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A SMART AMALGAMATION OF SPECTRAL NEURAL ALGORITHM FOR NONLINEAR LANE-EMDEN EQUATIONS WITH SIMULATED ANNEALING

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Abstract

The actual motivation of this paper is to develop a functional link between artificial neural network (ANN) with Legendre polynomials and simulated annealing termed as Legendre simulated annealing neural network (LSANN). To demonstrate the applicability, it is employed to study the nonlinear Lane-Emden singular initial value problem that governs the polytropic and isothermal gas spheres. In LSANN, minimization of error is performed by simulated annealing method while Legendre polynomials are used in hidden layer to control the singularity problem. Many illustrative examples of Lane-Emden type are discussed and results are compared with the formerly used algorithms. As well as with accuracy of results and tranquil implementation it provides the numerical solution over the entire finite domain.

Keywords: Lane-Emden equations, simulated annealing, legendre polynomials, neural network

1 Introduction

Polytropic models describe a major role in astronomical dynamics and the theory of stellar structure and as such the polytropic differential equations are considered as a criterion to check the strength of the recently-developed numerical methods. The well-known polytropic differential equations are Lane-Emden equations which describe the polytropes in hydrostatic equilibrium as simple models of stars. The generalized form of second order Lane-Emden equation is

$$\frac{d^2\phi}{d\xi^2}+\frac{\beta}{\xi}\frac{d\phi}{d\xi}+\omega g(\xi)h(\phi)=v(\xi), \tag{1}$$

$0<\xi<1, \beta>0$

with initial conditions

$$\phi(0)=\alpha\ ,\phi'(0)=\gamma, \tag{2}$$

or boundary conditions

$$\phi(0)=\alpha'\ ,\phi(1)=\gamma, \tag{3}$$

for $v(\xi)=0$, $\beta=2$, $\omega=1$,$h(\phi)=\phi^n$, $\alpha=1$, $\gamma=0$ and $g(\xi)=1$ in Eqs. (1)-(2) give

$$\frac{d^2\phi}{d\xi^2}+\frac{2}{\xi}\frac{d\phi}{d\xi}+\phi^n=0, \tag{4}$$

which represents the standard Lane-Emden equation of first kind with polytropic index n. The solutions of the Lane-Emden equation, which are known as polytropes, are functions of density versus radius expressed as $\phi(\xi)$.The index n determines the order of that solution. In particular, the solution only depends on n to give solutions for stars over a range of total mass and radius. For $n=0$ gives the solution for a constant density incompressible sphere i.e. $1-\frac{\xi^2}{6}$, $n=1$ to *1.5.* It approximates a

very cool and redder i.e. late type star such as K, M, S, or C class of stars with a surface temperature lower than that of the Sun and $n = 3$ is the Eddington approximation. There is no analytical solution for $n = 3$ but it is useful as it corresponds to the stellar model of the Sun. Polytropic index n can be any positive value but exact solutions exist at only $n = 0, 1$ and 5.

For $v(\xi) = 0$, $\alpha = 0$, $\gamma = 0$,$\beta = 2$, $\omega = 1$, $h(\phi) = e^{\phi}$ and $g(\xi) = 1$ in Eq. (1) gives

$$\frac{d^2\phi}{d\xi^2} + \frac{2}{\xi}\frac{d\phi}{d\xi} + e^{\phi} = 0. \tag{5}$$

It represents the standard Lane-Emden equation of second kind. That models the hydrostatic self-gravitating gas spheres and is well-known as Bonnor–Ebert gas sphere. If $h(\phi) = e^{-\phi}$ is replaced in Eq. (5) then the equation models from the Richardson's theory of thermionic currents, where one strives for the density and electric force of an electron gas in the region of a hot body in thermal equilibrium. Due to vast applicability of Lane-Emden equations, they are studied widely by different researchers with diverse analytical and numerical methods and in the last few years many numerical and analytical techniques are tested on these equations. Some of them are discussed below:

- Earlier than 2012 the noteworthy methods were Adomian decomposition method (ADM) [1] in 2001, by variational iteration method (VIM) [2] in 2008, by Rational Legendre pseudospectral [3] in 2009 and by sinc collocation [4] in 2010.
- In 2012, the well-known methods to study the different cases of Eq. (1) Boubakar polynomial expansion scheme [5], Legendre operational matrix of differenciation [6] and modified Legendre spectral method [7]
- In 2013, different cases of these polytropic equations are studied by second kind Chebyshev operational matrix [8], by Haar wavelet [9] and by homotopy perturbation method (HPM) and fourier transform [10].
- In 2014, Lane-Emden-Fowler equations were studied by Laguerre polynomial approach [11], Chebychev neural network (ChNN) [12] and by Differential transformation method (DTM) [13].
- In 2015, the equations were studied by (DTM) [14], Chebychev wavelet and finite difference method [15] and by rational approximation [16].

In recent years, the briskly growing field of connectionist networks took the attention of mathematicians and physicist to use the tools such as genetic learning systems, content-addressable memory and fuzzy systems to solve physical problems but the chief tool is ANN. It is a powerful computational tool having the capability of handling nonlinear and complex features of any physical process with a high degree of accuracy. It has been proven to be a versatile tool for approximating initial or boundary value problems due to its universal approximation capability. The ANN has many benefits over the traditionally used numerical methods for approximating the initial or boundary value problems such as the numerical approximation is continuous over the domain of the integration, a black box learning approach is followed and capable of approximating high nonlinear systems. Due to the described benefits a lot of attention is devoted to solve the ordinary, partial and fractional differential equations by using different kinds of ANN. Different Neural network architectures have been developed by altering the number of layers, activation function and training algorithms. The route of training a neural network comprises of modifying the values of network adaptive coefficients to boost network performance. Different researchers applied different neural architectures to solve differential equations such as Aarts and Veer [17] employed the neural algorithm to solve partial differential equations with multilayer neural structure, linear and logsigmoid activation function and evolutionary algorithm for training the weights. Numerical approximation of ordinary differential equations was proposed by Meade and Fernandez [18] by applying feed forward neural network method along with piecewise splines of Lagrange polynomials. Parisi et al. [19] applied feed forward neural network approach and in tandem with genetic algorithm for training of network to a non-steady fixed bed non-catalytic solid-gas reactor system. Lagrais et al. [20] implemented ANN coupled with Broyden–Fletcher–Goldfarb–Shanno algothim for training to solve ordinary and partial differential equations. Jianyu et al.[21] proposed radial basis function neural network with two stage gradient

descent strategy for partial differential equations. Malek and Beidokhti [22] presented hybrid neural network (HNN) by applying the Nelder Meade optimization techniques for numerical simulation of lower as well as higher order ordinary differential equations (ODEs). Fazayeli et al. proposed improvement in back propagation training algorithm [23]. Parand et al. [24] solved nonlinear Lane-Emden type equations by unsupervised neural network with stacked generalization and Levenberg-Marquardt training algorithms. Diverse neural network architectures with different training methods can be studied in [25-26].

In this paper, we propose an algorithm based on functional link neural network with Legendre polynomials and thermal minimizing methodology known as simulated annealing. Legendre neural network has already been implemented successfully on nonlinear channel equalization for wireless communication [27], for prediction of machinery noise in open cast mines [28] and for nonlinear active noise control [29] but here we developed the network for the numerical simulation of nonlinear ordinary differential equations with singularity while training of network adaptive coefficients is performed by simulated annealing. Simulated annealing is a probabilistic form of the gradient descent optimization method that can escape from local optima and go on to find the global optimum in a large search space un like other methods. As well as finding the global optimum it can handle the functions that have ridges and plateaus. With so many benefits over the other optimization methods it is largely independent of initial values and can optimize the unconstraint functions. For the case of Lane-Emden equations simulated annealing revealed the exceptional performance. The structure of LSANN is one input layer, one hidden layer with Legendre polynomials and one output layer with tangent hyperbolic as activation function.

2 Legendre Simulated Annealing Neural Network

2.1 Legendre neural network

LSANN is the modification of the functional link artificial neural network (FLANN) initially developed by Pao [30] to link the gap between the linearity in the single layer neural network and the computation exhaustive multilayer neural network. Here Legendre orthogonal polynomials are used to increase the nonlinear approximation capability and to overcome the singularity problem of Lane-Emden equations. Legendre polynomials are symbolically represented by $L_l(x)$and have the following recursive formula

$$L_{l+1} = \frac{(2l+1)}{(l+1)}\, x\, L_l(x) - \frac{l}{(l+1)}\, L_{l-1}(x), \quad (6)$$

for $l > 1$while $L_0(x) = 1$ and $L_1(x) = x$. Here Legendre polynomials are used for the expansion of discretized input array. Consider second order differential Eqs. (1)-(2) to implement the LSANN methodology we can write the Eq. (1) as

$$\nabla^2 \phi_t(\xi,\psi) - F(\xi, \phi_t(\xi,\psi), \nabla\phi_t(\xi,\psi)) = 0, \quad (7)$$

$\xi \in [0,1]$,

where $\nabla$is the differential operator, ψ are the network adaptive coefficients (weights) that work like dendrites in a biological network and $\phi_t(\xi,\psi)$is the trial solution of Eq. (1) that comprise of two parts first part satisfies the initial conditions in Eq. (2) and the second part contains the output of LSANN. The trial solution can be written as [20].

$$\phi_t(\xi) = \alpha + \xi\gamma + \xi^2 N(\xi,\psi), \quad (8)$$

where $N(\xi,\psi)$is the output of LSANN that can be described in the Figure 1 and can be calculated by

$$\mathrm{N}(\xi,\psi) = \tanh(\theta) = \frac{\mathrm{e}^{2\theta}-1}{e^{2\theta}+1}, \quad (9)$$

whereas,

$$\theta = \sum_{\mathrm{i}=1}^{\mathrm{m}} L_{i-1}(\xi)\, \psi_i. \quad (10)$$

From Eqs. (8)-(10) we get the trial solution in terms of weights and independent variable.

$$\phi_t(\xi) = \alpha + \xi\gamma + \xi^2 \frac{\mathrm{e}^{2\sum_{\mathrm{i}=1}^{\mathrm{m}} L_{i-1}(\xi)\,\psi_i} - 1}{e^{2\sum_{\mathrm{i}=1}^{\mathrm{m}} L_{i-1}(\xi)\,\psi_i} + 1}. \quad (11)$$

To train the network we will have to minimize the mean square error (MSE) defined as Eq. 12.

Training of network adaptive coefficients is performed by simulated annealing algorithm that is described in Section 2.2. After completing the training of weights the final values of weights is used in

$$E(\psi_i) = \sum_{j=1}^{n} \frac{1}{n} (\nabla^2 \phi_t(\xi_j, \psi_i) - F(\xi_j, \phi_t(\xi_j, \psi_i), \nabla \phi_t(\xi_j, \psi_i)))^2,\ \xi \in [0,1] \tag{12}$$

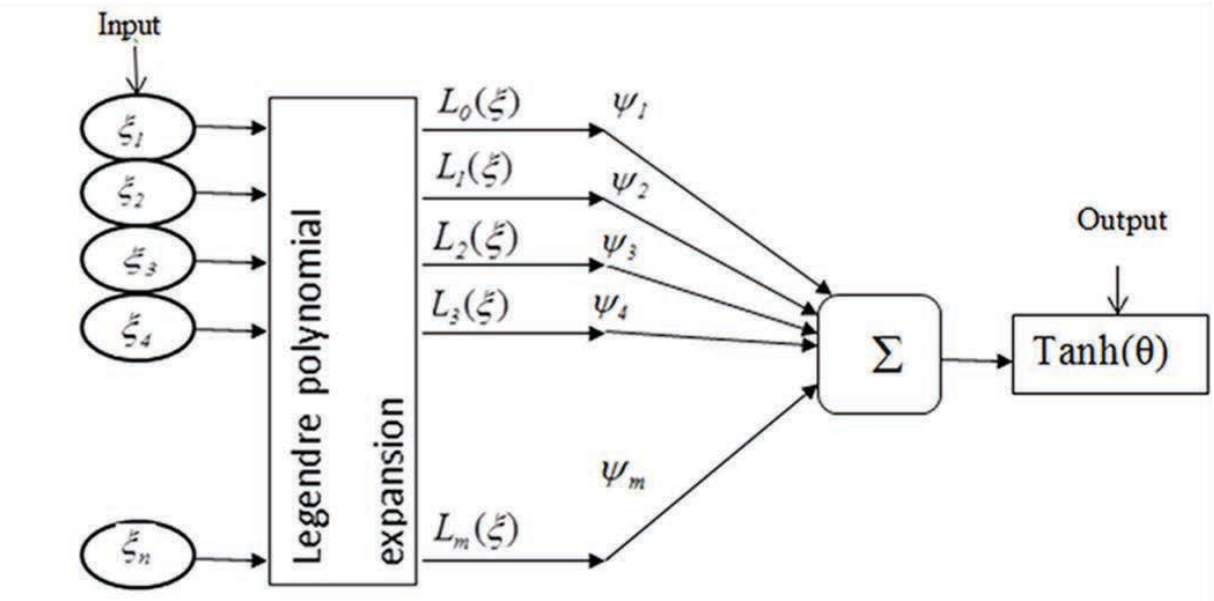

Figure 1. Structure of LSANN

Eq. (11) with a discretized equally spaced array of points for the independent variable. The array can be the same that was used in error minimization or it can be different over the same domain.

2.2 Simulated Annealing

Simulated annealing, a combinatorial optimization methodology, is inspired from the physical process of annealing, where a metal object is gradually cooled after heating it to an extremely high temperature. The comparison of thermo dynamical annealing and simulated annealing is shown in the table given in [31].

Table 1. The comparison of thermodynamical annealing and simulated annealing

Thermodynamic Simulation	**Combinatorial Optimisation**
System States	Feasible Solutions
Energy	Objective value
Change of State	Neighbouring Solutions
Temperature	Control Parameter
Frozen State	Optimal Solution

The procedure is comprised of perturbation and evaluation of the solution quality. Here objective function is mean square error, symbolically represented by$E$, that would be minimized by the stochastic algorithm and the calculated values of weights would be used to calculate the solutions. Let represent the problem for m weights as

$$E(\psi) = \{\psi_1, \psi_2, \psi_3, ..., \psi_m\}, \tag{13}$$

while T is the process temperature,

$$T = \{T_o, T_1, T_2, \ldots, T_h\}. \tag{14}$$

Here the discrete variable temperature has initial starting value 1.0 and is decreased at the end of each iteration by multiplying it by a constant called αtypical choices are between 0.8 and 0.9. Let q be the number of iterations performed at each temperature then Eq. (13) can be written as

$$E_i(\psi) = \{\psi_{1i}, \psi_{2i}, \psi_{3i}, ..., \psi_{mi}\},\ \mathrm{i} = 1,2,3,\ldots q, \tag{15}$$

where i is the applied number of perturbations to the solution and $\delta_1, \delta_2, \delta_3, \ldots$ are errors of$E_1, E_2, E_3, \ldots$ respectively. Probability is calculated by

$$P_a = \begin{cases} e^{\frac{-k\nabla\delta}{T}}, & \nabla\delta > 1 \\ 1, & \nabla\delta < 1 \end{cases}, \tag{16}$$

where $\nabla\delta$ is the difference between the solution error after and before the consecutive perturbation, T is the current temperature and k is the suitable con-

stant. Accepted probability would be in range between $0.8-0.9$while $\nabla\delta$ can be estimated as

$$\nabla\delta \approx \frac{1}{Q-1}\sum_{i=1}^{Q}\delta_i - \frac{1}{Q(Q-1)}\sum_{i=1}^{Q}(\delta_i)^2, \qquad (17)$$

where Q represents the number of perturbations at each value of temperature. At each$\omega = 1$ iteration, a new point is randomly generated in the neighborhood of the current point. The radius of the new point from the current point is based on probability given by Eq. (16). Implementation of algorithm is performed here by *Mathematica 10*.

3 Test Experiments

1- First kind Lane-Emden-Equations

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^m = 0, \qquad (18)$$

subject to:

$$\mathrm{y}'(0) = 1 \quad \textit{and} \quad \mathrm{y}(0) = 1. \qquad (19)$$

First kind Lane-Emden equations relate to the polytropic models and here m is the polytropic index which is related to the relation between pressure and density comprising the star. In galactic dynamics all the polytropes with $\mathrm{n} > 5$ have infinite radii while $\mathrm{n} < 5$ have a surface. The two cases most interesting for real stars are $\mathrm{n} = 1.5$ and $\mathrm{n} = 3$ which correspond to many astrophysical applications, no exact solution is present for these values but numerically they can be solved. We applied LSANN method on Eqs. (18)-(19) the results and absolute true error with one hidden layer and 5 weights for polytropic index $\mathrm{n} = 0, 1$and 5 are shown in Table 1. While numerical solutions for the two interesting cases are compared by the results in [1], shown in Table 2, that shows the achievements of implemented methodology. Figure 2 shows the comparison of ChNN and LSANN.

2-Second kind Lane-Emden equation (Bonnor-Ebert gas sphere)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^y = 0, \qquad (20)$$

subject to condition:

$$\mathrm{y}'(0) = 0 \quad \textit{and} \quad \mathrm{y}(0) = 0. \qquad (21)$$

We tested the suggested algorithm on the Bonnor-Ebert gas sphere equation with 20 equidistant points from 0 to 1 and 6 weights that attained the goal to minimize the mean square error upto 6.674×10^{-12}. The results are compared with ADM [1], squared remainder minimization method (SRM) [32], DTM [13], HNN [22] and ANN [33] that can be seen in Table 3. Graphical comparison of LSANN with ADM [1], HNN [22] and ANN [33] can be visualized in Figure 3. The values of weights after training are shown in Table 4.

3-Second kind Lane-Emden equation (Richardson theory of thermionic currents)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^{-y} = 0, \qquad (22)$$

subject to the condition

$$\mathrm{y}'(0) = 0 \quad \textit{and} \quad \mathrm{y}(0) = 0. \qquad (23)$$

The above equation is derived by Richardson in his study of thermionic distribution in the neighborhood of flat surfaces. Due to the nonlinear term e^{-y} the above equation was solved by different methods such as SRM [32], ADM [1] and DTM [13] as the equation has no exact solution so we compared our results with ADM [1], HNN [22] and ANN [33] to demonstrate the strength of proposed method and results are shown in Figure 4 and Table 5. Table 6 displayed the value of weights after training.

Table 5. Values of weights for Bonnor-Ebert gas sphere equation

w_1	-0.16552514387606254
w_2	0.000097451051157554
w_3	0.005309207540794966
w_4	0.000075101433142333
w_5	-0.00015349067560112
w_6	0.0000129981950430015

Table 7. Values of weights for Richardson theory of Thermionic currents model

w_1	-00.17103896487645
w_2	-0.000411042999503
w_3	-0.005622609128186
w_4	-0.000287719254200
w_5	-0.000015227536836
w_6	-0.000031991478252

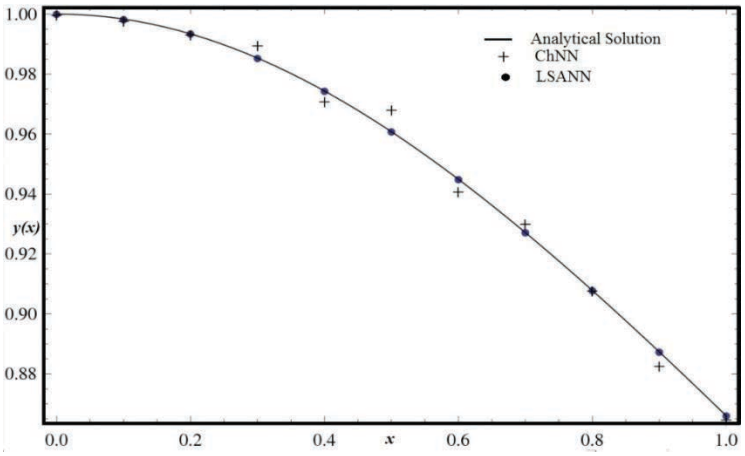

Figure 2. Comparision of ChNN and LSANN at m = 5for first kind Lane-Emden equation

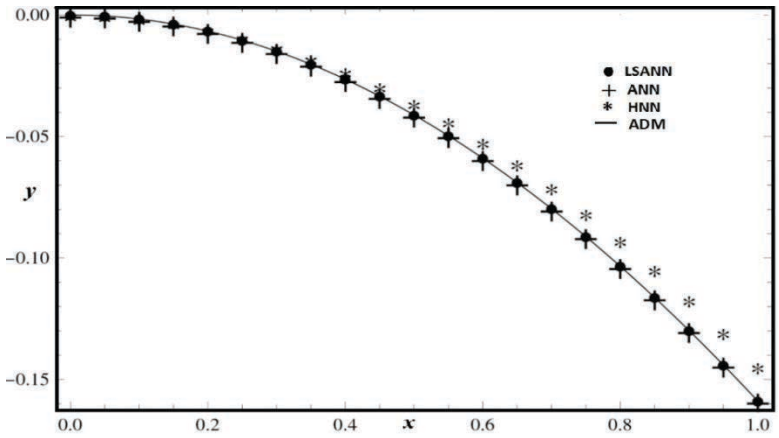

Figure 3. Comparison of LSANN with other methods for Bonnor-Ebert gas sphere equation

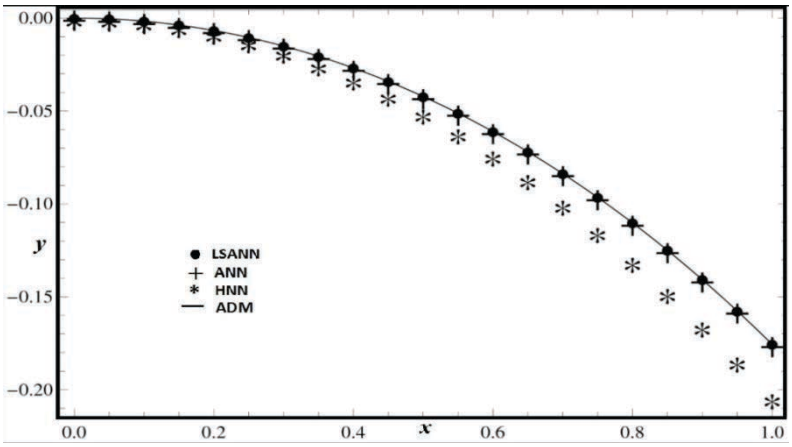

Figure 4. Comparison of LSANN and ADM for Richardson theory of thermionic current model

Table 2. LSANN results for first kind Lane-Emden equations

x	$y(x)$			Absolute True Error		
	$m=0$	$m=1$	$m=5$	$m=0$	$m=1$	$m=5$
0.1	0.98833	0.98334	0.99834	1.389×10^{-9}	7.381×10^{-9}	2.035×10^{-6}
0.2	0.99333	0.99335	0.99340	2.078×10^{-9}	1.274×10^{-9}	2.406×10^{-7}
0.3	0.98500	0.98507	0.98533	8.671×10^{-9}	4.840×10^{-9}	1.025×10^{-6}
0.4	0.97333	0.97354	0.97436	9.141×10^{-9}	1.069×10^{-9}	6.213×10^{-7}
0.5	0.95833	0.95885	0.96077	1.311×10^{-8}	1.936×10^{-9}	1.867×10^{-7}
0.6	0.94000	0.94107	0.94491	3.011×10^{-9}	9.578×10^{-8}	8.260×10^{-7}
0.7	0.91833	0.92031	0.92715	2.983×10^{-9}	1.652×10^{-8}	3.129×10^{-6}
0.8	0.89333	0.89670	0.90784	2.465×10^{-8}	3.936×10^{-9}	1.523×10^{-6}
0.9	0.86500	0.87036	0.88736	3.518×10^{-9}	3.265×10^{-9}	3.662×10^{-7}
1.0	0.83333	0.84147	0.86602	4.657×10^{-9}	3.091×10^{-9}	1.070×10^{-7}

Table 3. Comparison of LSANN result with results in [1] for first kind Lane-Emden equations

x	Present Method $y(x)$		Reference[1]$y(x)$	
	$m=1.5$	$m=3$	$m=1.5$	$m=3$
0.1	0.998335	0.998336	0.998336	0.998336
0.2	0.993353	0.993373	0.993353	0.993373
0.5	0.959104	0.959839	0.959104	0.959839
0.9	0.872846	0.879617	0.872849	0.879631
1.0	0.845170	0.855058	0.845182	0.855095

Table 4. Comparison of LSANN with other methods for Bonnor-Ebert gas sphere equation

x	Present method	ADM	SRM	DTM	ANN	HNN
0.1	-0.0016634	-0.0166583	-0.0016658	-0.0166583	-0.0004162	-0.0003607
0.2	-0.0066502	-0.0066533	-0.0066534	-0.0066533	-0.0016634	-0.0014426
0.3	-0.0149296	-0.0149329	-0.0149329	-0.0149329	-0.0037411	-0.0032459
0.4	-0.0264517	-0.0264555	-0.2645555	-0.0264555	-0.0066469	-0.0057706
0.5	-0.0411502	-0.0411540	-0.0411539	-0.0411540	-0.0103767	-0.0090167
0.6	-0.0585408	-0.0589441	-0.0589440	-0.0589441	-0.0149250	-0.0129842
0.7	-0.0797227	-0.0797260	-0.0797259	-0.0797260	-0.0202850	-0.0176730
0.8	-0.1033820	-0.1033860	-0.1033860	-0.1033860	-0.0264485	-0.0230833
0.9	-0.1297950	-0.1297980	-0.1297980	-0.1297980	-0.0334065	-0.0292149
1.0	-0.1588240	-0.1588270	-0.1558828	-0.1588270	-0.0411486	-0.0360680

Table 6. Comparison of LSANN with other methods for Richardson theory of thermionic current model

x	Present method	ANN	HNN	ADM
0.05	-0.0004167158	-0.0004168678	-0.0005112515	-0.0004167188
0.10	-0.0016674970	-0.0016681944	-0.0020449814	-0.0016675005
0.15	-0.0037542257	-0.0037554983	-0.0046011582	-0.0037542247
0.20	-0.0066800404	-0.0066817197	-0.0081797579	-0.0066800339
0.25	-0.0104493569	-0.0104512070	-0.0127807628	-0.0104493484
0.30	-0.0150678925	-0.0150697040	-0.0184041611	-0.0150678882
0.35	-0.0205426944	-0.0205443377	-0.0250499454	-0.0205426999
0.40	-0.0268821742	-0.0268836154	-0.0327181128	-0.0268821920
0.45	-0.0340961475	-0.0340974347	-0.0414086634	-0.0340961759
0.50	-0.0421958821	-0.0421971122	-0.0511216002	-0.0421959153
0.55	-0.0511941535	-0.0511954312	-0.0618569276	-0.0511941831
0.60	-0.0611053099	-0.0611067108	-0.0736146515	-0.0611053269
0.65	-0.0719453473	-0.0719468942	-0.0863947781	-0.0719453438
0.70	-0.0837319953	-0.0837336544	-0.1001973132	-0.0837319659
0.75	-0.0964848154	-0.0964865135	-0.1150222621	-0.0964847552
0.80	-0.1102253112	-0.1102269705	-0.1308696283	-0.1102252120
0.85	-0.1249770528	-0.1249786326	-0.1477394132	-0.1249768957
0.90	-0.1407658164	-0.1407673430	-0.1656316155	-0.1407655594
0.95	-0.1576197378	-0.1576212991	-0.1845462307	-0.1576193009
1.00	-0.1755694833	-0.1755711532	-0.2044832504	-0.1755687313

4 Conclusion

The LSANN method was developed and implemented on first and second kind Lane-Emden equations to verify the strength of proposed algorithm. To train the network thermal minimization methodology known as simulated annealing was used that made it possible to reduce the MSE that increased the accuracy of results. The most marvelous advantage of the method is the accuracy prediction of result that can be achieved by observing the MSE. In addition to other benefits, implementation of LSANN is very tranquil and the results can be obtained on entire finite domain. The benefit of LSANN over the previously used neural methods is the usage of less number of network parameters with better accuracy and global minimization of MSE. Comparison of numerical results with other artificial neural network methods showed the superiority in training of network while the comparison with other methods demonstrates the simplicity of the method with better accuracy. All the calculations were performed on Mathematica 10.

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