$\Psi_0(\kappa, v) = 1, \ \kappa \geq v$$

and

$$\zeta(m) = \min\{i \in \mathbb{N}_0 : i > m, \eta(i) > m - 1\}. \tag{1.16}$$

Very recently Attia and El-Matary [4] established the conditions

$$\limsup_{m\to\infty} \left(Q(m,l) + \sum_{j=\gamma(m)}^{m} b(j) \prod_{j_1=\eta(j)}^{\gamma(m)-1} \frac{1}{1 - b(j_1)U_l(j_1)} \right) > 1 \tag{1.17}$$

for some $l \in \mathbb{N}$, where $U_0(m) = 1$,

$$U_l(m) = \frac{\prod_{j=\eta(m)}^{\gamma(m)-1} \frac{1}{1-b(j)U_{l-1}(j)}}{1 - V_l(m)} \quad \text{for } l = 1, 2, \ldots,$$

$$Q(m,l) = \frac{\sum_{j=m+1}^{\zeta(m+1)-1} b(j) \sum_{j_1=\eta(j)}^{m} b(j_1) \prod_{j_2=\eta(j_1)}^{\gamma(j_1)-1} \frac{1}{1-b(j_2)U_l(j_2)}}{1 - \sum_{j=m+1}^{\zeta(m+1)-1} b(j)}$$

and

$$V_l(m) = \sum_{k=1}^{l-1} \prod_{j=2}^{k} \frac{1}{1 - V_{l-1}(\gamma^{j-1}(m))} \sum_{j_1=\gamma(m)}^{m-1} b(j_1) \sum_{j_2=\eta(j_1)}^{\gamma(m)-1} b(j_k) \ldots \sum_{j_k=\eta(j_{k-1})}^{\gamma^{k-1}(m)-1} b(j_r) + \prod_{j=2}^{l} \frac{1}{1 - V_{l-1}(\gamma^{j-1}(m))} \sum_{j_1=\gamma(m)}^{m-1} b(j_1) \sum_{j_2=\eta(j_1)}^{\gamma(m)-1} b(j_2) \ldots \sum_{j_l=\eta(j_{l-1})}^{\gamma^{l-1}(m)-1} b(j_l) \prod_{j_{l+1}=\eta(j_l)}^{\gamma^{l}(m)-1} \frac{1}{1 - b(j_{l+1})U_{l-1}(j_{l+1})}$$

for $l = 1, 2, \ldots$, and

$$\limsup_{m\to\infty} \left(\frac{\sum_{s=m+1}^{\gamma(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\phi_l^{-1}(\gamma(s_1), \eta(s_1))}{1 - \sum_{s=m+1}^{\gamma(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\phi_l^{-1}(\gamma(m), \eta(s)) \right) > 1, \tag{1.18}$$

where $l \in \mathbb{N}$ and $\phi_n(\kappa, v)$ and $\gamma(m)$ are defined respectively by (1.11) and (1.16).

In this paper we establish many oscillation results for Eq. (1.1), which may be considered as analogues of some results of the delay differential equations that can be found in [2]. The improvement of some of our results is shown, especially for Eq. (1.4). Finally, we discuss the applicability and effectiveness of some of the obtained results using two numerical examples.

2. MAIN RESULTS

In the following, we first introduce three important properties for a positive solution $x(m)$ of Eq. (1.1), which have a fundamental role in establishing our main results. According to [16, Lemma 2.1] and [8, Lemma 3], we have respectively that

$$\liminf_{m\to\infty} \frac{x(m+1)}{x(\gamma(m))} \geq \omega(a^*) \tag{2.1}$$

and

$$\liminf_{m\to\infty} \frac{x(\theta(m))}{x(m)} \geq \lambda(a), \quad a > 0. \tag{2.2}$$

The third property is introduced in the following lemma. For this purpose, the sequences $\{\Lambda_l(m)\}_{l\geq 0}$ and $\{\Upsilon_l(\kappa, v)\}_{l\geq 1}$, $\kappa \geq v$ are defined by

$$\Lambda_l(m) = \frac{\Upsilon_l(\gamma(m), \eta(m))}{1 - \sum_{s=\gamma(m)}^{m-1} b(s)\Upsilon_l(\gamma(m), \eta(s))}, \quad l = 1, 2, \ldots,$$

where

$$\Lambda_0(m) = \rho = \begin{cases} 1, & a = 0, \\ \lambda(a), & a > 0 \end{cases} \tag{2.3}$$

for $m \geq n_0$, $n_0 \in \mathbb{N}_0$ and

$$\Upsilon_l(\kappa, v) = \prod_{s=v}^{\kappa-1} \frac{1}{1 - b(s)\Lambda_{l-1}(s)}, \quad l = 1, 2, \ldots$$

Lemma 2.1. *Assume that $l \in \mathbb{N}_0$. Then, $b(m)\Lambda_l^\epsilon(m) < 1$ and*

$$\frac{x(\eta(m))}{x(m)} \geq \Lambda_l^\epsilon(m) \quad \textit{for all sufficiently large } m, \tag{2.4}$$

where

$$\Lambda_0^\epsilon(m) = \rho(\epsilon) = \begin{cases} 1, & a = 0, \\ \lambda(a) - \epsilon, & \epsilon \in (0, \lambda(a)), \quad a > 0 \end{cases} \tag{2.5}$$

and $\Lambda_l^\epsilon(m) = \Lambda_l(m)$, $\Upsilon_l^\epsilon(\kappa, v) = \prod_{s=v}^{\kappa-1} \frac{1}{1-b(s)\Lambda_{l-1}^\epsilon(s)}$ for $l = 1, 2, \ldots$

Proof. In view of the nonincreasing nature of $x(m)$ and (2.2), it follows for sufficiently small $\epsilon > 0$ that

$$\frac{x(\eta(m))}{x(m)} \geq \Lambda_0^\epsilon(m) \quad \text{for all sufficiently large } m. \tag{2.6}$$

Dividing Eq. (1.1) by $x(m)$, and then taking the product, from u to $v-1$, $v \geq u$, we get

$$0 < \frac{x(v)}{x(u)} = \prod_{s=u}^{v-1} \frac{x(s+1)}{x(s)} = \prod_{s=u}^{v-1} \left(1 - b(s)\frac{x(\eta(s))}{x(s)}\right), \tag{2.7}$$

that is,

$$x(u) = x(v) \prod_{s=u}^{v-1} \frac{1}{1 - b(s)\frac{x(\eta(s))}{x(s)}} \quad \text{for all } v \geq u. \tag{2.8}$$

Summing up Eq. (1.1) from u_1 to $v_1 - 1$, we get

$$x(v_1) - x(u_1) + \sum_{s=u_1}^{v_1-1} b(s)x(\eta(s)) = 0 \quad \text{for all } v_1 \geq u_1. \tag{2.9}$$

Therefore,

$$x(m) - x(\gamma(m)) + \sum_{s=\gamma(m)}^{m-1} b(s)x(\eta(s)) = 0.$$

Using (2.8) and $\eta(s) \le \gamma(s) \le \gamma(m)$ for $\gamma(m) \le s \le m-1$, the last equation takes the form

$$x(m) - x(\gamma(m)) + x(\gamma(m)) \sum_{s=\gamma(m)}^{m-1} b(s) \prod_{s_1=\eta(s)}^{\gamma(m)-1} \frac{1}{1 - b(s_1)\frac{x(\eta(s_1))}{x(s_1)}} = 0,$$

which, in turn, gives

$$\frac{x(\gamma(m))}{x(m)} = \left(1 - \sum_{s=\gamma(m)}^{m-1} b(s) \prod_{s_1=\eta(s)}^{\gamma(m)-1} \frac{1}{1 - b(s_1)\frac{x(\eta(s_1))}{x(s_1)}}\right)^{-1} > 0,$$

which yields by (2.6) and (2.8) that

$$\begin{aligned} \frac{x(\eta(m))}{x(m)} = \frac{x(\eta(m))}{x(\gamma(m))}\frac{x(\gamma(m))}{x(m)} &\ge \frac{\prod_{s=\eta(m)}^{\gamma(m)-1} \frac{1}{1-b(s)\Lambda_0^\epsilon(s)}}{1 - \sum_{s=\gamma(m)}^{m-1} b(s) \prod_{s_1=\eta(s)}^{\gamma(m)-1} \frac{1}{1-b(s_1)\Lambda_0^\epsilon(s_1)}} \\ &= \frac{\Upsilon_1^\epsilon(\gamma(m), \eta(m))}{1 - \sum_{s=\gamma(m)}^{m-1} b(s)\Upsilon_1^\epsilon(\gamma(m), \eta(s))} = \Lambda_1^\epsilon(m). \end{aligned}$$

Continuing in this manner, we get

$$\frac{x(\eta(m))}{x(m)} \ge \frac{\Upsilon_l^\epsilon(\gamma(m), \eta(m))}{1 - \sum_{s=\gamma(m)}^{m-1} b(s)\Upsilon_l^\epsilon(\gamma(m), \eta(s))} = \Lambda_l^\epsilon(m).$$

From this and (2.7), we obtain

$$b(m)\Lambda_l^\epsilon(m) < 1.$$

The proof is complete. □

Theorem 2.2. *If for some $l \in \mathbb{N}_0$ one of the following conditions is satisfied:*

(i)

$$b(\nu_s)\Lambda_l^\epsilon(\nu_s) \ge 1 \quad \textit{for all } s \in \mathbb{N}_0,$$

where $\{\nu_s\}_{s\ge 0}$ is an unbounded sequence of positive integers,

(ii)

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}(\gamma(m), \eta(s)) > 1 - \omega(a^*), \tag{2.10}$$

then Eq. (1.1) *is oscillatory.*

Proof. If not, there is no loss of generality to assume that there exists an eventually positive solution $x(m)$ of Eq. (1.1). Then according to Lemma 2.1, the proof of (i) is immediate. To complete the proof, we assume that (ii) is valid. In view of (2.9) from the proof of Lemma 2.1, we get

$$x(m+1) - x(\gamma(m)) + \sum_{s=\gamma(m)}^{m} b(s)x(\eta(s)) = 0. \tag{2.11}$$

Since $\eta(s) \leq \gamma(m)$ for $s \leq m$. Using (2.4) and (2.8), we have

$$x(\eta(s)) \geq x(\gamma(m)) \prod_{s_1=\eta(s)}^{\gamma(m)-1} \frac{1}{1 - b(s_1)\Lambda_l^{\epsilon}(s_1)} = x(\gamma(m))\Upsilon_{l+1}^{\epsilon}\left(\gamma(m), \eta(s)\right).$$

Substituting into (2.11), we obtain

$$x(m+1) - x(\gamma(m)) + x(\gamma(m)) \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(m), \eta(s)\right) \leq 0, \tag{2.12}$$

that is,

$$\sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(m), \eta(s)\right) \leq 1 - \frac{x(m+1)}{x(\gamma(m))}.$$

Consequently,

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(m), \eta(s)\right) \leq 1 - \liminf_{m\to\infty} \frac{x(m+1)}{x(\gamma(m))}.$$

Using (2.1) when $a^* > 0$ and the fact that $\frac{x(m+1)}{x(\gamma(m))} > 0$ when $a^* = 0$ imply that

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(m), \eta(s)\right) \leq 1 - \omega(a^*).$$

By letting $\epsilon \to 0$ the last inequality leads to

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}\left(\gamma(m), \eta(s)\right) \leq 1 - \omega(a^*).$$

This contradicts (2.10). □

Theorem 2.3. *Let $l \in \mathbb{N}_0$ and $\{\nu_i\}_{i\geq 0}$ be an unbounded sequence of positive integers. If one of the following conditions holds:*

(i)

$$\sum_{s=\gamma(\nu_i^*)}^{\gamma(\nu_i)-1} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(\nu_i^*),\eta(s)\right) \geq 1 \quad \text{for all } i\in\mathbb{N}_0 \text{ and all } \gamma(\nu_i)\leq \nu_i^* \leq \nu_i, \tag{2.13}$$

where $\Upsilon_l^{\epsilon}(\kappa, v)$ is defined as in Lemma (2.1),

(ii)

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} \frac{b(s)\Upsilon_{l+1}\left(\gamma(s),\eta(s)\right)}{1-\sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon_{l+1}\left(\gamma(s),\eta(s_1)\right)} > 1-\omega(a^*), \tag{2.14}$$

then Eq. (1.1) *is oscillatory.*

Proof. As before, let $x(m) > 0$ eventually for all sufficiently large m. Therefore, (2.9) implies that

$$x(m+1) - x(\gamma(m)) + \sum_{s=\gamma(m)}^{m} b(s)x(\eta(s)) = 0. \tag{2.15}$$

Since $\eta(s) \leq \gamma(s)$, then (2.4) and (2.8) imply that

$$x(\eta(s) \geq x(\gamma(s)) \prod_{s_1=\eta(s)}^{\gamma(s)-1} \frac{1}{1-b(s_1)\Lambda_l(s_1)} = x(\gamma(s))\Upsilon_{l+1}^{\epsilon}\left(\gamma(s),\eta(s)\right).$$

Substituting into (2.15), it follows that

$$x(m+1) - x(\gamma(m)) + \sum_{s=\gamma(m)}^{m} b(s)x(\gamma(s))\Upsilon_{l+1}^{\epsilon}\left(\gamma(s),\eta(s)\right) \leq 0. \tag{2.16}$$

Again, (2.9) for $m \geq s$ leads to

$$x(\gamma(m)) - x(\gamma(s)) + \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)x(\eta(s_1)) = 0. \tag{2.17}$$

Using the fact that $\gamma(s) \geq \eta(s_1)$ for $s \geq \gamma(m)$, $\gamma(m)-1 \geq s_1$, then (2.4) and (2.8), imply that

$$x(\eta(s_1)) \geq x(\gamma(s))\Upsilon_{l+1}^{\epsilon}\left(\gamma(s),\eta(s_1)\right).$$

From this and (2.17), we have

$$x(\gamma(m)) - x(\gamma(s)) + x(\gamma(s)) \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon_{l+1}^{\epsilon}\left(\gamma(s),\eta(s_1)\right) \leq 0,$$

or

$$0 < \frac{x(\gamma(m))}{x(\gamma(s))} \leq \left(1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s_1)\right)\right), \tag{2.18}$$

which leads to

$$x(\gamma(s)) \geq x(\gamma(m))\left(1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}(\gamma(s), \eta(s_1))\right)^{-1}.$$

Substituting into (2.16), we have

$$x(m+1) - x(\gamma(m)) + x(\gamma(m)) \sum_{s=\gamma(m)}^{m} \frac{b(s)\Upsilon_{l+1}\left(\gamma(s), \eta(s)\right)}{1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s_1)\right)} \leq 0.$$

Consequently,

$$\sum_{s=\gamma(m)}^{m} \frac{b(s)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s)\right)}{1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s_1)\right)} \leq 1 - \frac{x(m+1)}{x(\gamma(m))}.$$

Therefore,

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} \frac{b(s)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s)\right)}{1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s_1)\right)} \leq 1 - \omega(a^*).$$

Letting $\epsilon \to 0$ in the above inequality, we obtain

$$\limsup_{m\to\infty} \sum_{s=\gamma(m)}^{m} \frac{b(s)\Upsilon_{l+1}\left(\gamma(s), \eta(s)\right)}{1 - \sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon_{l+1}\left(\gamma(s), \eta(s_1)\right)} \leq 1 - \omega(a^*).$$

This contradicts (2.14), so the proof of (ii) is complete.

By (2.18),

$$\sum_{s_1=\gamma(s)}^{\gamma(m)-1} b(s_1)\Upsilon^{\epsilon}_{l+1}\left(\gamma(s), \eta(s_1)\right) < 1, \quad \gamma(m) \leq s \leq m$$

for all sufficiently large m, this contradicts (2.13) and completes the proof of (i). □

Theorem 2.4. *If for some* $l \in \mathbb{N}_0$,

$$\left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1(\gamma(m), \eta(s_1))}{1 - \sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}\left(\gamma(m), \eta(s)\right)\right) > 1, \tag{2.19}$$

where $\zeta(m)$ *is defined by* (1.16), *then Eq.* (1.1) *is oscillatory.*

Proof. As before, assume that $x(m) > 0$ for all sufficiently large m. From (2.12), we get

$$\frac{x(m+1)}{x(\gamma(m))} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}(\gamma(m),\eta(s)) \le 1. \tag{2.20}$$

Using (2.9) and the fact that $\zeta(m+1) > m+1$, we get

$$x(\zeta(m+1)) - x(m+1) + \sum_{s=m+1}^{\zeta(m+1)-1} b(s)x(\eta(s)) = 0. \tag{2.21}$$

Again (2.9) and the fact that $m \ge \eta(s)$ for $\zeta(m+1)-1 \ge s$, lead to

$$x(m+1) - x(\eta(s)) + \sum_{s_1=\eta(s)}^{m} b(s_1)x(\eta(s_1)) = 0.$$

Substituting into (2.21), we obtain

$$x(\zeta(m+1)) - x(m+1) + x(m+1)\sum_{s=m+1}^{\zeta(m+1)-1} b(s)$$
$$+ \sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)x(\eta(s_1)) = 0.$$

Therefore,

$$x(\zeta(m+1)) - x(m+1) + x(m+1)\sum_{s=m+1}^{\zeta(m+1)-1} b(s)$$
$$+ x(\gamma(m))\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\frac{x(\eta(s_1))}{x(\gamma(m))} = 0.$$

The positivity of $x(\zeta(m+1))$ leads to

$$\frac{x(m+1)}{x(\gamma(m))} > \frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\frac{x(\eta(s_1))}{x(\gamma(m))}}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} > 0. \tag{2.22}$$

Since $\eta(s_1) \le \gamma(m)$ for $s_1 \le m$, using (2.8) we obtain

$$\frac{x(m+1)}{x(\gamma(m))} > \frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1^{\epsilon}(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)}.$$

From this and (2.20), we get

$$\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1^{\epsilon}(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}(\gamma(m),\eta(s)) < 1.$$

It follows that

$$\limsup_{m\to\infty}\left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1^{\epsilon}(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}^{\epsilon}\left(\gamma(m),\eta(s)\right)\right)\leq 1.$$

As ϵ goes to zero, the above inequality gives

$$\limsup_{m\to\infty}\left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s)\sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_{l+1}\left(\gamma(m),\eta(s)\right)\right)\leq 1.$$

This contradicts with (2.19) and completes the proof. □

Corollary 2.5. *Let*

$$a_1^* = \liminf_{m\to\infty}\sum_{s=m-\varsigma}^{m-1} b(s)\sum_{s_1=s-\varsigma}^{m-\varsigma-1} b(s_1).$$

Equation (1.4) *is oscillatory, if*

$$\limsup_{m\to\infty}\sum_{s=m-\varsigma}^{m} b(s) > 1-\frac{a_1^*}{1-a}. \tag{2.23}$$

Proof. Since, $\eta(m)=m-\varsigma$ for Eq. (1.4), one can choose $\gamma(m)=m-\varsigma$, it follows that $\zeta(m)=m+\varsigma$. Let

$$B(m) = \frac{\sum\limits_{s=m+1}^{\zeta(m+1)-1} b(s)\sum\limits_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1(\gamma(m),\eta(s_1))}{1-\sum\limits_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_1\left(\gamma(m),\eta(s)\right).$$

Then

$$B(m)\geq\frac{\sum_{s=m+1}^{m+\varsigma} b(s)\sum_{s_1=s-\varsigma}^{m} b(s_1)}{1-\sum_{s=m+1}^{m+\varsigma} b(s)} + \sum_{s=m-\varsigma}^{m} b(s).$$

Therefore,

$$B(m)\geq\sum_{s=m-\varsigma}^{m} b(s)+\frac{a_1^*-\epsilon}{1-a-\epsilon}$$

for sufficiently small $\epsilon > 0$ and all sufficiently large m. Consequently,

$$\limsup_{m\to\infty} B(m) \geq \limsup_{m\to\infty} \sum_{s=m-\varsigma}^{m} b(s) + \frac{a_1^* - \epsilon}{1-a-\epsilon}.$$

As ϵ goes to zero, we get

$$\limsup_{m\to\infty} B(m) \geq \limsup_{m\to\infty} \sum_{s=m-\varsigma}^{m} b(s) + \frac{a_1^*}{1-a},$$

it follows from (2.23) that $\limsup_{m\to\infty} B(m) > 1$. A direct application of Theorem 2.4 completes the proof. □

Next, we define the sequence $\{\overline{\phi}_l(\kappa, v)\}_{l\geq 0}$ as follows:

$$\overline{\phi}_1(\kappa, v) = \prod_{s=v}^{\kappa-1} (1 - \rho b(s)), \quad \overline{\phi}_{l+1}(\kappa, v) = \prod_{s=v}^{\kappa-1} \left(1 - b(s)\overline{\phi}_l^{-1}(s, \eta(s))\right),$$

where ρ is defined by (2.3).

Theorem 2.6. *If for some* $n_1, n_2 \in \mathbb{N}$,

$$\limsup_{m\to\infty} \left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\overline{\phi}_{n_1}^{-1}(\gamma(m), \eta(s_1))}{1 - \sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\overline{\phi}_{n_2}^{-1}(\gamma(m), \eta(s)) \right) > 1, \quad (2.24)$$

where $\zeta(m)$ *is defined by* (1.16)*, then Eq.* (1.1) *is oscillatory.*

Proof. From (2.15), we have

$$\frac{x(m+1)}{x(\gamma(m))} + \sum_{s=\gamma(m)}^{m} b(s)\frac{x(\eta(s))}{x(\gamma(m))} = 1. \quad (2.25)$$

From (2.8), the nonincreasing nature of $x(m)$ and (2.2), we have

$$x(u) \geq x(v) \prod_{s=u}^{v-1} \frac{1}{1-\rho(\epsilon)b(s)} = \overline{\phi}_1^{-1}(v, u, \epsilon) \quad (2.26)$$

for all $v \geq u$, sufficiently large u and sufficiently small ϵ, where $\rho(\epsilon)$ is defined by (2.5).
Substituting form (2.26) into (2.8), we obtain

$$x(u) \geq x(v) \prod_{s=u}^{v-1} \frac{1}{1 - \overline{\phi}_1^{-1}(s, \eta(s), \epsilon)\, b(s)} = \overline{\phi}_2^{-1}(v, u, \epsilon) \quad \text{for all } v \geq u.$$

By repeating this argument n times, we obtain

$$x(u) \geq x(v) \prod_{s=u}^{v-1} \frac{1}{1-\overline{\phi}_{n-1}^{-1}(s,\eta(s),\epsilon)\, b(s)} = \overline{\phi}_{n}^{-1}(v,u,\epsilon) \quad \text{for all } v \geq u. \tag{2.27}$$

Since $\gamma(m) \geq \eta(s)$ for $m \geq s$, then we have

$$\frac{x(\eta(s))}{x(\gamma(m))} \geq \overline{\phi}_{n_2}^{-1}(\gamma(m),\eta(s),\epsilon), \quad m \geq s. \tag{2.28}$$

From (2.22) and (2.27), we obtain

$$\frac{x(m+1)}{x(\gamma(m))} > \frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\overline{\phi}_{n_1}^{-1}(\gamma(m),\eta(s_1),\epsilon)}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)}. \tag{2.29}$$

Substituting from (2.28) and (2.29) into (2.25), we get

$$\frac{\sum\limits_{s=m+1}^{\zeta(m+1)-1} b(s) \sum\limits_{s_1=\eta(s)}^{m} b(s_1)\overline{\phi}_{n_1}^{-1}(\gamma(m),\eta(s_1),\epsilon)}{1-\sum\limits_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\overline{\phi}_{n_2}^{-1}(\gamma(m),\eta(s),\epsilon) < 1.$$

Therefore,

$$\limsup_{m\to\infty} \left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\overline{\phi}_{n_1}^{-1}(\gamma(m),\eta(s_1),\epsilon)}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\overline{\phi}_{n_2}^{-1}(\gamma(m),\eta(s),\epsilon) \right) \leq 1.$$

As ϵ goes to zero, we have

$$\limsup_{m\to\infty} \left(\frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\overline{\phi}_{n_1}^{-1}(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\overline{\phi}_{n_2}^{-1}(\gamma(m),\eta(s)) \right) \leq 1,$$

which contradicts (2.24). □

Remark 2.7.
(1) Condition (2.24) improves and generalizes conditions (1.9), (1.12) and (1.18).
(2) Condition (2.23) improves condition (1.6) when $a_1^* \neq 0$.
(3) If $b(m) \geq \beta$ for all $m \geq m_0$, $m_0 \in \mathbb{N}_0$ and $a = \beta\varsigma$, then condition (2.23) becomes

$$\limsup_{m\to\infty} \sum_{s=m-\varsigma}^{m} b(s) > 1 - \frac{\left(a^2 + a\beta\right)}{2\,(1-a)}. \tag{2.30}$$

It is easy to show that condition (2.30) improves condition (1.8).

The following examples demonstrate the effectiveness and efficiency of some of our results over all the mentioned results. All calculations are made by using a code of Maple program.

Example 2.8. Consider the first-order difference equation

$$\Delta x(m) + b(m)x(\eta(m)) = 0, \quad n \geq 2, \tag{2.31}$$

where

$$b(m) = \begin{cases} \beta, & \text{if } m \in \{2m_r + 3,\ 2m_r + 2, \ldots, 2m_r - 8\}, \\ \delta, & \text{otherwise,} \end{cases}$$

where $\beta > \delta \geq 0$ and $\{m_r\}_{r\geq 0}$ is a sequence of positive integers such that $m_0 > 4$, $m_{r+1} > m_r + 7$ for all $r \in \mathbb{N}_0$, $\lim_{r\to\infty} m_r = \infty$, and

$$\eta(m) = \begin{cases} m-1, & \text{if } m = 2r, \\ m-3, & \text{if } m = 2r+1, \end{cases} \qquad r \in \mathbb{N}_0.$$

By (1.2) and (1.16), it is obvious that

$$\theta(m) = \begin{cases} m-1, & \text{if } m = 2r, \\ m-2, & \text{if } m = 2r+1, \end{cases} \qquad r \in \mathbb{N}_0$$

and

$$\zeta(m) = \begin{cases} m+2, & \text{if } m = 2r, \\ m+1, & \text{if } m = 2r+1, \end{cases} \qquad r \in \mathbb{N}_0,$$

(see Figure 1).

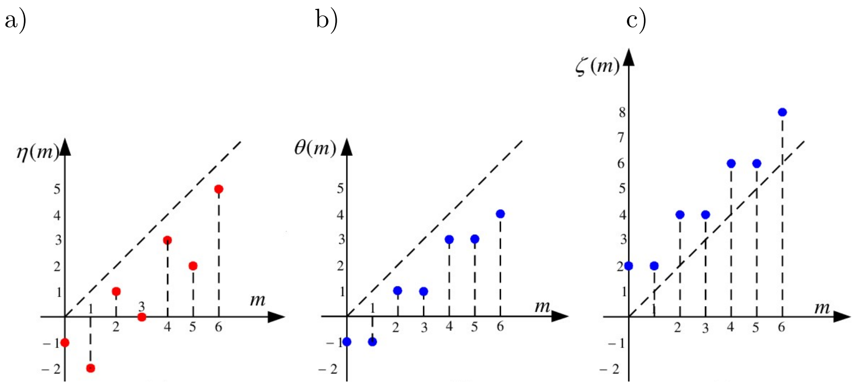

Fig. 1. The graphs of the sequences: a) $\eta(m)$, b) $\theta(m)$, and c) $\zeta(m)$

Let us assume that $\gamma(m) = \theta(m)$. Since

$$\sum_{s=\eta(m)}^{m-1} b(s) \geq \sum_{s=m-1}^{m-1} b(s) \geq \delta \quad \text{for all } m \in \mathbb{N}_0,$$

and

$$\sum_{s=\eta(2m_r+6)}^{2m_r+5} b(s) = \sum_{s=2m_r+5}^{2m_r+5} b(s) = \delta \quad \text{for all } r \in \mathbb{N}_0,$$

then $a = \liminf_{m\to\infty} \sum_{s=\eta(m)}^{m-1} b(s) = \delta$. Next, we study the cases $\delta = 0$ and $\delta = \frac{1}{\mathrm{e}}$.

Assume that $\delta = 0$, then $\rho = 1$ (that is defined by (2.3)), and hence conditions (1.8), (1.10), (1.13) and (1.14) can not be applied. Let

$$D(m) = \sum_{s=\theta(m)}^{m} b(s) \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1-b(s_1)}.$$

Then

$$D(2m_r+1) = \frac{\beta\left(3-\beta^3+4\beta^2-5\beta\right)}{(1-\beta)^3} < 0.9999, \quad r \in \mathbb{N}_0,$$

for all $\beta \in [0, 0.225]$, it follows that $\limsup_{m\to\infty} D(m) = \lim_{r\to\infty} D(2m_r+1) < 1$, so condition (1.9) is not satisfied. Also, for $\beta = 0.1621$, we have

$$\lim_{r\to\infty} \left(\sum_{s=\theta(2m_r+1)}^{2m_r+1} b(s)\phi_4^{-1}\left(\theta(2m_r+1), \eta(s)\right) \right) < 0.9 < 1,$$

and

$$\limsup_{m\to\infty}\left(\frac{\vartheta_l(m)}{1-\sum_{s=m+1}^{\zeta(m+1)-1}b(s)}+\sum_{s=\theta(m)}^{m}b(s)\prod_{s_1=\eta(s)}^{\theta(m)-1}\frac{1}{1-b(s_1)\Psi_l(s_1,\eta(s_1))}\right)$$
$$=\lim_{r\to\infty}\left(\frac{\vartheta_l(2m_r+1)}{1-\sum_{s=2m_r+2}^{\zeta(2m_r+2)-1}b(s)}+\sum_{s=\theta(2m_r+1)}^{2m_r+1}b(s)\prod_{s_1=\eta(s)}^{\theta(2m_r+1)-1}\frac{1}{1-b(s_1)\Psi_1(s_1,\eta(s_1))}\right)$$
$$<0.996,$$

therefore, conditions (1.12) with $l=4$ and (1.15) $l=1$ are not satisfied.

However, as we will show, our condition (2.24) is enough to guarantee the oscillation of Eq. (2.31) for all $\beta\in[0.1621,\ 0.19]$. Let

$$D_1(m)=\frac{\sum_{s=m+1}^{\zeta(m+1)-1}b(s)\sum_{s_1=\eta(s)}^{m}b(s_1)\overline{\phi_3}^{-1}(\gamma(m),\eta(s_1))}{1-\sum_{s=m+1}^{\zeta(m+1)-1}b(s)}+\sum_{s=\gamma(m)}^{m}b(s)\overline{\phi_4}^{-1}(\gamma(m),\eta(s)).$$

Then

$$D_1(2m_r+1)=\frac{\sum_{s=2m_r+2}^{\zeta(2m_r+2)-1}b(s)\sum_{s_1=\eta(s)}^{2m_r+1}b(s_1)\overline{\phi_3}^{-1}(\gamma(2m_r+1),\eta(s_1))}{1-\sum_{s=2m_r+2}^{\zeta(2m_r+2)-1}b(s)}+\sum_{s=\gamma(2m_r+1)}^{2m_r+1}b(s)\overline{\phi_4}^{-1}(\gamma(2m_r+1),\eta(s)).$$

Using Maple software, we get

$$D_1(2m_r+1)=\frac{A_1(\beta,1)}{(1-2\beta)}+\frac{\beta}{(1-\beta A_2(\beta,1))^2(1-\beta A_2(\beta,1)A_3(\beta,1))}+\beta+\frac{\beta}{1-\beta A_2(\beta,1)},$$

where

$$A_1(\beta,\rho)=\beta^2\left(1-\beta\left(1-\frac{\beta}{(-\beta\rho+1)^3}\right)^{-1}\right)^{-1}+\beta\left(\beta+\beta\left(1-\beta\left(1-\frac{\beta}{(-\beta\rho+1)^3}\right)^{-1}\right)^{-1}\right),$$

$$A_2(\beta,\rho)=\left(1-\frac{\beta}{\left(1-\frac{\beta}{1-\beta\rho}\right)^2\left(1-\frac{\beta}{(1-\beta\rho)^3}\right)}\right)^{-1}$$

and

$$A_3(\beta,\rho)=\left(1-\frac{\beta}{1-\frac{\beta}{(1-\beta\rho)^3}}\right)^{-2}.$$

Consequently, $D_1(2m_r+1)>1.0014$ for all $\beta\in[0.1621,\,0.19]$ and all $r\geq 0$. Therefore,

$$\lim_{r\to\infty} D_1(2m_r+1)>1 \quad \text{for all} \quad \beta\in[0.1621,\,0.19],$$

which, in view of condition (2.24) with $n_1=3$ and $n_2=4$, means that Eq. (2.31) is oscillatory for all $\beta\in[0.1621,\,0.19]$.

Now, we consider the case that $\delta=\frac{1}{\mathrm{e}}$, so $\rho=\mathrm{e}$ and $a=\frac{1}{\mathrm{e}}$. Therefore,

$$D_1(2m_r+1)=\frac{A_1(\beta,\frac{1}{\mathrm{e}})}{(1-2\,\beta)}+\frac{\beta}{\left(1-\beta\,A_2\left(\beta,\frac{1}{\mathrm{e}}\right)\right)^2\left(1-\beta\,A_2\left(\beta,\frac{1}{\mathrm{e}}\right)A_3\left(\beta,\frac{1}{\mathrm{e}}\right)\right)}+\beta$$
$$+\frac{\beta}{1-\beta\,A_2\left(\beta,\frac{1}{\mathrm{e}}\right)}>1 \quad \text{for } \beta\in[0.1367,\,0.142].$$

Then condition (2.24) with $n_1=3$ and $n_2=4$ is satisfied, and hence Eq. (2.31) is oscillatory for all $\beta\in[0.1367,\,0.142]$. Observe, however, that

$$\begin{aligned}F(2m_r+1)&=\frac{\sum_{s=2m_r+2}^{\zeta(2m_r+2)-1}b(s)\sum_{s_1=\eta(s)}^{2m_r+1}b(s_1)\phi_4^{-1}(\gamma(2m_r+1),\eta(s_1))}{1-\sum_{s=2m_r+2}^{\zeta(2m_r+2)-1}b(s)}\\&\quad+\sum_{s=\gamma(2m_r+1)}^{2m_r+1}b(s)\phi_4^{-1}\left(\gamma(2m_r+1),\eta(s)\right)\\&=\frac{\beta^2\left(2\left(1-\beta A_2(\beta,1)\right)^{-1}+1\right)}{1-2\beta}\\&\quad+\frac{\beta}{\left(1-\beta\,A_2\left(\beta,1\right)\right)^2\left(1-\beta\,A_2\left(\beta,1\right)A_3\left(\beta,1\right)\right)}\\&\quad+\beta+\frac{\beta}{1-\beta\,A_2\left(\beta,1\right)}<1 \quad \text{for } \beta\in[0,\,0.16]\end{aligned}$$

and

$$\begin{aligned}&\limsup_{m\to\infty}\sum_{s=\theta(m)}^{m}b(s)\mathrm{e}^{\sum_{s_1=\eta(s)}^{\theta(m)-1}b(s_1)\prod_{s_2=\eta(s_1)}^{s_1-1}\frac{1}{1-\Upsilon_l(s_2)}}\\&=\lim_{r\to\infty}\sum_{s=\theta(2m_r+1)}^{2m_r+1}b(s)\mathrm{e}^{\sum_{s_1=\eta(s)}^{\theta(2m_r+1)-1}b(s_1)\prod_{s_2=\eta(s_1)}^{s_1-1}\frac{1}{1-\Upsilon_l(s_2)}}<0.958\end{aligned}$$

for $\beta \in [0, 0.1525]$. Therefore, conditions (1.14) with $l = 1$ and (1.18) with $l = 4$ cannot be applied for $\beta \in [0, 0.1525]$ and $\beta \in [0, 0.16]$, respectively.

Example 2.9. Consider the first-order difference equation

$$\Delta x(m) + b(m)x(\eta(m)) = 0, \quad m \geq 2, \tag{2.32}$$

where

$$b(m) = \begin{cases} 0.15, & \text{if } m \in \{3m_r + 2,\ 3m_r + 1, \dots, 3m_r - 17\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{m_r\}_{r \geq 0}$ is a sequence of positive integers such that $m_0 > 10$, $m_{r+1} > m_r + 10$ for all $r \in \mathbb{N}_0$, $\lim_{r\to\infty} m_r = \infty$, and

$$\eta(m) = \begin{cases} m - 1, & \text{if } m = 3r, \\ m - 3, & \text{if } m = 3r + 1, \\ m - 1, & \text{if } m = 3r + 2, \end{cases} \quad r \in \mathbb{N}_0.$$

In view of (1.2) and (1.16), it follows that

$$\theta(m) = \begin{cases} m - 1, & \text{if } m = 3r, \\ m - 2, & \text{if } m = 3r + 1, \\ m - 1, & \text{if } m = 3r + 2, \end{cases} \quad r \in \mathbb{N}_0,$$

and

$$\zeta(m) = \begin{cases} m + 2, & \text{if } m = 3r, \\ m + 1, & \text{if } m = 3r + 1, \\ m + 1, & \text{if } m = 3r + 2, \end{cases} \quad r \in \mathbb{N}_0$$

(see Figure 2).

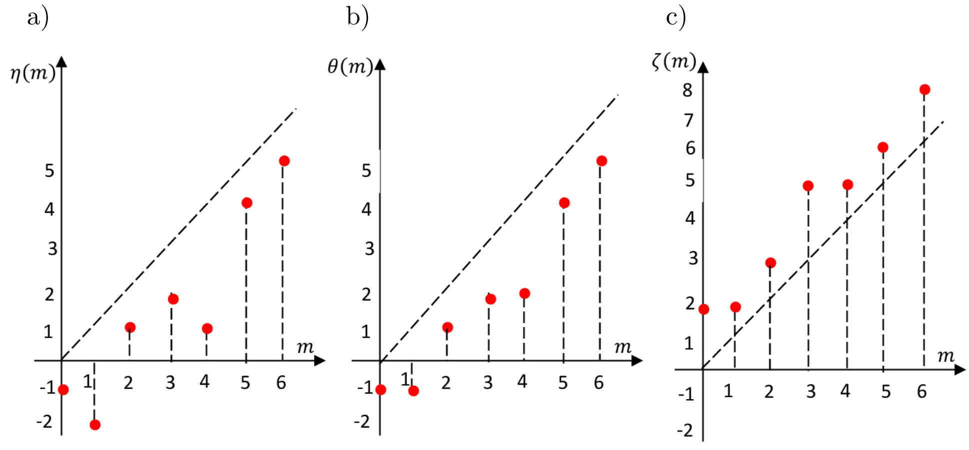

Fig. 2. The graphs of the sequences: a) $\eta(m)$, b) $\theta(m)$, and c) $\zeta(m)$

In the following we consider that $\gamma(m) = \theta(m)$. Clearly,

$$\sum_{s=\eta(m)}^{m-1} b(s) \geq 0 \quad \text{for all } m \in \mathbb{N}_0$$

and

$$\sum_{s=\eta(3m_r+5)}^{3m_r+4} b(s) = \sum_{s=3m_r+4}^{3m_r+4} b(s) = 0 \quad \text{for all } r \in \mathbb{N}_0,$$

then $a = \liminf_{m\to\infty} \sum_{s=\eta(m)}^{m-1} b(s) = 0$, and so conditions (1.8), (1.10), (1.13) and (1.14) fail to apply. Let

$$F(m) = \frac{\sum_{s=m+1}^{\zeta(m+1)-1} b(s) \sum_{s_1=\eta(s)}^{m} b(s_1)\Upsilon_1(\gamma(m), \eta(s_1))}{1 - \sum_{s=m+1}^{\zeta(m+1)-1} b(s)} + \sum_{s=\gamma(m)}^{m} b(s)\Upsilon_2\left(\gamma(m), \eta(s)\right).$$

Using Maple code, we obtain

$$\limsup_{m\to\infty} F(m) = \lim_{r\to\infty} F(3m_r + 1) > 1.02.$$

Consequently, condition (2.19) with $l = 1$ is satisfied and hence Eq. (2.32) is oscillatory. Observe that

$$\limsup_{m\to\infty} \left(\sum_{s=\theta(m)}^{m} 0.15 \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1-0.15} \right)$$
$$= \lim_{r\to\infty} \left(\sum_{s=\theta(3m_r+1)}^{3m_r+1} 0.15 \prod_{s_1=\eta(s)}^{\theta(3m_r+1)-1} \frac{1}{1-0.15} \right) < 0.6718,$$

$$\left(\frac{\vartheta_1(m)}{1 - \sum_{s=m+1}^{\zeta(m+1)-1} 0.15} + \sum_{s=\theta(m)}^{m} 0.15 \prod_{s_1=\eta(s)}^{\theta(m)-1} \frac{1}{1 - 0.15\Psi_1(s_1, \eta(s_1))} \right)$$
$$= \left(\frac{\vartheta_1(3m_r+1)}{1 - \sum_{s=3m_r+2}^{\zeta(3m_r+2)-1} 0.15} + \sum_{s=\theta(3m_r+1)}^{3m_r+1} 0.15 \prod_{s_1=\eta(s)}^{\theta(3m_r+1)-1} \frac{1}{1 - 0.15\Psi_1(s_1, \eta(s_1))} \right)$$
$$< 0.764,$$

$$\limsup_{m\to\infty} \left(Q(m,2) + \sum_{j=\gamma(m)}^{m} 0.15 \prod_{j_1=\eta(j)}^{\gamma(m)-1} \frac{1}{1-0.15U_2(j_1)} \right)$$
$$= \lim_{r\to\infty} \left(Q(3m_r+1, 2) + \sum_{j=\gamma(3m_r+1)}^{3m_r+1} 0.15 \prod_{j_1=\eta(j)}^{\gamma(3m_r+1)-1} \frac{1}{1-0.15U_2(j_1)} \right) < 0.99$$

and

$$\limsup_{m\to\infty}\left(\frac{\sum_{s=m+1}^{\gamma(m+1)-1}0.15\sum_{s_1=\eta(s)}^{m}0.15\phi_4^{-1}(\gamma(s_1),\eta(s_1))}{1-\sum_{s=m+1}^{\gamma(m+1)-1}0.15}+\sum_{s=\gamma(m)}^{m}0.15\phi_4^{-1}\left(\gamma(m),\eta(s)\right)\right)$$
$$=\lim_{r\to\infty}\left(\frac{\sum_{s=3m_r+2}^{\gamma(3m_r+2)-1}0.15\sum_{s_1=\eta(s)}^{3m_r+1}0.15\phi_4^{-1}(\gamma(s_1),\eta(s_1))}{1-\sum_{s=3m_r+2}^{\gamma(3m_r+2)-1}0.15}+\sum_{s=\gamma(3m_r+1)}^{3m_r+1}0.15\phi_4^{-1}\left(\gamma(3m_r+1),\eta(s)\right)\right)<0.99.$$

Therefore, none of the conditions (1.9), (1.15) with $l=1$, (1.17) with $l=2$, and (1.18) and (1.12) with $l=4$ can be applied to this equation.

3. CONCLUSION

In this paper, we investigated the oscillation problem of the retarded difference equation (1.1). Using the ideas of [2], many new oscillation criteria were established. We showed the improvement of our results over previous works, especially for (1.4).

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