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# SYSTEM VIBRATION CONTROL USING LINEAR QUADRATIC REGULATOR

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Balancing a bipedal robot movement against external perturbations is considered a challenging and complex topic. This paper discusses how the vibration caused by external disturbance has been tackled by a Linear Quadratic Regulator, which aims to provide optimal control to the system. A simulation was conducted on MATLAB in order to prove the concept. Results have shown that the linear quadratic regulator was successful in stabilizing the system efficiently.

**Key words:** linear quadratic regulator, inverted pendulum on cart

## 1. Introduction

Keeping the balance of a walking bi-legged robot is a challenging and complex task. A successful walking gait of a robot is to go from the start point to the destination point without falling. A Bi-Legged robot should have a smooth design close to human beings in order to have balance while standing. However, there are some issues such as the slope or the nature of the ground. The environment eventually causes friction and external disturbances that may suddenly act on the robot. It is therefore important to balance the bipedal robot statically and dynamically in order to make sure that it will not fall.

Researchers have tackled the control of sudden external perturbations applied to a bipedal robot by balancing strategy is such as ankle, hip and stepping strategies [1-4] by using the torque in order to balance the bipedal robot back to stability. In recent papers as well, it has been proposed to tackle this issue by capture point [5-6]. In our experiment at the laboratory of Tallinn University of Technology, the problem of the static and dynamic balance was tackled by keeping the Zero-Moment Point and the Ground projection of center of mass in the support polygon, during the single support phase and the double support phase, controlled by the model predictive controller [7-10].

There are various proposed types of control over vibration caused by sudden external perturbations to the system. Roose *et al.* [12] used fuzzy logic with a PID controller since PID stabilizes only linear systems. Fuzzy logic uses linguistic variables and their membership functions as rule-base to get the proper output. Performance of the fuzzy logic controller for balancing the vibration of the inverted pendulum was successful in providing better settling time and lowest overshoot. Abut et al. [13] defined fuzzy logic as an approach which uses approximate thinking instead of thinking based on exact values which is suitable for systems with a difficult mathematical model. It was concluded from the experiment of the PID- type fuzzy logic that the system had the lowest amplitude while reaching stability position and best performance for settling time. In a recent research, Hazem *et al.* [14] proved that combining fuzzy logic with the linear quadratic regulator improved the settling time, the peak overshoot, steady state error and the total root mean square error by high percentage if the model was applied on a double inverted pendulum.

In this article, it will be proposed to use a linear quadratic regulator on an inverted pendulum on cart in order to tackle the vibration caused by external perturbations applied on the robot pelvis laterally.

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In Section 2, the chosen model and derivation of the equation of motion using the Lagrange equations is discussed. Section 3 illustrates the results obtained in our simulation. And finally, in section 4 simulation results and conclusions are presented.

## 2. Proposed model and controller

### 2.1. Inverted pendulum on cart

Simplification is a common method when starting to investigate any technical or scientific problem. It was therefore decided to choose an inverted pendulum on cart as the model since it is considered a benchmark tool for testing control techniques.

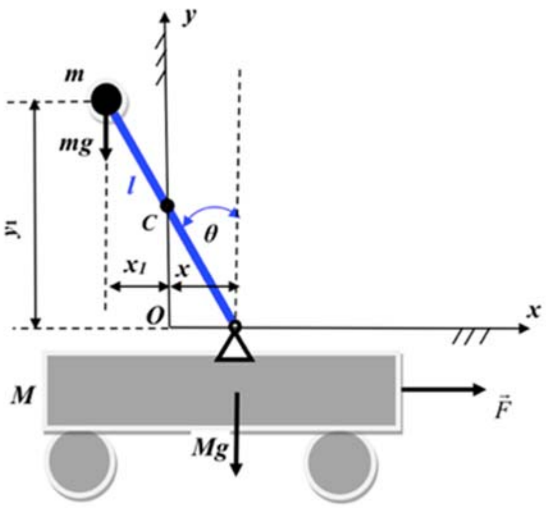


Fig.1. Inverted pendulum on cart.

The cart moves in a horizontal way in the $x$ direction with a mass $M$, while the inverted pendulum right on top of the cart is attached to the cart with a massless rigid rod (Fig.1, [11]). The pendulum has a mass $m$ and rod length $l$. The gravity forces are $mg$ for the pendulum and $Mg$ for the cart respectively, where $g$ is the acceleration of gravity. It is assumed that there is no friction on the ground. The external force $F$ is applied to the cart in the $x$-axis direction. The pendulum is rotating from the vertical by angle $\theta$. It is important to understand the inverted pendulum in order to make it stable in the upright position.

Let us choose the fixed axes of coordinates $x$ and $y$, assuming that the axis $Oy$ passes through the initial position of the center of gravity $C$ of the system shown in Fig.1. The position of the system relative to the fixed axes $xOy$ is determined by the coordinate of the center of gravity of the cart $x$ and the angle of rotation of the pendulum $\theta$ relative to the vertical. That is, the given system has two degrees of freedom.

Let us take the coordinate $x$ and angle $\theta$ as generalized coordinates. Then the dependence of the coordinates center of gravity of the inverted pendulum $x_1$ and $y_1$ on the generalized coordinates $x$ and $\theta$ will have the form:

$$x_1 = x - l \cdot \sin\theta, \tag{2.1}$$

$$y_1 = l \cdot \cos\theta. \tag{2.2}$$

Looking to the velocity direction, it can be calculated as the first derivate of $x_1$ and $y_1$ in respect to time as:

$$\dot{x}_1 = \dot{x} - l \cdot \dot{\theta} \cdot \cos\theta \tag{2.3}$$

$$\dot{y}_1 = -l \cdot \dot{\theta} \cdot \sin\theta \tag{2.4}$$

So, in order to find the equation of motion, the derivation of the Lagrange equations is based on the kinetic energy and potential energy. The potential energy of the pendulum is expressed as:

$$V = m \cdot g \cdot y_1 = m \cdot g \cdot l \cdot \cos\theta . \tag{2.5}$$

The kinetic energy of the cart can be expressed as:

$$T_{cart} = \frac{1}{2} M \cdot v^2 = \frac{1}{2} M \cdot \dot{x}^2 \tag{2.6}$$

The kinetic energy of the pendulum is expressed as:

$$T_{pend} = \frac{1}{2}\, m \cdot v_1^2 = \frac{1}{2} m \cdot \left( \dot{x}_1^2 + \dot{y}_1^2 \right) \tag{2.7}$$

Taking into account expressions (2.3) and (2.4), we obtain the kinetic energy of the entire system:

$$\begin{aligned} T &= \frac{1}{2} M \cdot \dot{x}^2 + \frac{1}{2} m \cdot \left( \dot{x}^2 - 2 \cdot l \cdot \dot{x} \cdot \dot{\theta} \cdot \cos\theta + l^2 \cdot \dot{\theta}^2 \cdot \cos^2\theta + l^2 \cdot \dot{\theta}^2 \cdot \sin^2\theta \right) = \\ &= \frac{1}{2}(M + m) \cdot \dot{x}^2 - m \cdot l \cdot \dot{x} \cdot \dot{\theta} \cdot \cos\theta + + \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 . \end{aligned} \tag{2.8}$$

The kinetic (generalized) potential of the system will be:

$$L = T - V = \frac{1}{2}(M + m) \cdot \dot{x}^2 - m \cdot l \cdot \dot{x} \cdot \dot{\theta} \cdot \cos\theta + \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 - m \cdot g \cdot l \cdot \cos\theta . \tag{2.9}$$

The Lagrange equations have the form:

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}} \right) - \left( \frac{\partial L}{\partial x} \right) = Q_x = F(t) , \tag{2.10}$$

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{\theta}} \right) - \left( \frac{\partial L}{\partial \theta} \right) = Q_\theta = 0 , \tag{2.11}$$

$Q_x$ and $Q_\theta$ are generalized forces which indicate the external forces at the (generalized) coordinates $x$ and $\theta$.

The first equation of motion will be in the $x$ direction taking into account the kinetic (generalized) potential (2.9). According to the Lagrange equation (2.10), we will get

$$(M + m) \cdot \ddot{x} - m \cdot l \cdot \ddot{\theta} \cdot \cos\theta + m \cdot l \cdot \dot{\theta}^2 \cdot \sin\theta = F(t) . \tag{2.12}$$

The second equation of motion will be according to the Lagrange equation (2.11). Taking into account the kinetic (generalized) potential (2.9), we will obtain:

$$l \cdot \ddot{\theta} - \ddot{x} \cdot \cos\theta - g \cdot \sin\theta = 0 \tag{2.13}$$

After deriving the Lagrange equations of motion, the next step is to derive the state space of the inverted pendulum on cart in order to get the matrices A and B that will be used as input in the linear quadratic regulator matrix calculations.

**2.2. Linear quadratic regulator**

The Linear Quadratic Regulator (LQR) aims to optimize the cost function in a system when it is in a nonlinear state. It is used as a state feedback gain. Unlike the PIDs, the LQR studies the system behavior through the state space form. Using a trial and error procedure, the linear quadratic regulator matrices Q and R which are the state weighting matrix and control weighting matrix, respectively, should be tuned in order to get the optimal control over the external perturbations and minimize the overshooting and settling time.

After deriving the equation of motion of the inverted pendulum on cart and linearizing it, it is needed to get the state space which is expressed as follows:

$$\dot{x} = Ax + Bu \tag{2.14}$$

$$y = Cx + Du \tag{2.15}$$

$x$ is the vector state, $u$ is the control input which is the force, $A$ is the system matrix, $B$ is the input matrix, $C$ is the output matrix and $y$ is the output vector.

The linear quadratic regulator that was used in the model in order to control the inverted pendulum on cart is depending on $Q$ and $R$ weight matrices to get the most optimal response with the purpose of regulating the system to make the output $y$ be zero with minimum input. The LQR gives the full state vector, which is $x$ derived from the state space in Eq. (2.14) and multiplies it by the matrix gain $K$ and subtracts it from the scaled reference gain in order to get the optimal output. The purpose is to get the optimal $K$ by choosing the optimal characteristics through performance and effort. The LQR problem is as follows:

$$j = \int_0^\infty [x^T(t) Q x(t) + u^T(t) R u(t)] \tag{2.16}$$

$Q$ is the performance weight matrix and $R$ is the energy control. The feedback control law that can minimize the value of cost is:

$$u = -Kx \tag{2.17}$$

The optimal gain of feedback $K$ can be expressed as:

$$K = R^{-1} B^T P \tag{2.18}$$

$P$ is found by solving Riccati equation:

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \tag{2.19}$$

### 2.3. Simulation results

In MATLAB simulation of vibration control of the inverted pendulum, it is necessary to apply the trial and error procedurein order to find the optimal control to the system while receiving external perturbation. In this simulation, it had to compromise between the performance $Q$ and the actuator effort, which is $R$ in order to have the optimal gain $K$. In that case, it has given equal priority to linear displacement of the cart $q_1$, same as the velocity of the cart $\dot{q}_1$, same as the angular displacement link $q_2$, same as angular velocity of the link $\dot{q}_2$. So, the obtained results are based on penalizing all these important factors the same way in the matrix giving them the same priority. The identity matrix (eye) set is as follows:

$$\text{eye}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.1}$$

Table 1 shows different variables of the inverted pendulum on cart that were used in the simulation.

Table 1. Inverted pendulum on cart specifications.

| Pendulum specification | |
|---|---|
| Variable | Value |
| $M_{Cart}$ | *1.5 kg* |
| $m_{Pend}$ | *0.5 kg* |
| $l_{Rod}$ | *1 m* |

Table 2. The different K gains results

| *No* | *K1* | *K2* | *K3* | *K4* | *Q* | *R* |
|---|---|---|---|---|---|---|
| *1* | *-14.14* | *171.88* | *– 25.27* | *54.75* | *1 * eye* | *1* |
| *2* | *– 0.99* | *53.46* | *– 2.81* | *15.38* | *1 * eye* | *1* |
| *3* | *– 1.00* | *53.46* | *– 2.81* | *15.38* | *10 * eye* | *10* |
| *4* | *– 1.00* | *53.46* | *– 2.81* | *15.38* | *10 * eye* | *10* |

As a trial and error of tuning $Q$ and $R$ in the simulation as shown above, those were the different gain $K$s to minimize the peak overshoot and settling time of the curve.

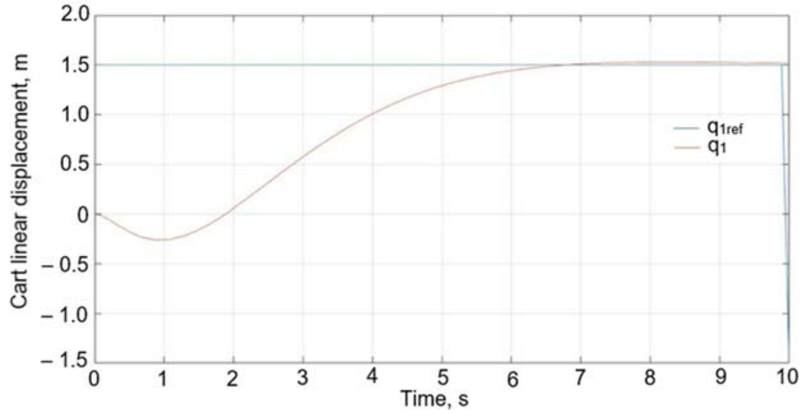


Fig.2. Linear displacement of the cart.

As a result, the optimal control for the cart linear displacement, as shown in Fig.2, has a small peak overshoot and the time the cart took to settle and stabilize was *6.7* seconds. The optimal control to make the link of the pendulum angular displacement go back to zero was *9.3* seconds (Fig.3).

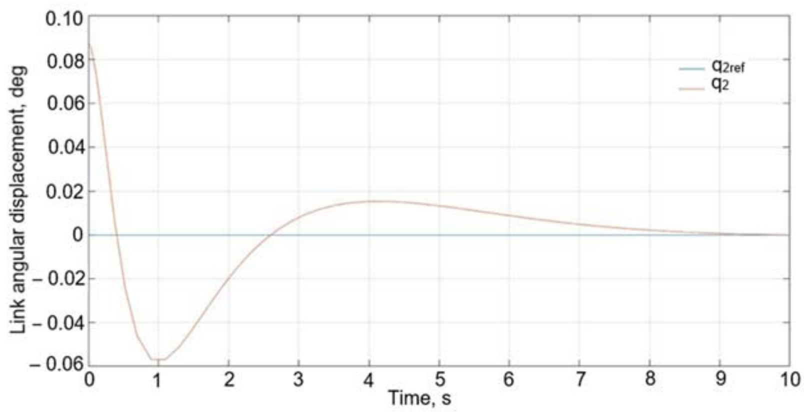


Fig.3. Angular displacement of the link.

## 3. Conclusions

Tackling the vibration caused by external perturbations on the system is quite important to stabilize the system while moving on uneven terrain or slope. With all the complexity and unstable nature of the inverted pendulum on cart, the linear quadratic regulator controller was successful in controlling the vibration caused by sudden external disturbance to the system.

After deriving the Lagrange equation of motion of the proposed model and linearization, the LQR could stabilize the system on MATLAB simulation with small overshoot and minimal settling time. This research aims at getting a much smaller peak overshoot and settling time. Further studies will be conducted using the method of a fuzzy linear quadratic regulator.

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## Nomenclature

$A$ – system matrix
$B$ – input matrix
$C$ – output matrix
$D$ – direct feedthrough matrix
$F$ – external force
$K$ – matrix gain
$L$ – kinetic potential of the system
$M$ – cart mass
$P$ – positive definite symmetric solution
$Q$ – state weighting matrix
$Q_x$, $Q_\theta$ – generalized forces
$R$ – control weighting matrix
$T_{cart}$ – kinetic energy of the cart
$T_{pend}$ – kinetic energy of the pendulum
$V$ – potential energy of the pendulum
$g$ – gravity acceleration
$l$ – rod length
$m$ – pendulum mass
$u$ – control input
$v$ – velocity of the cart
$v_1$ – velocity of the pendulum
$x$ – state vector
$x_1$ – coordinate of the gravity center of the inverted pendulum
$y$ – output vector
$y_1$ – coordinate of the gravity center of the inverted pendulum
$\theta$ – angle of pendulum rotation

## References

[1] Bahdi E. (2018): *Development of a Locomotion and Balancing Strategy for Humanoid Robots*.– M.S. Thesis, Computer Engineering, University of Denver, p.102.

[2] Zhang L. and Fu C. (2018): *Predicting foot placement for balance through a simple model with swing leg dynamics*.– Journal of Biomechanics, vol.77, pp.155-162.

[3] Reimann H., Fettrow T. and Jeka J. J. (2018): *Strategies for the control of balance during locomotion*.– Kinesiology Review, vol.7, No.1, pp.18-25.

[4] Shafiee-Ashtiani M., Yousefi-Koma, A., Shariat-Panahi, M. and Khadiv, M. (2017): *Push recovery of a humanoid robot based on model predictive control and capture point*.– 4-th RSI International Conference on Robotics and Mechatronics, ICRoM 2016, pp.433-438.

[5] Vaz J. C. and Oh P. (2020): *Material handling by humanoid robot while pushing carts using a walking pattern based on capture point*.– Proceedings - IEEE International Conference on Robotics and Automation, pp.9796-9801.

[6] Jo H.M. and Oh J.H. (2018): *Balance recovery through model predictive control based on capture point dynamics for biped walking robot*.– Robotics and Autonomous Systems, No.105, pp.1-10.

[7] Araffa K. and Tkach M. (2019): *Implementation and simulation a model predictive control for motion generation of biped robot*.– Adaptive Systems of Automatic Control, vol.2, No.35, pp.3-12.

[8] White J., Swart D. Hubicki C. (2020): *Force-based control of bipedal balancing on dynamic terrain with the tallahassee cassie robotic platform*.– Proceedings - IEEE International Conference on Robotics and Automation, pp.6618-6624.

[9] Al-Shuka H.F.N., Corves B. J., Vanderborght B. and Zhu W.-H. (2014): *Zero-Moment Point-Based Biped Robot with Different Walking Patterns*.– International Journal of Intelligent Systems and Applications, vol.7, No.1, pp.31-41.

[10] DeHart B. J. (2019): *Dynamic Balance and Gait Metrics for Robotic Bipeds*.– Thesis for PhD in electrical and Computer Engineering, University of Waterloo, p.153.

[11] Araffa K. and Tkach M. (2019): *Implementation and simulation a model predictive control for motion generation of biped robot*.– Adaptive Systems of Automatic Control, vol.2, No.35, pp.3-12.

[12] Roose A.I., Yahya S. and Al-Rizzo H. (2017): *Fuzzy-logic control of an inverted pendulum on a cart*.– Computers and Electrical Engineering, No.61, pp.31-47, https://doi.org/10.1016/j.compeleceng.2017.05.016.

[13] Abut T. and Soyguder S. (2019): *Real-time control and application with self-tuning PID-type fuzzy adaptive controller of an inverted pendulum*. Industrial Robot, vol.46, No.1, pp.159-170.

[14] Hazem Z. Ben, Fotuhi M.J. and Bingül Z. (2020): *Development of a Fuzzy-LQR and Fuzzy-LQG stability control for a double link rotary inverted pendulum*. Journal of the Franklin Institute, vol.357, No.15, pp.10529-10556.

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