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ASYMPTOTIC EXPANSIONS FOR THE FIRST HITTING TIMES OF BESSEL PROCESSES

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Abstract. We study a precise asymptotic behavior of the tail probability of the first hitting time of the Bessel process. We deduce the order of the third term and decide the explicit form of its coefficient.

Keywords: Bessel process, hitting time, tail probability, modified Bessel function, asymptotic expansion, Laplace transform.

Mathematics Subject Classification: 60G40, 60J60, 41A60, 30C10.

1. INTRODUCTION

A stock price processes in the Black–Scholes model, which is one of basic models in mathematical finance, is based on a geometric Brownian motion, which can be represented by a time change of a Bessel process. Moreover the Bessel process is useful for the study of the CIR model, which describes the evolution of instant interest rates.

Random time (e.g, default time and optimal stopping time) is one of the most principal objects in mathematical finance. The first hitting time in particular of an asset price process works essentially in the theory of American and exotic derivative securities when the model of a stock price is described by a geometric Brownian motion or a Bessel process. The probability that the first hitting time of the Bessel process is larger than a given time is significant for evaluating the possibility that a financial instrument (e.g., the stock price and the dangerous asset price) firstly reaches a given value after the given time. For applications of Bessel processes and first hitting times to mathematical finance, see [4, 18] for example.

We write $\tau_{a,b}^{(\nu)}$ for the first hitting time to b of the Bessel process with index ν starting from a. When $0 \leqq a < b$, the probability density function of $\tau_{a,b}^{(\nu)}$ is given explicitly (see [2, 15]). In the case that $0 \leqq b < a$, it is in [1, 6]. We can trivially deduce

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the formula for $P(t < \tau_{a,b}^{(\nu)} < \infty)$, which is denoted by $T_{a,b}^{(\nu)}(t)$ and is called the tail probability of $\tau_{a,b}^{(\nu)}$, from the form of the density function if $0 \leqq a < b$. In the case that $0 \leqq b < a$, although $T_{a,b}^{(\nu)}(t)$ is not easily calculated by integrating the density function of $\tau_{a,b}^{(\nu)}$, we can obtain a formula for the distribution function of $\tau_{a,b}^{(\nu)}$ by inverting its Laplace transform (cf. [7]). When $0 = b < a$, we can find that the distribution function of $\tau_{a,0}^{(\nu)}$ admits an asymptotic expansion in powers of $1/t$ for large t.

We concentrate on the case that $0 < b < a$. In the case that $\nu = 0$, we can give the asymptotic expansion of $T_{a,b}^{(\nu)}(t)$ by the same method which has been applied in [6]. Since $T_{a,b}^{(-\nu)}(t)$ can be represented by $T_{a,b}^{(\nu)}(t)$ for $\nu > 0$, it is sufficient to consider the case that $\nu > 0$. By a formula for the distribution function of $\tau_{a,b}^{(\nu)}$, we have asymptotic behavior of $T_{a,b}^{(\nu)}(t)$ for large t in [7]. Moreover an improvement of order of the error term is given in [8].

An interesting subject of the tail probability is to derive its term of higher order or an asymptotic expansion. The second term of $T_{a,b}^{(\nu)}(t)$ is discussed in [11] and decided explicitly in [3, 10]. Our purpose of this paper is to give an explicit form of the third term of the tail probability of $\tau_{a,b}^{(\nu)}$ in the case when $0 < b < a$ and $\nu > 0$. We can determine the order of its third term and its coefficient explicitly. Although an explicit form of $T_{a,b}^{(\nu)}(t)$ is given in [7], it is not useful for giving the higher term. The formula given in [3] is appropriate for calculations.

This article is organized as follows. Section 2 is devoted to giving several known results on $T_{a,b}^{(\nu)}(t)$. Section 3 gives an explicit form of $T_{a,b}^{(\nu)}(t)$ in terms of Bessel functions and asymptotic behavior of $T_{a,b}^{(\nu)}(t)$ which is our new result. In Section 4 we discuss asymptotic behavior of a functional of Bessel functions which plays an important role to prove our result provided in Section 3. Section 5 deals with showing that each coefficient is not zero and Section 6 is devoted to the case of the special value of ν.

2. PRELIMINARIES

For $\nu \in \mathbb{R}$ a Bessel process with index ν is the one-dimensional diffusion process with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}.$$

When $2\nu + 2$ is a positive integer, the law of the Bessel process coincides with that of radial part of a $2\nu + 2$ dimensional Brownian motion. This is the reason why $2\nu + 2$ is called the dimension of the Bessel process. We need to remark the classification of boundary points. The endpoint ∞ is natural for any $\nu \in \mathbb{R}$. The origin 0 is an entrance and not exit boundary for $\nu \geqq 0$ and is a regular boundary, which we make instantaneously reflecting, for $-1 < \nu < 0$. Moreover 0 is an exit but not entrance boundary for $\nu \leqq -1$. For more details, see [13, 14, 17].

For $a, b \geqq 0$ we use the notation $\tau_{a,b}^{(\nu)}$ for the first hitting time to b of the Bessel process with index ν starting at a. When $0 \leqq a < b$, the density function of $\tau_{a,b}^{(\nu)}$ is given explicitly in [2, 15]. Hence we discuss the case that $0 \leqq b < a$ in this paper. This case is interesting since we need to consider the natural boundary ∞. It is known that the Laplace transform of $\tau_{a,b}^{(\nu)}$ is represented in the form of a ratio of modified Bessel functions (cf. [14]). We obtain that, for $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} P(\tau_{a,b}^{(\nu)} \leqq t) dt = \begin{cases} \dfrac{2^{\nu+1}}{\Gamma(|\nu|)} \dfrac{K_\nu(a\sqrt{2\lambda})}{\lambda(a\sqrt{2\lambda})^\nu} & \text{if } 0 = b < a \text{ and } \nu < 0, \\ \left(\dfrac{b}{a}\right)^\nu \dfrac{K_\nu(a\sqrt{2\lambda})}{\lambda K_\nu(b\sqrt{2\lambda})} & \text{if } 0 < b < a \text{ and } \nu \in \mathbb{R}, \end{cases} \tag{2.1}$$

where Γ and K_ν are the gamma function and the modified Bessel function of the second kind of order ν, respectively (cf. [7]). This formula yields the well-known fact that

$$P(\tau_{a,b}^{(\nu)} < \infty) = \begin{cases} \left(\dfrac{b}{a}\right)^{2\nu} & \text{if } 0 < b < a \text{ and } \nu > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

Indeed, we can obviously have (2.2) by the Tauberian theorem with the help of the formula $K_\nu = K_{-\nu}$ for $\nu > 0$ and

$$K_\nu(x) = \begin{cases} \left(\log \dfrac{1}{x}\right)\{1 + o(1)\} & \text{if } \nu = 0, \\ \dfrac{\Gamma(\nu)}{2} \left(\dfrac{2}{x}\right)^\nu \{1 + o(1)\} & \text{if } \nu > 0 \end{cases}$$

as $x \downarrow 0$ (cf. [16, p. 111]). In addition, an explicit form of $P(\tau_{a,b}^{(\nu)} \leqq t)$ is given in [7].

In this paper we consider asymptotic behavior of the tail probability of $\tau_{a,b}^{(\nu)}$ and let

$$T_{a,b}^{(\nu)}(t) = P(t < \tau_{a,b}^{(\nu)} < \infty).$$

In the case that $0 = b < a$ and $\nu < 0$ we can invert the right hand side of (2.1) and obtain that

$$T_{a,0}^{(\nu)}(t) = \frac{2^\nu}{\Gamma(|\nu|) a^{2\nu}} \int_t^\infty s^{\nu-1} e^{-\frac{a^2}{2s}} ds = \frac{1}{\Gamma(|\nu|)} \int_0^{\frac{a^2}{2t}} x^{|\nu|-1} e^{-x} dx$$

in virtue of (2.2) (cf. [7]). Hence, for $t > 0$, we easily obtain that

$$T_{a,0}^{(\nu)}(t) = \frac{a^{2|\nu|}}{\Gamma(|\nu|) t^{|\nu|}} \sum_{n=0}^\infty \frac{(-a^2)^n}{(|\nu| + n) n!} \frac{1}{t^n}.$$

In the case when $0 < b < a$ and $\nu = 0$, we can conclude that $T_{a,b}^{(0)}(t)$ admits the asymptotic expansion in powers of $1/\log t$ by the same calculation applied in [6]. In addition, it follows from (2.1) and (2.2) that

$$T_{a,b}^{(\nu)}(t) = \left(\frac{b}{a}\right)^{2\nu} T_{a,b}^{(-\nu)}(t)$$

for $0 < b < a$ and $\nu < 0$ (cf. [11, 17]). Therefore it is sufficient to consider only the case that $0 < b < a$ and $\nu > 0$.

The remainder of this section is devoted to giving a known result on asymptotic behavior of $T_{a,b}^{(\nu)}(t)$. It is convenient to use the notation $\rho(t) = t/2b^2$ and $c = a/b$. We define five constants in order to give the coefficients of the first and the second terms of $T_{a,b}^{(\nu)}(t)$. For $\mu > 0$ and $r > 1$, let

$$\alpha_{\mu,r} = r^{\mu} - r^{-\mu}, \quad \beta_{\mu,r} = \frac{r}{4}\left(\frac{\alpha_{\mu-1,r}}{\mu-1} - \frac{\alpha_{\mu+1,r}}{\mu+1}\right), \quad \kappa_{\mu,r} = \frac{1-r^{-2\mu}}{2^{2\mu}\Gamma(\mu)\Gamma(\mu+1)},$$
$$\eta_{\mu} = \frac{1}{2(1-\mu)}, \quad \xi_{\mu} = \frac{\Gamma(1-\mu)\cos(\pi\mu)}{2^{2\mu-1}\Gamma(\mu+1)}.$$

It has been shown that $T_{a,b}^{(\nu)}(t)$ is asymptotically equal to $\kappa_{\nu,c}\Gamma(\nu)\rho(t)^{-\nu}$ (cf. [7, 8]). In addition, the second term of $T_{a,b}^{(\nu)}(t)$ is provided in [3, 10, 11]. Let $\kappa_{\nu,c}^{(1)} = \kappa_{\nu}\Gamma(\nu)$ and

$$\kappa_{\nu,c}^{(2)} = \begin{cases} \kappa_{\nu,c}\xi_{\nu}\Gamma(2\nu) & \text{if } 0 < \nu < 1 \text{ and } \nu \neq 1/2, \\ -\dfrac{(c-1)^3}{12\sqrt{\pi c}} & \text{if } \nu = 1/2, \\ -\dfrac{1-c^{-2}}{8} & \text{if } \nu = 1, \\ \kappa_{\nu,c}\left(\eta_{\nu} + \dfrac{\beta_{\nu,c}}{\alpha_{\nu,c}}\right)\Gamma(\nu+1) & \text{if } \nu > 1. \end{cases}$$

We have that

$$T_{a,b}^{(\nu)}(t) \sim \kappa_{\nu,c}^{(1)}\rho(t)^{-\nu} + \begin{cases} \kappa_{\nu,c}^{(2)}\rho(t)^{-2\nu} & \text{if } 0 < \nu < 1 \text{ and } \nu \neq 1/2, \\ \kappa_{\nu,c}^{(2)}\rho(t)^{-\nu-1} & \text{if } \nu = 1/2 \text{ or } \nu > 1, \\ \kappa_{1,c}^{(2)}\rho(t)^{-2}\log\rho(t) & \text{if } \nu = 1, \end{cases}$$

where $f(t) \sim g(t)$ means that $f(t) = g(t)(1 + o[1])$ as $t \to \infty$.

3. ASYMPTOTIC EXPANSION FOR $T_{a,b}^{(\nu)}(t)$

In this section we consider only the case when $\nu > 0$ and $0 < b < a$. Our purpose of this paper is to give the third term of $T_{a,b}^{(\nu)}(t)$ for latge t.

We define three more constants. For $\mu > 0$ and $r > 1$, let

$$\zeta_{\mu,r} = \frac{r^2}{32}\left(\frac{\alpha_{\mu-2,r}}{\mu-2} - \frac{\alpha_{\mu+2,r}}{\mu+2} + \frac{\alpha_{\mu,r}-\alpha_{\mu-2,r}}{\mu-1} + \frac{\alpha_{\mu+2,r}-\alpha_{\mu,r}}{\mu+1}\right),$$
$$\phi_\mu = \frac{\Gamma(1-\mu)}{2^{2\mu}\Gamma(\mu+1)}, \quad \sigma_\mu = \frac{2\mu-3}{16(\mu-2)(\mu-1)^2}$$

and then we have that $\xi_\mu = 2\phi_\mu \cos(\pi\mu)$. The third term of $T_{a,b}^{(\nu)}(t)$ is more complicated than lower terms.

Theorem 3.1. *We have the following asymptotic behavior for $T_{a,b}^{(\nu)}(t)$ as $t \to \infty$.*

(1) *If $0 < \nu < 1/3$,*

$$\begin{aligned} T_{a,b}^{(\nu)}(t) &= \kappa_{\nu,c}^{(1)}\rho(t)^{-\nu} + \kappa_{\nu,c}^{(2)}\rho(t)^{-2\nu} \\ &\quad + \kappa_{\nu,c}(\xi_\nu^2 - \phi_\nu^2)\Gamma(3\nu)\rho(t)^{-3\nu} + O[t^{-4\nu}]. \end{aligned} \tag{3.1}$$

(2) *If $\nu = 1/3$,*

$$\begin{aligned} T_{a,b}^{(1/3)}(t) &= \kappa_{1/3,c}^{(1)}\rho(t)^{-1/3} + \kappa_{1/3,c}^{(2)}\rho(t)^{-2/3} \\ &\quad + \kappa_{1/3,c}\left(\eta_{1/3} - \xi_{1/3}^3 + \frac{\beta_{1/3,c}}{\alpha_{1/3,c}}\right)\Gamma\left(\frac{4}{3}\right)\rho(t)^{-4/3} + O[t^{-5/3}]. \end{aligned} \tag{3.2}$$

(3) *If $1/3 < \nu < 1/2$,*

$$\begin{aligned} T_{a,b}^{(\nu)}(t) &= \kappa_{\nu,c}^{(1)}\rho(t)^{-\nu} + \kappa_{\nu,c}^{(2)}\rho(t)^{-2\nu} \\ &\quad + \kappa_{\nu,c}(\xi_\nu^2 - \phi_\nu^2)\Gamma(3\nu)\rho(t)^{-3\nu} + O[t^{-\nu-1}]. \end{aligned}$$

(4) *If $\nu = 1/2$,*

$$T_{a,b}^{(1/2)}(t) = \kappa_{1/2,c}^{(1)}\rho(t)^{-1/2} + \kappa_{1/2,c}^{(2)}\rho(t)^{-3/2} + \frac{(c-1)^5}{160\sqrt{\pi}\,c}\rho(t)^{-5/2} + O[t^{-7/2}].$$

(5) *If $1/2 < \nu < 1$,*

$$\begin{aligned} T_{a,b}^{(\nu)}(t) &= \kappa_{\nu,c}^{(1)}\rho(t)^{-\nu} + \kappa_{\nu,c}^{(2)}\rho(t)^{-2\nu} \\ &\quad + \kappa_{\nu,c}\left(\eta_\nu + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}}\right)\Gamma(\nu+1)\rho(t)^{-\nu-1} + O[t^{-3\nu}]. \end{aligned} \tag{3.3}$$

(6) *If $\nu = 1$,*

$$\begin{aligned} T_{a,b}^{(1)}(t) &= \kappa_{1,c}^{(1)}\rho(t)^{-1} + \kappa_{1,c}^{(2)}\rho(t)^{-2}\log\rho(t) \\ &\quad + \frac{1-c^{-2}}{4}\left(\frac{1}{2}\gamma - \log 2 + \frac{4c\log c - \alpha_{2,c}c}{8\alpha_{1,c}}\right)\rho(t)^{-2} \\ &\quad + O[t^{-3}\log t], \end{aligned} \tag{3.4}$$

where γ is the Euler constant.

(7) *If* $1 < \nu < 2$ *and* $\nu \neq 3/2$,

$$T^{(\nu)}_{a,b}(t) = \kappa^{(1)}_{\nu,c}\rho(t)^{-\nu} + \kappa^{(2)}_{\nu,c}\rho(t)^{-\nu-1} + \kappa_{\nu,c}\xi_\nu \Gamma(2\nu)\rho(t)^{-2\nu} + O[t^{-\nu-2}]. \tag{3.5}$$

(8) *If* $\nu = 2$,

$$T^{(2)}_{a,b}(t) = \kappa^{(1)}_{2,c}\rho(t)^{-2} + \kappa^{(2)}_{2,c}\rho(t)^{-3} + \frac{3(1-c^{-4})}{256}\rho(t)^{-4}\log\rho(t) + O[t^{-4}]. \tag{3.6}$$

(9) *If* $\nu > 2$ *or* $\nu = 3/2$,

$$\begin{aligned} T^{(\nu)}_{a,b}(t) &= \kappa^{(1)}_{\nu,c}\rho(t)^{-\nu} + \kappa^{(2)}_{\nu,c}\rho(t)^{-\nu-1} \\ &\quad + \kappa_{\nu,c}\left(\eta_\nu^2 - \sigma_\nu + \frac{\beta_{\nu,c}\eta_\nu + \zeta_{\nu,c}}{\alpha_{\nu,c}}\right)\Gamma(\nu+2)\rho(t)^{-\nu-2} + O[R_\nu(t)], \end{aligned} \tag{3.7}$$

where

$$R_\nu(t) = \begin{cases} t^{-2\nu} & \textit{if } 2 < \nu < 3, \\ t^{-6}\log t & \textit{if } \nu = 3, \\ t^{-\nu-3} & \textit{if } \nu > 3 \textit{ or } \nu = 3/2. \end{cases}$$

Before establishing Theorem 3.1, we start to treat an integral representation of $T^{(\nu)}_{a,b}(t)$. A formula for $T^{(\nu)}_{a,b}(t)$ can be deduced from (2.2) and the explicit form of the distribution function of $\tau^{(\nu)}_{a,b}$ given in [7]. However the formula is not useful for calculating higher terms of $T^{(\nu)}_{a,b}(t)$ and hence we will use another formula for $T^{(\nu)}_{a,b}(t)$ described in [3]. For $\mu > 0$, $r > 1$ and $x > 0$ let

$$L_{\mu,r}(x) = \frac{J_\mu(x)Y_\mu(rx) - J_\mu(rx)Y_\mu(x)}{J_\mu(x)^2 + Y_\mu(x)^2},$$

where J_μ and Y_μ are Bessel functions of the first and the second kinds of order μ, respectively.

Lemma 3.2. *We have that*

$$T^{(\nu)}_{a,b}(t) = \frac{c^{-\nu}}{\pi}\int_0^1 \frac{L_{\nu,c}(\sqrt{x})}{x} e^{-\rho(t)x}dx + O[t^{-1}e^{-\rho(t)}] \tag{3.8}$$

as $t \to \infty$.

Proof. We will use C_1, C_2, C_3, C_4 for suitable constants which are independent of variables.

It has been obtained in [3] that

$$T^{(\nu)}_{a,b}(t) = \frac{c^{-\nu}}{\pi}\int_0^\infty \frac{L_{\nu,c}(b\sqrt{u})}{u} e^{-\frac{tu}{2}}du. \tag{3.9}$$

We first give a proof of (3.9) because no paper shows it except for [3] written in Japanese. The proof which we aim to give is based on the original proof.

Note that the Laplace transform of $T_{a,b}^{(\nu)}(t)$ can be represented by $L_{\nu,c}$. Indeed, we can easily obtain by (2.1) and (2.2) that

$$\int_0^\infty e^{-\lambda t} T_{a,b}^{(\nu)}(t)dt = \frac{c^{-2\nu}}{\lambda} - \frac{c^{-\nu}K_\nu(a\sqrt{2\lambda})}{\lambda K_\nu(b\sqrt{2\lambda})}$$

and thus the formula

$$\frac{K_\nu(a\sqrt{x})}{xK_\nu(b\sqrt{x})} = \frac{c^{-\nu}}{x} - \frac{1}{\pi}\int_0^\infty \frac{L_{\nu,c}(b\sqrt{y})}{y(x+y)}dy$$

for $x > 0$ (cf. [12]) gives that

$$\int_0^\infty e^{-\lambda t} T_{a,b}^{(\nu)}(t)dt = \frac{2c^{-\nu}}{\pi}\int_0^\infty \frac{L_{\nu,c}(b\sqrt{y})}{y(2\lambda+y)}dy.$$

Hence it is sufficient to see that

$$\int_0^\infty dt\, e^{-\lambda t}\int_0^\infty \frac{L_{\nu,c}(b\sqrt{u})}{u}e^{-\frac{tu}{2}}du = 2\int_0^\infty \frac{L_{\nu,c}(b\sqrt{y})}{y(2\lambda+y)}dy. \tag{3.10}$$

In order to obtain (3.10), we need to change the order of the double integral in the left hand side of (3.10). In other words, we should show that the Fubini theorem can be applied to (3.10).

It is well-known that

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)}(1+o[1]), \tag{3.11}$$

$$Y_\nu(x) = -\frac{2^\nu \Gamma(\nu)}{\pi x^\nu}(1+o[1]) \tag{3.12}$$

as $x \downarrow 0$ (cf. [16, pp. 134–135]) and a simple calculation shows that

$$J_\nu(x)Y_\nu(cx) - J_\nu(cx)Y_\nu(x) = \frac{\alpha_{\nu,c}}{\pi\nu}(1+o[1]),$$

$$J_\nu(x)^2 + Y_\nu(x)^2 = \frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2 x^{2\nu}}(1+o[1])$$

as $x \downarrow 0$. Hence we have that

$$L_{\nu,c}(x) = \frac{\pi\alpha_{\nu,c}}{2^{2\nu}\Gamma(\nu)\Gamma(\nu+1)}x^{2\nu}(1+o[1]). \tag{3.13}$$

This immediately yields that

$$\sup_{0<x\leqq 1} |L_{\nu,c}(x)| \leqq C_1 x^{2\nu}. \tag{3.14}$$

Since

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right)(1 + O[x^{-1}]),$$
$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right)(1 + O[x^{-1}])$$

as $x \to \infty$ (cf. [16, pp. 134–135]), it can be easily obtained that

$$L_{\nu,c}(x) = \frac{\sin(c-1)x}{\sqrt{c}}(1 + O[x^{-1}])$$

as $x \to \infty$, which gives that

$$\sup_{x\geqq 1} |L_{\nu,c}(x)| \leqq C_2. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{aligned}
\int_0^\infty \frac{|L_{\nu,c}(b\sqrt{u})|}{u} e^{-\frac{tu}{2}} du &= 2\int_0^\infty \frac{|L_{\nu,c}(x)|}{x} e^{-\rho(t)x^2} dx \\
&\leqq C_1 \int_0^1 x^{2\nu-1} e^{-\rho(t)x^2} dx + C_2 \int_1^\infty \frac{1}{x} e^{-\rho(t)x^2} dx \\
&\leqq C_1 \int_0^1 x^{2\nu-1} dx + C_2 \int_0^\infty e^{-\rho(t)x^2} dx \\
&\leqq C_3(1 + t^{-1/2}).
\end{aligned}$$

Since $(1 + t^{-1/2})e^{-\lambda t}$ is an integrable function of t on $(0, \infty)$ for each $\lambda > 0$, we can apply the Fubini theorem to the left hand side of (3.10). Hence we finish the proof of (3.10) and can conclude that (3.9) holds.

We are ready to see (3.8). A change of variables from u to x given by $x = b^2 u$ gives that the right hand side of (3.9) is equal to

$$\frac{c^{-\nu}}{\pi} \int_0^\infty \frac{L_{\nu,c}(\sqrt{x})}{x} e^{-\rho(t)x} dx.$$

Applying (3.15) again, we have that

$$\left| \int_1^\infty \frac{L_{\nu,c}(\sqrt{x})}{x} e^{-\rho(t)x} dx \right| \leqq C_4 \int_1^\infty \frac{1}{x} e^{-\rho(t)x} dx \leqq C_4 \int_1^\infty e^{-\rho(t)x} dx,$$

which is of order $t^{-1}e^{-\rho(t)}$ and hence the proof of (3.8) is completed. □

Lemma 3.2 implies that large time asymptotics of $T_{a,b}^{(\nu)}(t)$ can be determined by asymptotic behavior of $L_{\nu,c}(x)$ as $x \downarrow 0$. In order to show Theorem 3.1, we need to improve (3.13) in the following way.

Proposition 3.3. *We have the following asymptotic behavior for $L_{\nu,c}(x)$ as $x \downarrow 0$.*

(1) *If $0 < \nu < 1/2$ and $\nu \neq 1/3$,*

$$L_{\nu,c}(x) = \pi c^{\nu} \kappa_{\nu,c} x^{2\nu} \{1 + \xi_\nu x^{2\nu} + (\xi_\nu^2 - \phi_\nu^2) x^{4\nu} + O[Q_\nu(x)]\}, \tag{3.16}$$

where

$$Q_\nu(x) = \begin{cases} x^{6\nu} & \text{if } 0 < \nu < 1/3, \\ x^2 & \text{if } 1/3 < \nu < 1/2. \end{cases}$$

(2) *If $\nu = 1/3$,*

$$\begin{aligned} L_{1/3,c}(x) = \pi c^{1/3} \kappa_{1/3,c} x^{2/3} \Big\{ & 1 + \xi_{1/3} x^{2/3} \\ & + \left(\eta_{1/3} - \xi_{1/3}^3 + \frac{\beta_{1/3,c}}{\alpha_{1/3,c}} \right) x^2 + O[x^{8/3}] \Big\}. \end{aligned} \tag{3.17}$$

(3) *If $\nu = 1/2$,*

$$L_{1/2,c}(x) = \alpha_{1/2,c} x \left\{ 1 - \frac{(c-1)^2}{6} x^2 + \frac{(c-1)^4}{120} x^4 + O[x^6] \right\}. \tag{3.18}$$

(4) *If $1/2 < \nu < 1$,*

$$L_{\nu,c}(x) = \pi c^{\nu} \kappa_{\nu,c} x^{2\nu} \left\{ 1 + \xi_\nu x^{2\nu} + \left(\eta_\nu + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}} \right) x^2 + O[x^{4\nu}] \right\}. \tag{3.19}$$

(5) *If $\nu = 1$,*

$$\begin{aligned} L_{1,c}(x) = \frac{1}{4} \pi \alpha_{1,c} x^2 \Big\{ & 1 + x^2 \log x \\ & + \left(\gamma_0 + \frac{4c \log c - \alpha_{2,c} c}{8\alpha_{1,c}} \right) x^2 + O[x^4 (\log x)^2] \Big\}, \end{aligned} \tag{3.20}$$

where $\gamma_0 = \gamma - \log 2 - 1/2$.

(6) *If $1 < \nu < 2$ and $\nu \neq 3/2$,*

$$L_{\nu,c}(x) = \pi c^{\nu} \kappa_{\nu,c} x^{2\nu} \left\{ 1 + \left(\eta_\nu + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}} \right) x^2 + \xi_\nu x^{2\nu} + O[x^4] \right\}. \tag{3.21}$$

(7) *If $\nu = 2$,*

$$L_{2,c}(x) = \frac{1}{32} \pi \alpha_{2,c} x^4 \left\{ 1 + \left(\eta_2 + \frac{\beta_{2,c}}{\alpha_{2,c}} \right) x^2 + \frac{1}{8} x^4 \log x + O[x^4] \right\}. \tag{3.22}$$

(8) *If* $\nu > 2$ *or* $\nu = 3/2$,

$$L_{\nu,c}(x) = \pi c^{\nu}\kappa_{\nu,c}x^{2\nu}\Big\{1 + \Big(\eta_\nu + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}}\Big)x^2 + \Big(\eta_\nu^2 - \sigma_\nu + \frac{\beta_{\nu,c}\eta_\nu + \zeta_{\nu,c}}{\alpha_{\nu,c}}\Big)x^4 + O[Q_\nu(x)]\Big\}, \tag{3.23}$$

where

$$Q_\nu(x) = \begin{cases} x^{2\nu} & \text{if } 2 < \nu < 3, \\ x^6 \log x & \text{if } \nu = 3, \\ x^6 & \text{if } \nu > 3 \text{ or } \nu = 3/2. \end{cases}$$

The proof of Proposition 3.3 is deferred to the next section and we try to complete the proof of Theorem 3.1. For $p > 0$, $t > 0$ and an integer $m \geqq 1$ let

$$F_p(t) = \int_0^1 e^{-\rho(t)x}x^{p-1}dx, \quad G_p^m(t) = \int_0^1 e^{-\rho(t)x}x^{p-1}(\log x)^m\, dx.$$

Lemma 3.2 and Proposition 3.3 imply that we need to give asymptotic behavior of $F_p(t)$ and $G_p^m(t)$ as $t \to \infty$ in order to show Theorem 3.1.

Lemma 3.4. *We have that*

$$F_p(t) = \Gamma(p)\rho(t)^{-p} + O[t^{-1}e^{-\rho(t)}], \tag{3.24}$$

$$G_p^1(t) = \Gamma(p)\rho(t)^{-p}\{\psi(p) - \log\rho(t)\} + O[t^{-2}e^{-\rho(t)}], \tag{3.25}$$

$$G_p^2(t) = O[t^{-p}(\log t)^2] \tag{3.26}$$

as $t \to \infty$, *where* ψ *is the logarithmic derivative of the gamma function.*

Proof. Recall that $\rho(t) = t/2b^2$. Let

$$f_p(t) = \int_1^\infty e^{-\rho(t)x}x^{p-1}dx$$

and then we have that

$$F_p(t) = \int_0^\infty e^{-\rho(t)x}x^{p-1}dx - f_p(t) = \Gamma(p)\rho(t)^{-p} - f_p(t).$$

It is sufficient to see that, for $t > 4b^2p$

$$0 \leqq f_p(t) \leqq 2\rho(t)^{-1}e^{-\rho(t)}. \tag{3.27}$$

However its proof is easy. Indeed, applying the integration of by parts, we have that, for $t > 4b^2p$

$$0 \leqq f_p(t) \leqq \int_1^\infty e^{-\rho(t)x} x^p dx = \rho(t)^{-1} e^{-\rho(t)} + p\rho(t)^{-1} \int_1^\infty e^{-\rho(t)x} x^{p-1} dx$$
$$\leqq \rho(t)^{-1} e^{-\rho(t)} + \frac{1}{2} f_p(t),$$

which gives (3.27).

We next try to prove (3.25). Let

$$g_p(t) = \int_1^\infty e^{-\rho(t)x} x^{p-1} \log x \, dx$$

and then we have that

$$G_p^1(t) = \int_0^\infty e^{-\rho(t)x} x^{p-1} \log x \, dx - g_p(t).$$

It is known that the integral in the right hand side coincides with

$$\Gamma(p)\rho(t)^{-p}\{\psi(p) - \log \rho(t)\}$$

(cf. [5, p. 573]). The way of the estimate of $g_p(t)$ is similar to $f_p(t)$. The integration by parts gives that $g_p(t)$ is dominated by

$$\int_1^\infty e^{-\rho(t)x} x^p \log x \, dx = p\rho(t)^{-1} g_p(t) + \rho(t)^{-1} f_p(t).$$

Hence it follows from (3.27) that $0 \leqq g_p(t) \leqq 2\rho(t)^{-1} f_p(t) \leqq 4\rho(t)^{-2} e^{-\rho(t)}$ for $t > 4b^2 p$ and we can conclude (3.25).

The estimate of $G_p^2(t)$ can be obtained easily. We actually can deduce (3.26) from

$$0 \leqq G_p^2(t) \leqq \int_0^\infty e^{-\rho(t)x} x^{p-1} (\log x)^2 \, dx$$
$$= \Gamma(p)\rho(t)^{-p} \left[\{\psi(p) - \log \rho(t)\}^2 + \sum_{n=0}^\infty \frac{1}{(n+p)^2} \right]$$

(cf. [5, p. 576]). □

We are now ready to prove Theorem 3.1. Lemma 3.2 yields that it is sufficient to give asymptotic behavior of

$$\int_0^1 \frac{L_{\nu,c}(\sqrt{x})}{x} e^{-\rho(t)x} dx. \tag{3.28}$$

We will consider only the following three cases:

$$(1)\quad 0<\nu<1/3 \qquad (2)\quad \nu=1 \qquad (3)\quad \nu=2$$

since the way of calculation in other cases is the same as the case (1).

For $0<\nu<1/3$ we obtain by (3.16) and (3.24) that (3.28) is $\pi c^{\nu}\kappa_\nu$ multiple of

$$\begin{aligned}&F_\nu(t)+\xi_\nu F_{2\nu}(t)+(\xi_\nu^2-\phi_\nu^2)F_{3\nu}(t)+O\left[F_{4\nu}(t)\right]\\&=\Gamma(\nu)\rho(t)^{-\nu}+\xi_\nu\Gamma(2\nu)\rho(t)^{-2\nu}+(\xi_\nu^2-\phi_\nu^2)\Gamma(3\nu)\rho(t)^{-3\nu}+O[t^{-4\nu}].\end{aligned}$$

Hence (3.1) can be easily derived by (3.8).

We next consider the case that $\nu=1$. In virtue of (3.20), (3.24) and (3.25), we obtain that (3.28) for $\nu=1$ is $\pi\alpha_{1,c}/4$ multiple of

$$\begin{aligned}&F_1(t)+\frac{1}{2}G_2^1(t)+\left(\gamma_0+\frac{4c\log c-\alpha_{2,c}c}{8\alpha_{1,c}}\right)F_2(t)+O[G_3^2(t)]\\&=\rho(t)^{-1}-\frac{1}{2}\rho(t)^{-2}\log\rho(t)+\left(\frac{1}{2}\gamma-\log 2+\frac{4c\log c-\alpha_{2,c}c}{8\alpha_{1,c}}\right)\rho(t)^{-2}\\&\quad+O[t^{-3}(\log t)^2],\end{aligned}$$

where we have applied that $\psi(2)=-\gamma+1$. This yields (3.3).

Moreover it follows from (3.22) that (3.28) for $\nu=2$ is $\pi\alpha_{2,c}/32$ multiple of

$$\begin{aligned}&F_2(t)+\left(\eta_2+\frac{\beta_{2,c}}{\alpha_{2,c}}\right)F_3(t)+\frac{1}{16}G_4^1(t)+O[F_4(t)]\\&=\rho(t)^{-2}+2\left(\eta_2+\frac{\beta_{2,c}}{\alpha_{2,c}}\right)\rho(t)^{-3}-\frac{3}{8}\rho(t)^{-4}\log\rho(t)+O[t^{-4}].\end{aligned}$$

Hence we obtain (3.6) and finish the proof of Theorem 3.1.

4. ASYMPTOTIC BEHAVIOR OF $L_{\nu,c}(x)$ FOR SMALL x

Recall that $c=a/b>1$ and that $L_{\nu,c}(x)=L_{\nu,c}^0(x)/L_\nu^1(x)$, where

$$\begin{aligned}L_{\nu,c}^0(x)&=J_\nu(x)Y_\nu(cx)-J_\nu(cx)Y_\nu(x),\\L_\nu^1(x)&=J_\nu(x)^2+Y_\nu(x)^2.\end{aligned}$$

Our goal of this section is to show Proposition 3.3, which gives asymptotic behavior of $L_{\nu,c}(x)$ as $x\downarrow 0$.

The calculation for $\nu=1/2$ is quite easy. Indeed, the well-known formulas

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x,\quad Y_{1/2}(x)=-\sqrt{\frac{2}{\pi x}}\cos x$$

(cf. [16, pp. 111–112]) give that

$$L^0_{1/2,c}(x) = \frac{1}{\pi x\sqrt{c}}\sin(c-1)x, \quad L^1_{1/2}(x) = \frac{1}{\pi x},$$

which yields that $L_{1/2,c}(x) = \sin(c-1)x/\sqrt{c}$. This immediately gives (3.18).

In order to prove Proposition 3.3 for $\nu \neq 1/2$, we need to derive higher terms of the right hand sides of (3.11) and (3.12). For $\mu \in \mathbb{R}$ the definition of J_μ yields that

$$J_\mu(x) = \frac{x^\mu}{2^\mu \Gamma(\mu+1)}\left\{1 - \frac{x^2}{4(\mu+1)} + \frac{x^4}{32(\mu+1)(\mu+2)} + O[x^6]\right\} \tag{4.1}$$

as $x \downarrow 0$ (cf. [16, p. 102]) and thus we need to improve (3.12). Note that

$$Y_\mu(x) = \frac{J_\mu(x)\cos(\pi\mu) - J_{-\mu}(x)}{\sin(\pi\mu)} \tag{4.2}$$

if μ is not integer (cf. [16, p. 104]) and that

$$\begin{aligned} Y_n(x) = &-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \\ &+\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{n+2k}}{k!(n+k)!}\left\{2\log\frac{x}{2} - \psi(k+1) - \psi(k+n+1)\right\} \end{aligned} \tag{4.3}$$

for an integer $n \geqq 1$ (cf. [16, p. 107]). Recall that we have put

$$\gamma_0 = \gamma - \log 2 - 1/2.$$

The following lemma is a straightforward consequence of (4.3) and thus we omit its proof.

Lemma 4.1. *It follows that*

$$Y_1(x) = -\frac{2}{\pi x}\left(1 - \frac{1}{2}x^2\log x - \frac{1}{2}\gamma_0 x^2 + O[x^4\log x]\right), \tag{4.4}$$

$$Y_2(x) = -\frac{4}{\pi x^2}\left(1 + \frac{1}{4}x^2 - \frac{1}{16}x^4\log x + O[x^4]\right) \tag{4.5}$$

and that

$$Y_n = -\frac{2^n\Gamma(n)}{\pi x^n}\left\{1 + \frac{1}{4(n-1)}x^2 + \frac{1}{32(n-1)(n-2)}x^4 + O[Q_n(x)]\right\}$$

for an integer $n \geqq 3$.

We concentrate on considering the case that ν is not an integer. In this case we can derive asymptotic behavior of $Y_\nu(x)$ for small x by (4.1) and (4.2).

Lemma 4.2. *Assume that ν is not an integer. We have the following asymptotic behavior of $Y_\nu(x)$ as $x \downarrow 0$.*

(1) *If $0 < \nu < 1$,*

$$Y_\nu(x) = -\frac{2^\nu \Gamma(\nu)}{\pi x^\nu}\left\{1 - \frac{1}{2}\xi_\nu x^{2\nu} - \frac{1}{2}\eta_\nu x^2 + \frac{1}{8(\nu+1)}\xi_\nu x^{2\nu+2} + O[x^4]\right\}. \tag{4.6}$$

(2) *If $1 < \nu < 2$ and $\nu \neq 3/2$,*

$$Y_\nu(x) = -\frac{2^\nu \Gamma(\nu)}{\pi x^\nu}\left(1 - \frac{1}{2}\eta_\nu x^2 - \frac{1}{2}\xi_\nu x^{2\nu} + O[x^4]\right). \tag{4.7}$$

(3) *If $\nu > 2$ or $\nu = 3/2$,*

$$Y_\nu(x) = -\frac{2^\nu \Gamma(\nu)}{\pi x^\nu}\left\{1 - \frac{1}{2}\eta_\nu x^2 + \frac{1}{32(2-\nu)(1-\nu)}x^4 + O[Q_\nu(x)]\right\}. \tag{4.8}$$

Proof. Note that (4.1) gives

$$J_{-\nu}(x) = \frac{2^\nu}{\Gamma(-\nu+1)}x^{-\nu}\left\{1 - \frac{x^2}{4(1-\nu)} + \frac{x^4}{32(2-\nu)(1-\nu)} + O[x^6]\right\}. \tag{4.9}$$

It follows from (4.2) that $Y_{3/2}(x) = -J_{-3/2}(x)$ and hence we can easily deduce (4.8) for $\nu = 3/2$ in virtue of the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{4.10}$$

(cf. [16, p. 3]). Since

$$-\nu < \nu < -\nu + 2 < \nu + 2 < -\nu + 4 < \nu + 4 < -\nu + 6 < \nu + 6$$

if $0 < \nu < 1$, we have by (4.1), (4.2) and (4.9) that

$$\begin{aligned} Y_\nu(x) = &- \frac{2^\nu}{\Gamma(1-\nu)\sin(\pi\nu)}x^{-\nu} + \frac{\cos(\pi\nu)}{2^\nu \Gamma(\nu+1)\sin(\pi\nu)}x^\nu \\ &+ \frac{2^\nu}{4(1-\nu)\Gamma(1-\nu)\sin(\pi\nu)}x^{-\nu+2} - \frac{\cos(\pi\nu)}{2^{\nu+2}(\nu+1)\Gamma(\nu+1)\sin(\pi\nu)}x^{\nu+2} \\ &+ O[x^{-\nu+4}]. \end{aligned}$$

The formula (4.10) yields that

$$\begin{aligned} Y_\nu(x) = -\frac{2^\nu \Gamma(\nu)}{\pi x^\nu}\Bigg\{1 &- \frac{\Gamma(1-\nu)\cos(\pi\nu)}{2^{2\nu}\Gamma(\nu+1)}x^{2\nu} \\ &- \frac{1}{4(1-\nu)}x^2 + \frac{\Gamma(1-\nu)\cos(\pi\nu)}{2^{2\nu+2}\Gamma(\nu+2)}x^{2\nu} + O[x^4]\Bigg\}, \end{aligned}$$

which gives (4.6).

Note that

$$\begin{cases} -\nu < -\nu+2 < \nu < -\nu+4 < \nu+2 & \text{if } 1<\nu<2, \\ -\nu < -\nu+2 < -\nu+4 < \nu < -\nu+6 & \text{if } 2<\nu<3, \\ -\nu < -\nu+2 < -\nu+4 < -\nu+6 < \nu & \text{if } \nu>3. \end{cases}$$

We can derive (4.7) and (4.8) in the same fashion as (4.6) and then omit their calculations. □

We are now ready to show Proposition 3.3 and first consider the case that $\nu = 1, 2$. It follows from (4.1) and (4.4) that the numerator of $L_{1,c}(x)$ is

$$\frac{c}{\pi}\left\{1 - \frac{1}{2}x^2 \log x - \frac{1}{8}(c^2 + 4\gamma_0)x^2 + O[x^4 \log x]\right\}$$
$$- \frac{c^{-1}}{\pi}\left\{1 - \frac{1}{2}c^2x^2 \log(cx) - \frac{1}{8}(1 + 4\gamma_0 c^2)x^2 + O[x^4 \log x]\right\}.$$

This immediately implies that

$$L^0_{1,c}(x) = \frac{\alpha_{1,c}}{\pi}\left\{1 + \frac{4c\log c - \alpha_{2,c}c}{8\alpha_{1,c}}x^2 + O[x^4 \log x]\right\}.$$

Applying (4.1) and (4.4) again, we obtain that

$$\begin{aligned} L^1_1(x) &= \frac{4}{\pi^2 x^2}\left\{1 - \frac{1}{2}x^2 \log x - \frac{1}{2}\gamma_0 x^2 + O[x^4 \log x]\right\}^2 + O[x^2] \\ &= \frac{4}{\pi^2 x^2}\left\{1 - x^2 \log x - \gamma_0 x^2 + O[x^4 (\log x)^2]\right\}, \end{aligned}$$

which yields that

$$\frac{1}{L^1_1(x)} = \frac{1}{4}\pi^2 x^2 \left\{1 + x^2 \log x + \gamma_0 x^2 + O[x^4 (\log x)^2]\right\}.$$

Therefore we can conclude that

$$L_{1,c}(x) = \frac{1}{4}\pi\alpha_{1,c}x^2\left\{1 + x^2\log x + \left(\gamma_0 + \frac{4c\log c - \alpha_{2,c}c}{8\alpha_{1,c}}\right)x^2 + O[x^4(\log x)^2]\right\},$$

which gives (3.20).

Similarly we obtain by (4.1) and (4.5) that $L^0_{2,c}(x)$ is

$$\begin{aligned} &\frac{c^2}{2\pi}\left(1 + \frac{3-c^2}{12}x^2 - \frac{1}{16}x^4 \log x + O[x^4]\right) \\ &- \frac{c^{-2}}{2\pi}\left(1 + \frac{3c^2-1}{12}x^2 - \frac{1}{16}c^4 x^4 \log x + O[x^4]\right) \\ &= \frac{1}{2\pi}(\alpha_{2,c} + \beta_{2,c}x^2 + O[x^4]) \end{aligned}$$

and that $L_2^1(x)$ is

$$\frac{16}{\pi^2 x^4}\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4\log x+O[x^4]\right).$$

Hence we have that

$$L_{2,x}(x)=\frac{1}{32}\pi\alpha_{2,c}x^4\left\{1+\left(\frac{\beta_{2,c}}{\alpha_{2,c}}-\frac{1}{2}\right)x^2+\frac{1}{8}x^4\log x+O[x^4]\right\}$$

and then finish to show (3.22) since $\eta_2=-1/2$.

Before showing Proposition 3.3 in the other cases, we will give a formula for the numerator of $L_{\nu,c}(x)$.

Lemma 4.3. *We have that*

$$L^0_{\nu,c}(x)=\frac{1}{\pi\nu}(\alpha_{\nu,c}+\beta_{\nu,c}x^2+O[x^4]) \tag{4.11}$$

in the case that $0<\nu<2$ *except for* $\nu=1,3/2$ *and that*

$$L^0_{\nu,c}(x)=\frac{1}{\pi\nu}(\alpha_{\nu,c}+\beta_{\nu,c}x^2+\zeta_{\nu,c}x^4+O[G_\nu(x)]) \tag{4.12}$$

in the case that $\nu>2$ *or* $\nu=3/2$.

Proof. When $0<\nu<1$, it follows from (4.1) and (4.6) that $L^0_{\nu,c}(x)$ is

$$\begin{aligned}&\frac{c^\nu}{\pi\nu}\left\{1-\frac{1}{2}\xi_\nu x^{2\nu}-\frac{1}{4}\left(2\eta_\nu+\frac{c^2}{\nu+1}\right)x^2+\frac{c^2+1}{8(\nu+1)}\xi_\nu x^{2\nu+2}+O[x^4]\right\}\\&-\frac{c^{-\nu}}{\pi\nu}\left\{1-\frac{1}{2}c^{2\nu}\xi_\nu x^{2\nu}-\frac{1}{2}\left(2c^2\eta_\nu+\frac{1}{\nu+1}\right)x^2+\frac{c^{2\nu}(c^2+1)}{8(\nu+1)}\xi_\nu x^{2\nu+2}+O[x^4]\right\}\\&=\frac{1}{\pi\nu}\left\{\alpha_{\nu,c}+\frac{1}{4}\left(2c^{-\nu+2}\eta_\nu+\frac{c^{-\nu}}{\nu+1}-2c^\nu\eta_\nu-\frac{c^{\nu+2}}{\nu+1}\right)x^2+O[x^4]\right\},\end{aligned}$$

which can be represented by the right hand side of (4.11). In the case when $1<\nu<2$ and $\nu\neq 3/2$, we can deduce (4.11) from (4.1) and (4.8) in the same way as the case that $0<\nu<1$ and thus omit its proof.

In the case that $\nu>2$ or $\nu=3/2$, it follows from (4.1) and (4.8) that $L^0_{\nu,c}(x)$ is

$$\begin{aligned}&\frac{c^\nu}{\pi\nu}\left[1-\frac{1}{4}\left(\frac{1}{1-\nu}+\frac{c^2}{\nu+1}\right)x^2\right.\\&\qquad\left.+\frac{1}{32}\left\{\frac{1}{(\nu-2)(\nu-1)}+\frac{2c^2}{(\nu+1)(1-\nu)}+\frac{c^4}{(\nu+1)(\nu+2)}\right\}x^4+O[Q_\nu(x)]\right]\\&-\frac{c^{-\nu}}{\pi\nu}\left[1-\frac{1}{4}\left(\frac{c^2}{1-\nu}+\frac{1}{\nu+1}\right)x^2\right.\\&\qquad\left.+\frac{1}{32}\left\{\frac{c^4}{(\nu-2)(\nu-1)}+\frac{2c^2}{(\nu+1)(1-\nu)}+\frac{1}{(\nu+1)(\nu+2)}\right\}x^4+O[Q_\nu(x)]\right].\end{aligned}$$

which is $1/\pi\nu$ multiple of

$$\alpha_{\nu,c} + \frac{1}{4}\left(\frac{c^{-\nu+2} - c^{\nu}}{1-\nu} - \frac{c^{\nu+2} - c^{-\nu}}{\nu+1}\right)x^2 + \frac{1}{32}\left\{\frac{c^{\nu} - c^{-\nu+4}}{(\nu-2)(\nu-1)} + \frac{2(c^{\nu+2} - c^{-\nu+2})}{(\nu+1)(\nu-1)} + \frac{c^{\nu+4} - c^{-\nu}}{(\nu+1)(\nu+2)}\right\}x^4 + O[Q_\nu(x)].$$

Since the coefficients of x^2 and x^4 coincide with $\beta_{\nu,c}$ and $\zeta_{\nu,c}$, respectively, we can conclude that (4.12) holds. □

We consider the case that $0 < \nu < 2$ but $\nu \neq 3/2$. For proofs of (3.16) and (3.17) we need to calculate the denominator of $L_{\nu,c}(x)$. When $0 < \nu < 1/2$, we have by (4.1) and (4.6) that

$$L^1_\nu(x) = \frac{x^{2\nu}}{2^{2\nu}\Gamma(\nu+1)^2}(1 + O[x^2]) + \frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2 x^{2\nu}}\left(1 - \xi_\nu x^{2\nu} + \frac{1}{4}\xi_\nu^2 x^{4\nu} - \eta_\nu x^2 + O[x^{2\nu+2}]\right),$$

which is equal to

$$\frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2 x^{2\nu}}\left[1 - \xi_\nu x^{2\nu} + \left\{\frac{1}{4}\xi_\nu^2 + \frac{\pi^2}{2^{4\nu}\Gamma(\nu)^2\Gamma(\nu+1)^2}\right\}x^{4\nu} - \eta_\nu x^2 + O[x^{2\nu+2}]\right].$$

The coefficient of $x^{4\nu}$ coincides with

$$\phi_\nu^2 \cos^2(\pi\nu) + \frac{\pi^2}{2^{4\nu}\Gamma(\nu)^2\Gamma(\nu+1)^2} = \phi_\nu^2,$$

where (4.10) has been applied. This yields that

$$\begin{aligned}\frac{1}{L^1_\nu(x)} &= \frac{\pi^2 x^{2\nu}}{2^{2\nu}\Gamma(\nu)^2}\frac{1}{1 - (\xi_\nu x^{2\nu} - \phi_\nu^2 x^{4\nu} + \eta_\nu x^2 + O[x^{2\nu+2}])} \\ &= \frac{\pi^2 x^{2\nu}}{2^{2\nu}\Gamma(\nu)^2}\{1 + \xi_\nu x^{2\nu} + (\xi_\nu^2 - \phi_\nu^2)x^{4\nu} + \eta_\nu x^2 \\ &\qquad - 2\xi_\nu\phi_\nu^2 x^{6\nu} + O[x^{2\nu+2}] + O[x^{8\nu}]\}.\end{aligned}$$

Note that $\xi_\nu^2 = \phi_\nu^2$ if and only if $\nu = 1/3$ in the case that $0 < \nu < 1/2$. Hence, if $0 < \nu < 1/2$ but $\nu \neq 1/3$, we have by (4.11) that

$$\begin{aligned}L_{\nu,c}(x) &= \frac{\pi x^{2\nu}}{2^{2\nu}\nu\Gamma(\nu)^2}(\alpha_{\nu,c} + O[x^2])\{1 + \xi_\nu + (\xi_\nu^2 - \phi_\nu^2)x^{4\nu} + O[x^{6\nu}]\} \\ &= \pi c^\nu \kappa_{\nu,c} x^{2\nu}\{1 + \xi_\nu x^{2\nu} + (\xi_\nu^2 - \phi_\nu^2)x^{4\nu} + O[x^{6\nu}] + O[x^2]\},\end{aligned}$$

which gives (3.16). When $\nu = 1/3$, noting that $\xi_\nu = \phi_\nu$, we deduce from (4.11) that

$$\begin{aligned}L_{\nu,c}(x) &= \frac{\pi x^{2\nu}}{2^{2\nu}\nu\Gamma(\nu)^2}(\alpha_{\nu,c} + \beta_{\nu,c}x^2 + O[x^4]) \\ &\qquad\qquad \times \{1 + \xi_\nu x^{2\nu} + (\eta_\nu - 2\xi_\nu\phi_\nu^2)x^{6\nu} + O[x^{8\nu}]\} \\ &= \pi c^\nu \kappa_{\nu,c} x^{2\nu}\left\{1 + \xi_\nu x^{2\nu} + \left(\eta_\nu - 2\xi_\nu\phi_\nu^2 + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}}\right)x^{6\nu} + O[x^{8\nu}]\right\}.\end{aligned}$$

This implies (3.17). The calculations for (3.19) and (3.21) is the case as that for (3.17) and then we omit them.

It remains the case that $\nu > 2$ or $\nu = 3/2$, however the calculation is easy. In virtue of (4.1) and (4.8) we can show that $L_\nu^1(x)$ is

$$\frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2 x^{2\nu}}(1 - \eta_\nu x^2 + \sigma_\nu x^4 + O[Q_\nu(x)]).$$

which yields that

$$\frac{1}{L_\nu^1(x)} = \frac{\pi^2 x^{2\nu}}{2^{2\nu}\Gamma(\nu)^2}\{1 + \eta_\nu x^2 + (\eta_\nu^2 - \sigma_\nu)x^4 + O[Q_\nu(x)]\}.$$

We combine it with (4.12) and hence can easily derive (3.23). The proof of Proposition 3.3 is finished.

Remark 4.4. Calculating $L_{1,c}^0(x)$ carefully, we can obtain that

$$L_{1,c}^0(x) = \frac{\alpha_{1,c}}{\pi}\left\{1 + \frac{4c\log c - \alpha_{2,c}c}{8\alpha_{1,c}}x^2 + O[x^4]\right\}.$$

However this is of no effect on improvement of the error term of (3.20).

5. THE COEFFICIENTS OF $T_{a,b}^{(\nu)}(t)$

In Section 3 we have established Theorem 3.1. However we did not discuss whether each coefficient in the formula for $T_{a,b}^{(\nu)}(t)$ is zero or not. This section is devoted to considering the coefficients of $T_{a,b}^{(\nu)}(t)$.

Proposition 5.1. *We have that*

$$4\alpha_{1,c}(\gamma - 2\log 2) + 4c\log c - \alpha_{2,c}c < 0. \tag{5.1}$$

Especially the third term of (3.4) *is negative.*

Proof. For $x > 0$ let

$$p(x) = 4(x^2 - 1)(\gamma - 2\log 2) + 4x^2\log x - x^4 + 1.$$

For a proof of (5.1) it is sufficient to see that $p(c) < 0$. Since $\log x \leqq x - 1$ for $x > 0$, we have that

$$\begin{aligned} p'(x) &= 4x\{2(\gamma - 2\log 2) + 1 + 2\log x - x^2\} \\ &\leqq 4x\{2(\gamma - 2\log 2) - (x-1)^2\}. \end{aligned}$$

Note that

$$\gamma - 2\log 2 = -0.809\,07\cdots,$$

which yields that $p'(x) < 0$. Since $p(1) = 0$, it follows that $p(x) < 0$ for $x > 1$ and hence we conclude (5.1). □

We can deduce from the following proposition that the third term of (3.7) are positive except for the case when $2 < \nu < 3$.

Proposition 5.2. *We have that*

$$\alpha_{\nu,c}(\eta_\nu^2 - \sigma_\nu) + \beta_{\nu,c}\eta_\nu + \zeta_{\nu,c} > 0$$

if $\nu \geqq 3$ *or* $\nu = 3/2$.

Proof. For simplicity let $\Lambda_{\nu,c} = c^\nu\{\alpha_{\nu,c}(\eta_\nu^2 - \sigma_\nu) + \beta_{\nu,c}\eta_\nu + \zeta_{\nu,c}\}$. Note that

$$\eta_\nu^2 - \sigma_\nu = \frac{2\nu - 5}{16(\nu-1)^2(\nu-2)}$$

and thus

$$c^\nu \alpha_{\nu,c}(\eta_\nu^2 - \sigma_\nu) = \frac{c^{2\nu} - 1}{8(\nu-1)^2} - \frac{c^{2\nu} - 1}{16(\nu-1)^2(\nu-2)}$$

holds. Since

$$c^\nu \beta_{\nu,c}\eta_\nu = \frac{1}{8(1-\nu)}\left(\frac{c^{2\nu} - c^2}{\nu - 1} - \frac{c^{2\nu+2} - 1}{\nu + 1}\right),$$

we obtain that

$$\begin{aligned} &c^\nu\{\alpha_{\nu,c}(\eta_\nu^2 - \sigma_\nu) + \beta_{\nu,c}\eta_\nu\} \\ &= \frac{c^2 - 1}{8(\nu-1)^2} - \frac{c^{2\nu} - 1}{16(\nu-1)^2(\nu-2)} + \frac{c^{2\nu+2} - 1}{8(\nu-1)(\nu+1)}. \end{aligned} \tag{5.2}$$

Moreover a simple calculation shows that

$$c^\nu \zeta_{\nu,c} = \frac{c^{2\nu} - c^4}{32(\nu-2)(\nu-1)} + \frac{c^{2\nu+4} - 1}{32(\nu+2)(\nu+1)} + \frac{c^{2\nu+2} - c^2}{16(\nu-1)(\nu+1)}. \tag{5.3}$$

In the case that $\nu = 3/2$, the right hand sides of (5.2) and (5.3) are

$$\frac{c^2 - 1}{2} + \frac{c^3 - 1}{2} + \frac{c^5 - 1}{10}$$

and

$$\frac{c^4 - c^3}{8} + \frac{c^7 - 1}{280} + \frac{c^5 - c^2}{20},$$

respectively. Hence we obtain that $\Lambda_{3/2,c} > 0$.

We next consider the case that $\nu \geqq 3$. Reducing denominators of all fractions in (5.2) and (5.3) to a common denominator, we can derive that the numerator of $\Lambda_{\nu,c}$ coincides with

$$\begin{aligned}&4(\nu-2)(\nu+1)(\nu+2)(c^2-1)-2(\nu+1)(\nu+2)(c^{2\nu}-1)\\&+4(\nu-1)(\nu-2)(\nu+2)(c^{2\nu+2}-1)+(\nu-1)(\nu+1)(\nu+2)(c^{2\nu}-c^4)\\&+(\nu-1)^2(\nu-2)(c^{2\nu+4}-1)+2(\nu-1)(\nu-2)(\nu+2)(c^{2\nu+2}-c^2),\end{aligned}$$

which is equal to

$$\begin{aligned}&(\nu-1)^2(\nu-2)c^{2\nu+4}+6(\nu-2)(\nu-1)(\nu+2)c^{2\nu+2}\\&+(\nu+1)(\nu+2)(\nu-3)c^{2\nu}-(\nu-1)(\nu+1)(\nu+2)c^4\\&+2(\nu-2)(\nu+2)(\nu+3)c^2-3(3\nu^3-2\nu^2-11\nu-2).\end{aligned} \tag{5.4}$$

If $\nu \geqq 3$, it follows that the summation of the first four terms in (5.4) is larger than $(7\nu^3-12\nu^2-25\nu+18)c^4$. Note that it is positive and thus we have that (5.4) is larger than $3(3\nu^3-2\nu^2-11\nu-2)(c^2-1)>0$. This yields that $\Lambda_{\nu,c}>0$. □

The following proposition deals with the case that $\nu=1/3$ and shows that the third terms of (3.17) and also (3.2) are negative.

Proposition 5.3. *We have that both $\eta_{1/3}-\xi_{1/3}^3$ and $\beta_{1/3,c}$ are negative. In particular, the third term of the right hand side of* (3.2) *is negative.*

Proof. The definitions of η_ν and ξ_ν give that

$$\eta_{1/3}-\xi_{1/3}^3=\frac{1}{4}\left\{3-\frac{\Gamma(2/3)^3}{\Gamma(4/3)^3}\right\}.$$

The formula

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\sqrt{\pi}\Gamma(2z)$$

(cf. [16, p. 3]) yields that

$$\frac{\Gamma(2/3)}{\Gamma(4/3)}=\frac{\sqrt{\pi}}{2^{1/3}\Gamma(7/6)}.$$

For a proof of $\eta_{1/3}-\xi_{1/3}^3<0$ it is sufficient to show that

$$\frac{\pi^{3/2}}{\Gamma(7/6)^3}>6. \tag{5.5}$$

Since

$$\Gamma''(x)=\int_0^\infty e^{-t}t^{x-1}(\log t)^2dt>0$$

for $x>0$ (cf. [5, p. 577]), we have that Γ is convex on $(0,\infty)$. Hence it follows that

$$\Gamma(px+(1-p)y)\leqq p\Gamma(x)+(1-p)\Gamma(y) \tag{5.6}$$

for $x, y > 0$ and $p \in [0,1]$. We apply (5.6) for $x = 1/2$, $y = 1$ and $p = 1/3$ and then obtain that

$$\Gamma\left(\frac{7}{6}\right) \leqq \frac{1}{3}\Gamma\left(\frac{3}{2}\right) + \frac{2}{3}\Gamma(1) = \frac{\sqrt{\pi}+4}{6}.$$

This gives that

$$\frac{\pi^{3/2}}{\Gamma(7/6)^3} \geqq \left(\frac{6\sqrt{\pi}}{\sqrt{\pi}+4}\right)^3. \tag{5.7}$$

Note that the function $x \mapsto x/(x+4)$ is increasing on $(0,\infty)$. Since $\sqrt{\pi} \geqq 1.77$, we have that

$$\frac{\pi^{3/2}}{\Gamma(7/6)^3} \geqq \left(\frac{6 \times 1.77}{1.77+4}\right)^3 > 6.2,$$

which immediately implies (5.5).

The proof for $\beta_{1/3,c} < 0$ is easy. Indeed, we deduce from the definition of $\beta_{\mu,c}$ that

$$\beta_{1/3,c} = \frac{3}{4}\left(\frac{c^{5/3}-c^{1/3}}{2} + \frac{c^{-1/3}-c^{7/3}}{4}\right) = -\frac{3}{16c^{1/3}}(c^{2/3}-1)^3(c^{2/3}+1).$$

Since $c > 1$, we have that $\beta_{1/3,c} < 0$. □

Remark 5.4. We can numerically compute the both hand sides of (5.7) by *Mathematica* and have that

$$\frac{\pi^{3/2}}{\Gamma(7/6)^3} = 6.973\,888\,298\cdots, \qquad \left(\frac{6\sqrt{\pi}}{\sqrt{\pi}+4}\right)^3 = 6.253\,125\,593\cdots.$$

The estimate (5.7) is sufficient for a proof of Proposition 5.3 but not sharp.

Proposition 5.5. *When $1 < c \leqq 1+\sqrt{6}$, we have the followings:*

(1) $\alpha_{\nu,c}\eta_\nu + \beta_{\nu,c} > 0$ *if* $1/2 < \nu < 1$.

(2) $\alpha_{\nu,c}\eta_\nu + \beta_{\nu,c} < 0$ *if* $\nu > 1$.

For each $c > 1+\sqrt{6}$ there exists a constant $\nu_0(c) \in (1/2,1)$ such that the followings hold:

(3) $\alpha_{\nu,c}\eta_\nu + \beta_{\nu,c} < 0$ *if* $1/2 < \nu < \nu_0(c)$ *or* $\nu > 1$.

(4) $\alpha_{\nu_0(c),c}\eta_{\nu_0(c)} + \beta_{\nu_0(c),c} = 0$.

(5) $\alpha_{\nu,c}\eta_\nu + \beta_{\nu,c} > 0$ *if* $\nu_0(c) < \nu < 1$.

This proposition yields that the second terms of (3.5), (3.6) and (3.7) are all not zero. Moreover we obtain that the third term of (3.3) coincides with zero if and only if $\nu = \nu_0(c)$. In the case that $\nu = \nu_0(c)$, we need to give an improvement of (3.3) and will discuss in the next section.

For $r > 1$ and $x > 0$ let

$$q_r(x) = (r^2+1)xr^{2x} - (r^2-1)r^{2x} + (r^2-3)x + r^2 - 1. \tag{5.8}$$

Since

$$\alpha_{\nu,c}\eta_\nu + \beta_{\nu,c} = \frac{q_c(\nu)}{4(1-\nu)(1+\nu)c^\nu}, \tag{5.9}$$

it is enough to consider the sign of $q_c(x)$ for $x > 1/2$. Proposition 5.5 is the straightforward consequence of the following lemma.

Lemma 5.6. *If* $1 < c \leqq 1 + \sqrt{6}$, *we have that* $q_c(x) > 0$ *for* $x > 1/2$. *For a given* $c > 1 + \sqrt{6}$ *we can choose a constant* $\nu_0(c) \in (1/2, 1)$ *satisfying that* $q_c(\nu_0(c)) = 0$ *and that*

$$\begin{cases} q_c(x) < 0 & \text{for } 1/2 < x < \nu_0(c), \\ q_c(x) > 0 & \text{for } x > \nu_0(c). \end{cases}$$

The remainder of this section is devoted to a proof of Lemma 5.6. For determining the sign of $q_c(x)$ for $x > 1/2$, we need to give several properties of the first and the second derivatives of q_c and an elementary calculation gives that

$$\begin{aligned} q_c'(x) &= (c^2+1)c^{2x} + 2(c^2+1)xc^{2x}\log c - 2(c^2-1)c^{2x}\log c + c^2 - 3, \\ q_c''(x) &= 4c^{2x}(\log c)\{(c^2+1)x\log c + c^2 + 1 - (c^2-1)\log c\}. \end{aligned}$$

We start to consider the second derivative and have that

$$q_c''(x) = 4c^{2x}(c^2+1)(\log c)^2\{x - w(c)\},$$

where

$$w(y) = 1 - \frac{2}{y^2+1} - \frac{1}{\log y}$$

for $y > 1$. The following lemma is useful for determining the sign of $q_c''(x)$.

Lemma 5.7. *We can choose a constant* $c_1 > 7$ *such that* $w(c_1) = 1/2$ *and*

$$\begin{cases} w(y) < 1/2 & \text{for } 1 < y < c_1, \\ w(y) > 1/2 & \text{for } y > c_1. \end{cases}$$

Moreover it follows that $w(y) < 1$ *for any* $y > 1$.

Proof. It is obvious that w is increasing on $(1, \infty)$ and that

$$\lim_{y \downarrow 1} w(y) = -\infty, \quad \lim_{y \to \infty} w(y) = 1.$$

Hence $w(y) < 1$ for any $y > 1$ and we can choose the constant c_1 satisfying the claim of the lemma. In addition, noting that $e^2 > 7$, we have that

$$w(7) = 1 - \frac{1}{25} - \frac{1}{\log 7} < \frac{1}{2}.$$

This immediately yields that $c_1 > 7$. □

For $y > 0$ let

$$\begin{aligned} u(y) &= y^4 + y^3 + y^2 - 3 + y^2\log y, \\ v(y) &= y^3 + y^2 + y - 3 + 3y\log y - y^3\log y. \end{aligned}$$

The formula

$$q_c'(x) = (c^2+1)c^{2x} + 2(c^2+1)xc^{2x}\log c - 2(c^2-1)c^{2x}\log c + c^2 - 3$$

gives that

$$q_c'(1) = u(c), \quad q_c'\left(\frac{1}{2}\right) = v(c).$$

The following lemma plays an important role to determine the sign of the first derivative of q_c.

Lemma 5.8. *We have that $u(y) > 0$ for $y > 1$ and can choose a constant $c_2 \in (4,5)$ satisfying that $v(c_2) = 0$ and*

$$\begin{cases} v(y) > 0 & \textit{for } 1 < y < c_2, \\ v(y) < 0 & \textit{for } y > c_2. \end{cases}$$

Proof. A proof of the first claim is quite easy. Indeed, since u is strictly increasing on $(1,\infty)$ and $u(1) = 0$, we have that $u(y) > 0$ for $y > 1$.

We now concentrate on proving the second claim. It is obvious that

$$\begin{aligned} v'(y) &= 2y^2 + 2y + 4 + 3\log y - 3y^2\log y \\ v''(y) &= y + 2 + \frac{3}{y} - 6y\log y. \end{aligned}$$

Since $v'''(y) = -5 - 3/y^2 - 6\log y < 0$ for $y > 1$, we have that v'' is decreasing on $(1,\infty)$. Note that $v''(1) = 6$ and $v''(2) = 11/2 - 6\log 4 < 0$. Hence there exists a constant $y_1 \in (1,2)$ such that $v''(y_1) = 0$ and that v'' is positive on $(1, y_1)$ and negative on (y_1,∞). These imply that v' takes the maximum at y_1 and that v' is increasing on $(1, y_1)$ and decreasing on (y_1,∞). Since $v'(1) = 8$, it follows that $v'(y_1) > 0$. In addition, since $v'(4) = 44 - 45\log 4 < 0$, there exists a constant $y_2 \in (y_1, 4)$ such that v is increasing on $(1, y_2)$ and deceasing on (y_2,∞). Moreover, we have that $v(1) = 0$, $v(4) = 81 - 52\log 4 > 3$ and $v(5) = 152 - 110\log 5 < -13$, where $4 < e^{3/2} < 5$ has been applied. Therefore we can choose a constant $c_2 \in (4,5)$ satisfying the claim of the lemma. □

The last lemma of this section is to give the signs of $q_c(1)$ and $q_c(1/2)$, which are significant for deciding the sign of $q_c(x)$ for $x > 1/2$.

Lemma 5.9. *We have that $q_c(1) > 0$ for any $c > 1$ and that*

$$\begin{cases} q_c(1/2) \geqq 0 & \textit{if } 1 < c \leqq 1+\sqrt{6}, \\ q_c(1/2) < 0 & \textit{if } c > 1+\sqrt{6}. \end{cases} \tag{5.10}$$

Proof. It follows from (5.8) that

$$q_c(1) = 4(c^2-1) > 0$$

and that

$$q_c\left(\frac{1}{2}\right) = -\frac{1}{2}(c-1)(c^2-2c-5).$$

It is easy to check (5.10). □

We are ready to give a proof of Lemma 5.6 and start to consider the case that $1 < c \leqq c_1$. Lemma 5.7 gives that $w(c) < 1/2$ and thus $x > w(c)$ holds for $x > 1/2$. This immediately yields that $q_c'' > 0$ on $(1/2, \infty)$, which is equivalent to that q_c' is increasing on $(1/2, \infty)$. Lemma 5.8 gives that the sign of $q_c'(1/2)$ depends on whether c is larger than c_2 or not. Recall that $c_1 > 7$ and $4 < c_2 < 5$ and hence it is convenient to consider the following two cases separately; (i) $1 < c \leqq c_2$, (ii) $c_2 < c \leqq c_1$.

In Case (i), Lemma 5.8 yields that $q_c'(1/2) = v(c) > 0$. Since $q_c''(x) > 0$ for $x > 1/2$, it follows that $q_c'(x) > 0$, which implies that q_c is increasing on $(1/2, \infty)$. Hence, if $1 < c \leqq 1 + \sqrt{6}$, we can deduce from (5.10) that $q_c(x) > 0$ for $x > 1/2$. In the case when $1 + \sqrt{6} < c \leqq c_2$, Lemma 5.9 yields that $q_c(1) > 0$ and $q_c(1/2) < 0$. Since q_c is increasing on $(1/2, 1)$, we can choose a suitable constant $\nu_1 \in (1/2, 1)$ with $q_c(\nu_1) = 0$. In addition, we can find that $q_c(x) < 0$ for $1/2 < x < \nu_1$ and that $q_c(x) > 0$ for $x > \nu_1$.

In Case (ii), we have by Lemma 5.8 that $q_c'(1/2) < 0 < q_c'(1)$ and thus can choose a constant ν_2 satisfying that $q_c'(\nu_2) = 0$, $q_c'(x) < 0$ for $1/2 < x < \nu_2$ and $q_c'(x) > 0$ for $x > \nu_2$. This implies that ν_2 is the minimum point of q_c' in $(1/2, \infty)$ and that q_c' is decreasing and increasing on $(1/2, \nu_2)$ and (ν_2, ∞), respectively. Moreover, since $c_2 > 1 + \sqrt{6}$, it follows from Lemma 5.9 that $q_c(1/2) < 0 < q_c(1)$. Therefore we can choose a constant $\nu_3 \in (\nu_2, 1)$ such that $q_c(\nu_3) = 0$, $q_c(x) < 0$ for $1/2 < x < \nu_3$ and $q_c(x) > 0$ for $x > \nu_3$.

We next discuss the case that $c > c_1$. Lemma 5.7 gives that $w(c) > 1/2$ and thus it follows that $q_c''(w(c)) = 0$ and that

$$\begin{cases} q_c''(x) < 0 & \text{for } 1/2 < x < w(c), \\ q_c''(x) > 0 & \text{for } x > w(c). \end{cases}$$

Since $c_1 > c_2$, Lemma 5.8 yields that $q_c'(1/2) < 0 < q_c'(1)$. Hence, with the help that $q_c''(x) > 0$ for $x > 1/2$, we can take $\nu_4 \in (w(c), 1)$ such that $q_c'(\nu_4) = 0$. Moreover we can see that $q_c'(x) < 0$ for $1/2 < x < \nu_4$ and that $q_c'(x) > 0$ for $x > \nu_4$. It follows from Lemma 5.9 that $q_c(1/2) < 0 < q_c(1)$ since $c_1 > 1 + \sqrt{6}$. Hence we can conclude that there exists a constant $\nu_5 \in (\nu_4, 1)$ such that $q_c(\nu_5) = 0$ and that

$$\begin{cases} q_c(x) < 0 & \text{for } 1/2 < x < \nu_5, \\ q_c(x) > 0 & \text{for } x > \nu_5. \end{cases}$$

We complete the proof of Lemma 5.6.

Remark 5.10. We tried to decide the sign of the third term of (3.7) for $2 < \nu < 3$. However we have no idea of its calculations.

6. PRECISE ASYMPTOTIC BEHAVIOR OF $T_{a,b}^{(\nu_0(c))}(t)$

If $c > 1+\sqrt{6}$ and $\nu = \nu_0(c)$, Proposition 5.5 gives that the third term of (3.19) and also that of (3.3) are zero. This section deals with the improvement of (3.3) in this case. We start to improve (3.19) in the case that $\nu = \nu_0(c)$.

Proposition 6.1. *If $c > 1+\sqrt{6}$, we have that*

$$L_{\nu_0(c),c}(x) = \pi c^{\nu_0(c)}\kappa_{\nu_0(c),c}x^{2\nu_0(c)} \times\Bigg[1+\xi_{\nu_0(c)}x^{2\nu_0(c)} + (\xi^2_{\nu_0(c)} - \phi^2_{\nu_0(c)})x^{4\nu_0(c)} + \left\{\frac{2\nu_0(c)+1}{2(1-\nu_0(c)^2)} + \frac{\beta_{\nu_0(c),c}}{\alpha_{\nu_0(c),c}}\right\}\xi_{\nu_0(c)}x^{2\nu_0(c)+2} + O[x^4+x^{6\nu}]\Bigg]$$

as $x \downarrow 0$.

Proof. Recall that Proposition 5.5 gives that $\alpha_{\nu_0(c),c}\eta_{\nu_0(c)} + \beta_{\nu_0(c),c} = 0$. It is sufficient to show that, if $1/2 < \nu < 1$,

$$L_{\nu,c}(x) = \pi c^{\nu}\kappa_{\nu,c}x^{2\nu}\Bigg[1+\xi_\nu x^{2\nu} + \left(\eta_\nu + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}}\right)x^2 + (\xi_\nu^2 - \phi_\nu^2)x^{4\nu} + \left\{\frac{2\nu+1}{2(1-\nu^2)} + \frac{\beta_{\nu,c}}{\alpha_{\nu,c}}\right\}\xi_\nu x^{2\nu+2} + O[x^4+x^{6\nu}]\Bigg] \tag{6.1}$$

as $x \downarrow 0$. It follows from (4.2) that

$$Y_\nu(x)^2 = \frac{2^{2\nu}\Gamma(\nu)^2}{\pi x^{2\nu}}\left\{1 - \xi_\nu x^{2\nu} - \eta_\nu x^2 + \frac{1}{4}\xi_\nu x^{4\nu} + \frac{1}{2(1-\nu^2)}\xi_\nu x^{2\nu+2} + O[x^4]\right\}.$$

Since

$$J_\nu(x)^2 = \frac{x^{2\nu}}{2^{2\nu}\Gamma(\nu+1)^2}(1+O[x^2]),$$

which is derived by (4.1), we have that

$$L_\nu^1(x) = \frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2 x^{2\nu}}\left\{1 - \xi_\nu x^{2\nu} - \eta_\nu x^2 + \phi_\nu^2 x^{4\nu} + \frac{1}{2(1-\nu^2)}\xi_\nu x^{2\nu+2} + O[x^4]\right\}, \tag{6.2}$$

where the formula

$$\phi_\nu^2 = \frac{1}{4}\xi_\nu^2 + \frac{\pi^2}{2^{4\nu}\Gamma(\nu)^2\Gamma(\nu+1)^2}$$

has been applied. Hence (6.2) yields that

$$\frac{1}{L_\nu^1(x)} = \frac{\pi^2 x^{2\nu}}{2^{2\nu}\Gamma(\nu)^2}\left\{1 + \xi_\nu x^{2\nu} + \eta_\nu x^2 + (\xi_\nu^2 - \phi_\nu^2)x^{4\nu} + \frac{2\nu+1}{2(1-\nu^2)}\xi_\nu x^{2\nu+2} + O[x^4+x^{6\nu}]\right\}.$$

Combining it with (4.11), we obtain (6.1) by a simple calculation. □

We need to discuss whether $\xi^2_{\nu_0(c)} - \phi^2_{\nu_0(c)}$ is zero or not.

Lemma 6.2. *We can choose a constant c_0 such that $\alpha_{2/3,c}\eta_{2/3} + \beta_{2/3,c} = 0$ if and only if $c = c_0$. Moreover we have that $c_0 > 4$.*

Proof. In virtue of (5.9) and (5.8), we should prove that there exists a constant c_0 satisfying that $c_0 > 1$ and

$$\frac{2}{3}(c_0^2+1)c_0^{4/3} - (c_0^2-1)c_0^{4/3} + \frac{2}{3}(c_0^2-3) + c_0^2 - 1 = 0. \tag{6.3}$$

For simplicity we put $x_0 = c_0^{2/3}$ and thus (6.3) gives that

$$x_0^5 - 5x_0^3 - 5x_0^2 + 9 = 0, \tag{6.4}$$

which is equivalent to

$$(x_0 - 1)(x_0^4 + x_0^3 - 4x_0^2 - 9x_0 - 9) = 0.$$

Hence it is sufficient to see that the equation

$$x^4 + x^3 - 4x^2 - 9x - 9 = 0 \tag{6.5}$$

has the unique solution on $(1, \infty)$. A simple calculation shows that (6.5) is equivalent to

$$\left\{x^2 + \frac{\sqrt{5}+1}{2}x + \frac{3(\sqrt{5}-1)}{2}\right\}\left\{x^2 - \frac{\sqrt{5}-1}{2}x - \frac{3(\sqrt{5}+1)}{2}\right\} = 0.$$

Since

$$x^2 + \frac{\sqrt{5}+1}{2}x + \frac{3(\sqrt{5}-1)}{2} > 0$$

for any $x > 1$, solutions of (6.5) coincide with those of

$$x^2 - \frac{\sqrt{5}-1}{2}x - \frac{3(\sqrt{5}+1)}{2} = 0.$$

This yields that x_0 is the solution of (6.5) which is larger than 1 if and only if

$$x_0 = \frac{\sqrt{5} - 1 + \sqrt{30 + 22\sqrt{5}}}{4}.$$

and thus we obtain that

$$c_0 = \sqrt{\frac{7 + \sqrt{5} + \sqrt{270 + 122\sqrt{5}}}{2}},$$

which is larger than 4. □

Lemma 6.3. *We have that c_0 is the unique constant satisfying that $\nu_0(c_0) = 2/3$.*

Proof. Lemma 6.2 gives that $\alpha_{2/3,c_0}\eta_{2/3} + \beta_{2/3,c_0} = 0$. Since Proposition 5.5 yields that $\alpha_{\nu,c_0}\eta_\nu + \beta_{\nu,c_0} = 0$ if and only if $\nu = \nu_0(c_0)$, we can conclude that $\nu_0(c_0) = 2/3$.

On the other hand, we assume that c_0' is another constant with $\nu_0(c_0') = 2/3$ and then have that

$$\alpha_{2/3,c_0'}\eta_{2/3} + \beta_{2/3,c_0'} = \alpha_{\nu_0(c_0'),c_0'} + \eta_{\nu_0(c_0')}\beta_{\nu_0(c_0'),c_0'} = 0$$

in virtue of Proposition 5.5. However this contradicts that $\alpha_{2/3,c_0'}\eta_{2/3} + \beta_{2/3,c_0'} \neq 0$, which is obtained by Lemma 6.2. This yields that $\nu_0(c_0') \neq 2/3$. □

Asymptotic behavior of $T_{a,b}^{(\nu_0(c))}(t)$ depends on the value of c.

Theorem 6.4. *If $c > 1 + \sqrt{6}$ and $c \neq c_0$, we have that*

$$\begin{aligned} T_{a,b}^{(\nu_0(c))}(t) &= \kappa_{\nu_0(c),c}^{(1)}\rho(t)^{-\nu_0(c)} + \kappa_{\nu_0(c),c}^{(2)}\rho(t)^{-2\nu_0(c)} \\ &\quad + \kappa_{\nu_0(c),c}(\xi_{\nu_0(c)}^2 - \phi_{\nu_0(c)}^2)\Gamma(3\nu_0(c))\rho(t)^{-3\nu_0(c)} + O[t^{-2\nu_0(c)-1}] \end{aligned} \tag{6.6}$$

as $t \to \infty$. In addition, if $a = c_0 b$ for $a, b > 0$, we have that

$$\begin{aligned} T_{a,b}^{(2/3)}(t) &= \kappa_{2/3,c_0}^{(1)}\rho(t)^{-2/3} + \kappa_{2/3,c_0}^{(2)}\rho(t)^{-4/3} \\ &\quad + \kappa_{2/3,c_0}^{(2)}\left(\frac{21}{10} + \frac{\beta_{2/3,c_0}}{\alpha_{2/3,c_0}}\right)\rho(t)^{-7/3} + O[t^{-8/3}] \end{aligned} \tag{6.7}$$

as $t \to \infty$ and that the third term of (6.7) *is positive.*

Proof. Lemmas 3.2 and 3.4 show that (6.6) and (6.7) are straightforward consequences of (6.1). Since calculations are similar to Theorem 3.1, we omit their calculations.

It remains to proving that the third term of (6.7) is positive. In the case that $c = c_0$, we put $x_0 = c_0^{2/3}$ and have that

$$\beta_{2/3,c_0} = -\frac{3}{20x_0}(x_0^5 - 5x_0^3 + 5x_0 - 1).$$

Since x_0 satisfies (6.4),

$$\beta_{2/3,c_0} = -\frac{3}{2x_0}(x_0^2 - 1).$$

Hence it follows that

$$\frac{21}{10}\alpha_{2/3,c_0} + \beta_{2/3,c_0} = \frac{3}{5x_0}(x_0^2 - 1),$$

which is positive since $x_0 > 1$. □

Remark 6.5. We numerically compute the constant c_0 by *Mathematica* and have that $c_0 = 4.033\,245\,7\ldots$

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