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INTRODUCTION TO DOMINATED EDGE CHROMATIC NUMBER OF A GRAPH

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Abstract. We introduce and study the dominated edge coloring of a graph. A dominated edge coloring of a graph G, is a proper edge coloring of G such that each color class is dominated by at least one edge of G. The minimum number of colors among all dominated edge coloring is called the dominated edge chromatic number, denoted by $\chi'_{dom}(G)$. We obtain some properties of $\chi'_{dom}(G)$ and compute it for specific graphs. Also examine the effects on $\chi'_{dom}(G)$, when G is modified by operations on vertex and edge of G. Finally, we consider the k-subdivision of G and study the dominated edge chromatic number of these kind of graphs.

Keywords: dominated edge chromatic number, subdivision, operation, corona.

Mathematics Subject Classification: 05C25.

1. INTRODUCTION AND DEFINITIONS

Graph coloring is one of the best known, popular and extensively researched subject in the field of graph theory, having many applications and conjectures, which are still open and studied by various mathematicians and computer scientists along the world. A proper vertex coloring problem for a given graph G is to color all the vertices of the graph with different colors in such a way that any two adjacent (having an edge connecting them) vertices of G have assigned different colors. Let $G = (V, E)$ be a simple graph and $\lambda \in \mathbb{N}$. If f is a proper coloring of G with the coloring classes $V_1, V_2, \ldots, V_\lambda$ such that every vertex in V_i has color i, then sometimes proper coloring write simply $f = (V_1, V_2, \ldots, V_\lambda)$. The chromatic number $\chi(G)$ of G is the minimum of colors needed in a proper coloring of a graph. There are various interesting types of vertex coloring other than the standard (proper) coloring, such as equitable vertex coloring [12], circular vertex coloring [16], acyclic vertex coloring [11] and distinguishing vertex coloring [5]. The other well-known and intensely studied type of graph coloring besides vertex coloring is the edge coloring. Analogically to the definition of the vertex

coloring, an edge coloring of a graph is an assignment of colors to the edges of the graph so that no two incident edges have the same color. Similar to proper vertex coloring, if f is a proper edge coloring of G with the coloring classes $E_1, E_2, \ldots, E_\lambda$ such that every edge in E_i has color i, write simply $f = (E_1, E_2, \ldots, E_\lambda)$. The edge chromatic number $\chi'(G)$ of G is the minimum of colors needed in a proper edge coloring of G. A review of edge-coloring problems and its applications was given by Nakano, Zhou and Nishizeki [17].

A dominator coloring of G is a proper coloring of G such that every vertex of G is adjacent to all vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [7]. For a graph G with no isolated vertex, the total dominator coloring is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TD-chromatic number, $\chi_d^t(G)$ of G is the minimum number of color classes in a TD-coloring of G. For more information see [8, 9].

A set T of vertices is a total dominating set of G if every vertex of G is adjacent to at least one vertex in T. The total domination number $\gamma_t(G)$ of G is the minimum number of vertices in a total dominating set of G. A set F of edges is an edge dominating set of G if every edge not in F is adjacent to at least one edge in F. A set D of edges is a total edge dominating set of G if every edge in G is adjacent to at least one edge in D. The edge domination number $\gamma'(G)$ and the total edge domination number $\gamma_t'(G)$ are the minimum number of edges in an edge dominating set and in a total edge dominating set of G, respectively [13].

Dominated coloring of a graph is a proper coloring in which each color class is dominated by a vertex. The least number of colors needed for a dominated coloring of G is called the dominated chromatic number of G and denoted by $\chi_{dom}(G)$ ([2, 6, 14]).

Motivated by dominated chromatic number of graphs, we consider the proper edge coloring of G and introduce the dominated edge chromatic number of G, $\chi'_{dom}(G)$, obtain some properties of $\chi'_{dom}(G)$ and compute this parameter for specific graphs, in the next section. In Section 3, we examine the effects on $\chi'_{dom}(G)$ when G is modified by operations on vertex and edge of G. Finally in Section 4, we study the dominated edge chromatic number of k-subdivision of graphs.

2. INTRODUCTION TO DOMINATED EDGE CHROMATIC NUMBER

First we need to introduce some additional but standard notation and definitions. The maximum degree of a graph G is denoted by $\Delta(G)$. The open and closed neighborhood of a vertex $x \in V$ are denoted by $N(x)$ and $N[x]$, respectively. The open neighborhood of an edge $e \in E$ is $N(e) = \{e' \in E : e' \text{ is adjacent to } e\}$. We denote by P_n the path on n vertices and by C_n the cycle on n vertices. The complete graph on n vertices is denoted by K_n. The complete bipartite graph with parts of orders r and s is denoted by $K_{r,s}$ and the star is the complete bipartite graph $K_{1,k}$ with $k \geq 1$. A bi-star $B_{p,q}$ is a graph formed by two stars S_p and S_q by adding an edge between

their center vertices. The join of two graphs G and H, denoted by $G + H$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.

In this section, we state the definition of dominated edge chromatic number and obtain this parameter for some specific graphs.

Definition 2.1. A dominated edge coloring of graph G is a proper edge coloring of G in which every color class is dominated by at least one edge of G. More precisely, a k-dominated edge coloring of G, is a proper k-coloring $\{C_1, C_2, \ldots, C_k\}$ of G in which for every $i \in \{1, 2, \ldots, k\}$, there exists an edge $e \in E$ such that $C_i \subseteq N(e)$. The dominated edge chromatic number of G, $\chi'_{dom}(G)$, is the minimum number of color among all dominated edge coloring of G.

By the definition, the dominated edge chromatic number of a graph G equals the dominated chromatic number of the line graph $L(G)$. Observe that $\chi'_{dom}(G) \geq 1$, and $\chi'_{dom}(G) = 1$ if and only if G is K_2. Also $\chi'_{dom}(G) \geqslant \gamma'_t(G)$, where $\gamma'_t(G)$ is total edge domination number of G. To see the reason, consider a minimum dominated edge coloring of G.

For constructing a total edge dominating set of G, D'_t, from each color class we take one of its dominating edges. The set D'_t is an edge dominating set with cardinality $\chi'_{dom}(G)$. Moreover, D'_t is a total edge dominating set, since each of its edges has a color and is therefore dominated by some other edge of D'_t.

Remark 2.2. For every graph G, $\chi'_{dom}(G) \geqslant \Delta(G)$ and this inequality is sharp. As an example, for the star graph $K_{1,n}$, $\chi'_{dom}(K_{1,n}) = n$.

Remark 2.3. In a dominated edge coloring, every color can be used at most twice. So if $\{C_1, \ldots, C_t\}$ is color classes of dominated edge coloring, then for every $1 \leqslant i \leqslant t$, $|C_i| = 1$ or $|C_i| = 2$. Therefore for every graph G of size m, $\chi'_{dom}(G) \geq \lceil \frac{m}{2} \rceil$.

Let us consider the following dominated edge colorings illustrated in Figures 1–3.

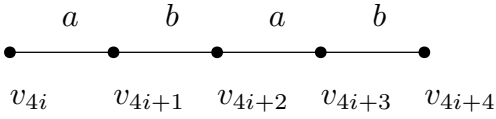

Fig. 1. Dominated edge coloring for P_5

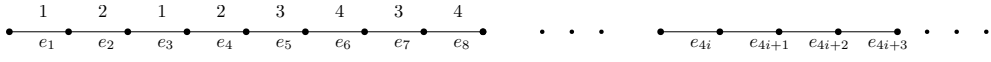

Fig. 2. Dominated edge coloring for P_n

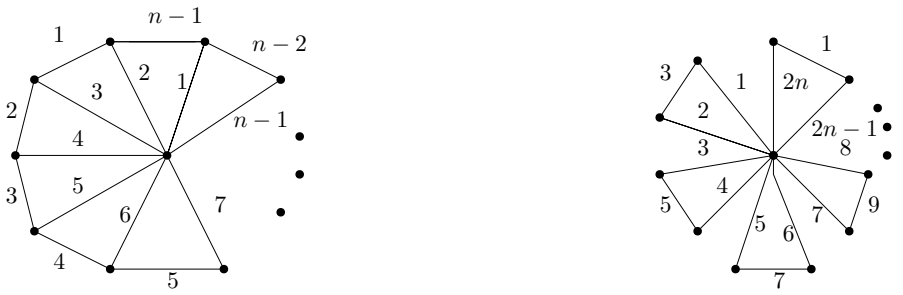

Fig. 3. Dominated edge coloring of wheel W_n and friendship graph F_n, respectively

By Remark 2.3, Figures 1, 2 and 3 we have the following theorem which is about the dominated edge chromatic number of path P_n, cycle C_n, complete graph K_n, complete bipartite graph $K_{m,n}$, wheel graph W_n and friendship graph $F_n := K_1 + nK_2$:

Theorem 2.4.

(i) *For every natural number* $n \geq 4$,

$$\chi'_{dom}(P_{n+1}) = \chi'_{dom}(C_n) = \begin{cases} \dfrac{n}{2} & \text{if } n \equiv 0 \ (mod\ 4), \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

(ii) *For every* $n \geq 4$,

$$\chi'_{dom}(K_n) = \left\lceil \frac{n(n-1)}{4} \right\rceil.$$

(iii) *For every* $m, n \geq 2$,

$$\chi'_{dom}(K_{m,n}) = \left\lceil \frac{mn}{2} \right\rceil.$$

(iv) *For any* $n \geq 3$,

$$\chi'_{dom}(W_n) = n - 1.$$

(v) *For* $n \geq 2$,

$$\chi'_{dom}(F_n) = 2n.$$

Theorem 2.5. *If G is a connected graph containing P_7 as an induced subgraph, then $\chi'_{dom}(G) \geq \Delta(G) + 2$. More generally, if the graph P_n is an induced subgraph of G, then $\chi'_{dom}(G) \geq \Delta(G) + \chi'_{dom}(P_{n-4})$.*

Proof. We assign $\Delta(G)$ colors to the edges which are incident to the vertex with maximum degree $\Delta(G)$. Now we consider P_7 as induced subgraph of G. Since we need two new colors for each four consecutive edges, so we have $\chi'_{dom}(G) \geq \Delta(G) + 2$. The proof of $\chi'_{dom}(G) \geq \Delta(G) + \chi'_{dom}(P_{n-4})$ is similar. □

Remark 2.6. The graph G in Figure 4 and its coloring shows that the lower bound in Theorem 2.5 is sharp.

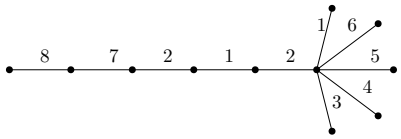

Fig. 4. Graph G with $\chi'_{dom} = \Delta(G) + 2$

Theorem 2.7.

(i) *If a and b are two integers with $a \geq b \geq 2$ such that $a \geq 2b$, then there exists a graph G with dominated edge chromatic number $\chi'_{dom}(G) = a$ and total edge domination number $\gamma'_t(G) = b$.*

(ii) *If a and b are two integers with $a \geq b \geq 2$, then there exists a graph G with dominated edge chromatic number $\chi'_{dom}(G) = a$ and total edge domination number $\gamma'_t(G) = b$.*

Proof. (i) If $a = 2b$, then we consider the friendship graph F_b. Therefore $\gamma'_t(F_b) = b$ and $\chi'_{dom}(F_b) = 2b$. Now if $a > 2b$, then we add $m = a - 2b$ pendant edges to center of the friendship graph F_b and call this new graph G (Figure 5). So $\chi'_{dom}(G) = a$ and $\gamma'_t(G) = b$.

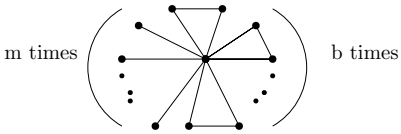

Fig. 5. The graph G in Theorem 2.7 (i)

(ii) Consider the graph $K_{1,a}$ and bK_1. If G is a graph which obtain by connecting the b vertices of bK_1 to b leaves of $K_{1,a}$ (Figure 6), then $\chi'_{dom}(G) = a$ and $\gamma'_t(G) = b$.

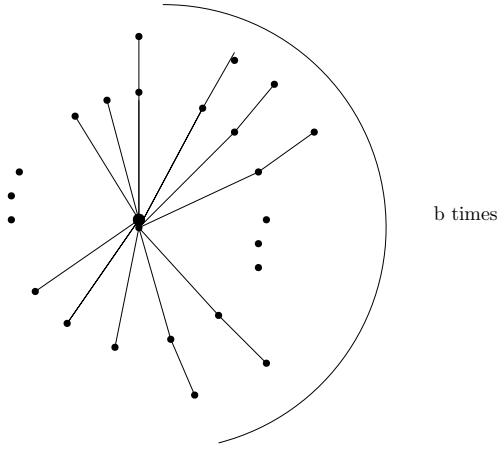

Fig. 6. The graph G in Theorem 2.7 (ii)

□

The corona of G and H is denoted by $G \circ H$, is a graph made by a copy of G (which has n vertices) and n copy of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The following theorem gives the lower and the upper bounds for the dominated edge chromatic number of $G \circ H$.

Theorem 2.8. *Let G and H be two graphs of orders n and k, respectively.*

(i) *For even n,*

$$\left\lceil \frac{|E(G)| + |E(H)| + nk}{2} \right\rceil \leq \chi'_{dom}(G \circ H) \leq \chi'_{dom}(G) + n\chi'_{dom}(H) + \frac{nk}{2},$$

(ii) *For odd n,*

$$\left\lceil \frac{|E(G)| + |E(H)| + nk}{2} \right\rceil \leq \chi'_{dom}(G \circ H) \leq \chi'_{dom}(G) + n\chi'_{dom}(H) + \frac{(n-1)k}{2} + k.$$

Proof. The left inequality follows as Remark 2.3. For the right inequality, we consider a dominated edge coloring for G and H and color G and n copy of H with $\chi'_{dom}(G) + n\chi'_{dom}(H)$ colors. Then, we color the other edges with $\frac{nk}{2}$ colors, if n is even and with $\frac{(n-1)k}{2} + k$ colors, if n is odd. This is a dominated edge coloring for $G \circ H$ and so we have the result. □

Theorem 2.9. *Suppose that in a dominated edge coloring (with minimum number of colors) of two graphs G and H of orders n and k, respectively, there is no color class with size one. Then*

(i) *for even n,*

$$\chi'_{dom}(G \circ H) = \chi'_{dom}(G) + n\chi'_{dom}(H) + \frac{nk}{2},$$

(ii) *for odd n,*

$$\chi'_{dom}(G \circ H) = \chi'_{dom}(G) + n\chi'_{dom}(H) + \frac{(n-1)k}{2} + k.$$

Proof. It is similar to the proof of Theorem 2.8. □

Remark 2.10. If t is the number of color classes of size one in a dominated edge coloring of two graphs G and H of order n and k, respectively, then

$$\chi'_{dom}(G \circ H) \leq \chi'_{dom}(G) + n\chi'_{dom}(H) + \left\lfloor \frac{n}{2} \right\rfloor k - t.$$

The following theorem gives an upper bound for $\chi'_{dom}(G + H)$.

Theorem 2.11. *For two connected graph G and H,*

$$\chi'_{dom}(G + H) \leq \chi'_{dom}(G) + \chi'_{dom}(H) + \left\lceil \frac{|V(G)| \times |V(H)|}{2} \right\rceil.$$

Proof. We color the graph G with $\chi'_{dom}(G)$ and the graph H with $\chi'_{dom}(H)$ colors and other edge with $\left\lceil \frac{|V(G)| \times |V(H)|}{2} \right\rceil$ new colors. So this is a dominated edge coloring. Note that if $\chi'_{dom}(G) = \frac{|E(G)|}{2}$ and $\chi'_{dom}(H) = \frac{|E(H)|}{2}$, then the inequality is sharp. □

3. DOMINATED EDGE CHROMATIC NUMBER OF SOME OPERATIONS ON A GRAPH

In this section, we examine the effects on $\chi'_{dom}(G)$ when G is modified by operations on vertex and edge of G. The graph $G-v$ is a graph that is made by deleting the vertex v and all edges incident to v from the graph G and the graph $G-e$ is a graph that obtained from G by simply removing the edge e. We present bounds for dominated edge chromatic number of $G-v$ and $G-e$. We begin with $G-v$.

Theorem 3.1. *If G is a connected graph and $v \in V(G)$ is not a cut vertex of G, then*

$$\chi'_{dom}(G) - deg(v) \leq \chi'_{dom}(G-v) \leq \chi'_{dom}(G) + deg(v).$$

Proof. First we prove the left inequality. We give a dominated edge coloring to $G-v$, add v and all the corresponding edges. Then we assign $deg(v)$ new colors to these edges and do not changes the color of other edges. So this is a dominated edge coloring of G and $\chi'_{dom}(G) \leq \chi'_{dom}(G-v) + deg(v)$.

For the right inequality, first we give a dominated edge coloring to G. In this case, since v is not a cut vertex, each edges which is adjacent to an edge with endpoint v has an other adjacent edge too. We change the color of this edge to a new color and do this $deg(v)$ times and do not change the color of the other edges. So this is an edge dominated edge coloring of $G-v$ and $\chi'_{dom}(G-v) \leq \chi'_{dom}(G) + deg(v)$. Therefore we have the result. □

Remark 3.2. The upper bound in Theorem 3.1 is sharp. Consider the graph G in Figure 7.

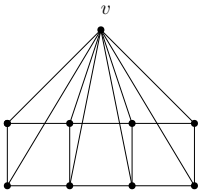

Fig. 7. The graph G in Theorem 3.1

By considering the graph in Figure 8, we have the following result.

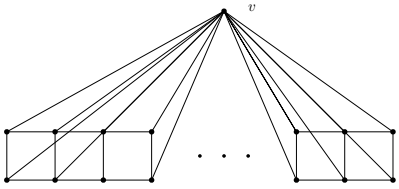

Fig. 8. The graph G in Theorem 3.3

Theorem 3.3. *There is a connected graph G and a vertex $v \in V(G)$ which is not a cut vertex of G such that $|\chi'_{dom}(G) - \chi'_{dom}(G-v)|$ can be arbitrarily large.*

Theorem 3.4. *If G is a connected graph, and $e = uv \in E(G)$ is not a bridge of G, then*

$$\chi'_{dom}(G) - 1 \leq \chi'_{dom}(G-e) \leq \chi'_{dom}(G) + deg(v) - 2.$$

Proof. First we prove the left inequality. We give a dominated edge coloring to $G-e$, then we add edge e. If we can give one of the previous colors to e, then $\chi'_{dom}(G-e) = \chi'_{dom}(G)$. Otherwise, we assign new color i to edge e. So we have a dominated edge coloring for G and $\chi'_{dom}(G) \leq \chi'_{dom}(G-e) + 1$.

Now we prove the right inequality. Let $e = uv$ and $deg(u) \geq deg(v)$. If e just dominates its color class, then $\chi'_{dom}(G-e) = \chi'_{dom}(G)$. If e is the only edge that dominates all adjacent color classes, then by removing e, some of these edges will not dominated or previous coloring with removing e is not a dominated edge coloring. Then we have to add new colors. Since e is not a bridge of G, there is a path between u and v other than e. In this case, we need to add at least $deg(v) - 2$ color. We can use two previous colors, the color of edge e and one of the previous colors (by Remark 2.3 and Theorem 2.4). So $\chi'_{dom}(G-e) \leq \chi'_{dom}(G) + deg(v) - 2$ and therefore we have the result. □

Remark 3.5. The bounds in Theorem 3.4 are sharp. For the upper lower consider the graph in Figure 9 and for the upper bound consider the graph in Figure 10.

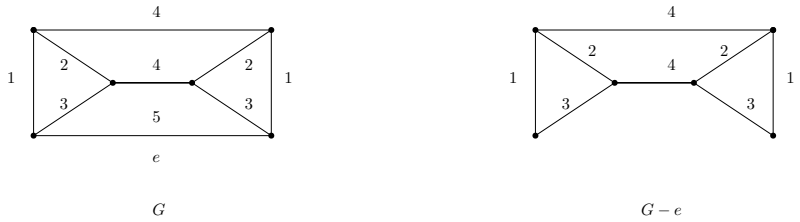

Fig. 9. The graph G in Theorem 3.4

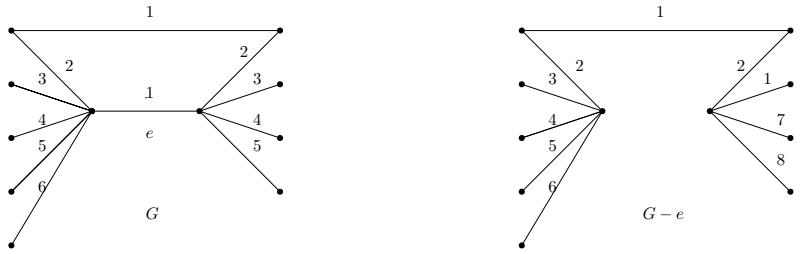

Fig. 10. The graph G in Theorem 3.4

Theorem 3.6. *There is a connected graph G and a vertex $v \in V(G)$ which is not a cut vertex of G such that $|\chi'_{dom}(G) - \chi'_{dom}(G-e)|$ can be arbitrarily large.*

Proof. We consider the graph in Figure 10 with $e = uv$ such that $deg(u)$ and $deg(v)$ are large enough. □

In a graph G, contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph G/e has one less edge than G. We denote this graph by G/e. We end this section with the following theorem which gives bounds for $\chi'_{dom}(G/e)$.

Theorem 3.7. *Let G be a connected graph and $e = uv \in E(G)$. Then we have*

$$\chi'_{dom}(G) - 1 \leq \chi'_{dom}(G/e) \leq \chi'_{dom}(G) + min\{deg(u), deg(v)\} - 1.$$

Proof. First we prove the left inequality. We give a dominated edge coloring to G/e, add e and assign it a new color, say i. This is a dominated edge coloring of G. So we have $\chi'_{dom}(G) \leq \chi'_{dom}(G/e) + 1$.

For the right inequality, we give a dominated edge coloring to G. Suppose that $min\{deg(u), deg(v)\} = deg(u)$. Now we make G/e and change the color of adjacent edge of e with endpoint u to new colors. So we have the result. □

Remark 3.8. The bounds in Theorem 3.7 are sharp. For the upper bound consider the graph bi-star $B_{p,q}$ and for the lower bound consider C_5 as G. Note that $\chi'_{dom}(C_5) = 3$ and $\chi'_{dom}(C_4) = 2$.

4. DOMINATED EDGE COLORING OF k-SUBDIVISION OF A GRAPH

The k-subdivision of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge v_iv_j of G with a path of length k, say $P^{\{v_i,v_j\}}$. These k-paths are called superedges, any new vertex is an internal vertex, and is denoted by $x_l^{\{v_i,v_j\}}$ if it belongs to the superedge $P^{\{v_i,v_j\}}, i < j$ with distance l from the vertex v_i, where $l \in \{1, 2, \ldots, k-1\}$. Note that for $k = 1$, we have $G^{\frac{1}{1}} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges. In this section, we study dominated edge coloring of k-subdivision of a graph ([10]). In particular, we obtain some bounds for $\chi'_{dom}(G^{\frac{1}{k}})$ and prove that for any $k \geq 2$, $\chi'_{dom}(G^{\frac{1}{k}}) \leq \chi'_{dom}(G^{\frac{1}{k+1}})$.

Theorem 4.1. *If G is a graph of size m, then $\chi'_{dom}(G^{\frac{1}{k}}) \geq m$, for $k \geq 3$.*

Proof. For $k = 3$, in any superedge $P^{\{v,w\}}$ such as $\{v, x_1^{\{v,w\}}, x_2^{\{v,w\}}, w\}$, the edge $x_1^{\{v,w\}}x_2^{\{v,w\}}$ need to use a new color in at least of its adjacent edges and we cannot use this color in any other superedges. So we have the result. □ □

Theorem 4.2. *If G is a connected graph of size m and $k \geq 2$, then*

$$\chi'_{dom}(P_{k+1}) \leq \chi'_{dom}(G^{\frac{1}{k}}) \leq m\chi'_{dom}(P_{k+1}).$$

Proof. First we prove the right inequality. Suppose that $e = uu_1$ be an arbitrary edge of G. This edge is replaced with the super edge $P^{\{u,u_1\}}$ in $G^{\frac{1}{k}}$, with vertices $\{u, x_1^{\{u,u_1\}}, \ldots, x_{k-1}^{\{u,u_1\}}, u_1\}$. We color this superedge with $\chi'_{dom}(P_{k+1})$ colors as a dominated edge coloring of P_{k+1}. We do this for all superedges. Thus we need at most $m\chi'_{dom}(P_{k+1})$ new colors for a dominated edge coloring of $G^{\frac{1}{k}}$.

For the left inequality, if G is a path the result is true. So we suppose that G is a connected graph which is not a path. Let c' be a dominated edge coloring of $G^{\frac{1}{k}}$. The restriction of c' to edges of $P^{\{u,v\}}$ is a dominated edge coloring and so we have the result. □

Remark 4.3. The lower bound of Theorem 4.2 is sharp for P_2 and by the following Theorem we show that the upper bound of this Theorem is sharp for $G = K_{1,n}$ and $k \geq 3$.

Theorem 4.4. *For $n \geq 3$ and $k \geq 3$, $\chi'_{dom}(K_{1,n}^{\frac{1}{k}}) = n\chi'_{dom}(P_{k+1})$.*

Proof. Color all path connected to the center vertex with $\chi'_{dom}(P_{k+1})$. This is a dominated edge coloring. Because the colors adjacent to the center were used twice, by the Remark 2.3, so we cannot use colors of one path to another path. Therefore we have the result. □ □

Theorem 4.5. *If G is a connected graph and $k \equiv 0(mod\,4)$, then $\chi'_{dom}(G^{\frac{1}{k}}) = m\chi'_{dom}(P_{k+1})$.*

Proof. By Remark 2.3 each color can be used at most twice. So we color each superedge with $\chi'_{dom}(P_{k+1})$ colors. Therefore $\chi'_{dom}(G^{\frac{1}{k}}) = m\chi'_{dom}(P_{k+1})$. □

Theorem 4.6. *If G is a graph of size m and $k \not\equiv 0(mod\,4)$ with $k \geq 5$, then*

(i) *if $k \equiv 1(mod\,4)$, then $\chi'_{dom}(G^{\frac{1}{k}}) \geq m\chi'_{dom}(P_k)$,*
(ii) *if $k \equiv 2(mod\,4)$, then $\chi'_{dom}(G^{\frac{1}{k}}) \geq m\chi'_{dom}(P_{k-1})$,*
(iii) *if $k \equiv (mod\,4)$, then $\chi'_{dom}(G^{\frac{1}{k}}) \geq m\chi'_{dom}(P_k)$.*

Proof. By Remark 2.3, in each superedge, each color can be used at most twice. So we can use some of colors that used once. If $k \equiv 1(mod\,4)$, we need at least $\chi'_{dom}(P_k)$ color in each superedges. If $k \equiv 2(mod\,4)$, we need at least $\chi'_{dom}(P_{k-1})$ color in each superedges and if $k \equiv 3(mod\,4)$, we need at least $\chi'_{dom}(P_k)$ color in each superedges. Therefore, we have the result. □

Theorem 4.7. *For any $k \geq 3$, $\chi'_{dom}(G^{\frac{1}{k}}) \leq \chi'_{dom}(G^{\frac{1}{k+1}})$.*

Proof. First we give a dominated edge coloring of $G^{\frac{1}{k}}$. Let $P^{\{u,v\}}$ be an arbitrary superedge of $G^{\frac{1}{k+1}}$ with vertex set $\{v, x_1^{\{v,w\}}, x_2^{\{v,w\}}, \ldots, x_k^{\{v,w\}}, w\}$. There exists an edge $u \in \{x_1^{\{v,w\}}x_2^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, x_k^{\{v,w\}}\}$. Consider the graph in Figure 11.

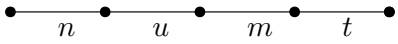

Fig. 11. A part of a superedge in the proof of Theorem 4.7

We have the following cases:

Case 1. Suppose that the edge u has color i and the edge n has color j and the edge m has color t. In this case, we make G/u and do not change the color of any edge. So without adding a new color we have a dominated edge coloring for this new graph.

Case 2. Suppose that the edge u has color i and the edge n has color j and the edge m has color j. In this case, we make G/u and change one of the edge with color j to color i such that two edge with color i not adjacent. So without adding a new color, we have a dominated edge coloring for this new graph.

Now we do the same algorithm for all superedges. So we have a dominated edge coloring again. □

Theorem 4.8. *For any graph G, $\chi'_{dom}(G^{\frac{1}{2}}) \leq \chi'_{dom}(G^{\frac{1}{3}})$.*

Proof. First we give a dominated edge coloring to the edges of $G^{\frac{1}{3}}$. Let $P^{\{w,z\}}$ be an arbitrary superedge of $G^{\frac{1}{3}}$ with edge set $\{s, v, u\}$ (see Figure 12) and suppose that edge v has the color i.

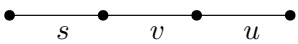

Fig. 12. A part of a superedge in the proof of Theorem 4.8

We have the following cases:

Case 1. The edge u has the color j and the edge s has the color t. In this case, we make G/v and don't change the color of any edges. So we have a dominated edge coloring for this new graph.

Case 2. The edge u and v have the color j. In this case, we make G/v and change one of the edge with color j to color i such that two edge with color i not adjacent. So without adding a new color we have a dominated edge coloring for this new graph.

Now we do the same algorithm for all superedge. Therefore we have a dominated edge coloring again. □

5. CONCLUSION

In this paper, we have introduced the dominated edge coloring of a graph and have obtained several results on the dominated edge chromatic number. In this respect, there are many problems that are open. Let us list a few of them. In Theorems 2.8, 2.9 and 2.11 we have studied the dominated edge chromatic number for corona and join products of two graphs. Now, we pose the following problem.

Problem 5.1. What is the dominated edge chromatic number of Cartesian, lexicographic and neighbourhood corona (see [3]) of two graphs.

In Section 4, we studied the dominated edge coloring of k-subdivision of a graph. The m^{th} power of G, is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length at most m between them in G. The fractional power of G, is m^{th} power of the n-subdivision of G, i.e., $(G^{\frac{1}{n}})^m$ or n-subdivision of m-th power of G, i.e., $(G^m)^{\frac{1}{n}}$ (see [4]). Note that the graphs $(G^{\frac{1}{n}})^m$ and $(G^m)^{\frac{1}{n}}$ are different graphs. In view of the results from Section 4 we pose another problem.

Problem 5.2. Study and investigate the dominated edge chromatic number for the fractional power of a graph.

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