10.24425/acs.2019.130205

Archives of Control Sciences
Volume 29(LXV), 2019
No. 3, *pages 549–567*

Stability of discrete-time fractional linear systems with delays

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The stability analysis for discrete-time fractional linear systems with delays is presented. The state-space model with a time shift in the difference is considered. Necessary and sufficient conditions for practical stability and for asymptotic stability have been established. The systems with only one matrix occurring in the state equation at a delayed moment have been also considered. In this case analytical conditions for asymptotic stability have been given. Moreover parametric descriptions of the boundary of practical stability and asymptotic stability regions have been presented.

Key words: fractional, discrete-time, stability, time-delay

1. Introduction

Integral and differential operators of noninteger order (fractional calculus) are used in many fields of science and engineering such as control systems, mechanics, chemistry, biology, electrical engineering and signal processing. Physical phenomena are often modelled by noninteger order differential or difference equations. Many books and articles present the state of the art of fractional modelling and applications, e.g. [1, 6, 8, 10, 11, 15, 16, 20, 23].

One of the most important issues in the theory of dynamical systems is the stability of dynamical model. The state-space model is very common. The model with a time shift in the difference is the most well-known in the case of discrete-time systems. The stability problem of such a model with Grünwald-Letnikov-type fractional-order difference has already been considered. The asymptotic stability conditions and the stability domains have been presented in [4, 5, 14,

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This work was supported by National Science Centre in Poland under work No. 2017/27/B/ST7/02443.
Received 14.03.2019. Revised 2.09.2019.

21, 22]. Moreover the so-called practical stability for a given length of practical implementation has been also analysed, e.g. [3, 7]. The stability testing of this model with state delays is less advanced. The necessary conditions for asymptotic stability have been given for scalar case in [2, 17–19] for the case with pure delay, one delay, two delays and multiple delays, respectively. The practical stability has been also considered in [2, 17–19]. The practical stability problem for the discrete-time fractional model of one dimensional heat transfer process has been presented in [13].

The discrete-time fractional order state-space model without a time shift in the difference has been introduced in [9]. The stability analysis of this model with delays and Grünwald-Letnikov-type fractional-order difference has been considered in [12], where the sufficient condition for asymptotic stability of this system with delays has been established. The problem of the practical stability has not been studied in [12].

In this paper the stability of fractional-order discrete-time system with delays (model with a time shift in the difference) will be investigated. New necessary and sufficient conditions for the asymptotic stability and the practical stability will be proposed.

2. Problem formulation

There are several definitions of fractional-order differential operator. The most popular are given by Riemann and Liouville, Caputo, and Grünwald and Letnikov. In this paper the Grünwald-Letnikov fractional order backward difference is used.

Definition 1 *[8, 15] The Grünwald-Letnikov fractional-order backward difference with fractional order $\alpha \in \Re$ and step $h > 0$ is given as*

$$\Delta^{\alpha} x(kh) = h^{-\alpha} \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} x((k-i)h), \qquad k = 0, 1, \ldots, \tag{1}$$

where

$$\binom{\alpha}{i} = \begin{cases} 1 & \text{for } i = 0, \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for } i = 1, 2, \ldots \end{cases} \tag{2}$$

Dynamical systems can be modeled by the state-space representation. There are state-space models with and without a time shift in the difference for noninteger order. In this paper the model with a time shift is used. This model is more popular.

Let us consider the fractional discrete-time linear system with multiple delays described by the homogeneous state equation

$$\Delta^{\alpha} x((k+1)h) = A_0 x(kh) + \sum_{r=1}^{q} A_r x((k-r)h), \qquad k = 0, 1, \ldots, \tag{3}$$

with the initial condition $x(-r) \in \Re^n$ $(r = 0, 1, \ldots, q)$, where q is positive number, $x(k) \in \Re^n$ is the state vector, α is the fractional order $\alpha \in (0, 1)$, $A_r \in \Re^{n\times n}$ $(r = 0, 1, \ldots, q)$ are matrices.

If $\Delta^{\alpha} x(kh)$ appears instead of $\Delta^{\alpha} x((k+1)h)$ on the left side of equation (3), then we get the state-space model without a time shift in the difference considered in [12].

Using fractional difference (1), equation (3) can be written in the following form

$$\begin{aligned} x((k+1)h) = {} & \left(h^{\alpha} A_0 + I\alpha\right) x(kh) + h^{\alpha} \sum_{r=1}^{q} A_r x((k-r)h) \\ & + \sum_{i=1}^{k} c_i(\alpha) x((k-i)h), \end{aligned} \tag{4}$$

where

$$c_i(\alpha) = (-1)^i \binom{\alpha}{i+1} \tag{5}$$

and $I \in \Re^{n\times n}$ is the identity matrix.

The sequence of coefficients (5) can be calculated by the following recursive formula [3]

$$c_{i+1}(\alpha) = c_i(\alpha) \frac{i+1-\alpha}{i+2}, \qquad i = 1, 2, \ldots, \tag{6}$$

where $c_1(\alpha) = 0.5\alpha(1-\alpha)$.

Note that coefficients (5) are positive for $\alpha \in (0, 1)$ and quickly decrease for increasing i. Therefore, in equation (4) we can limit upper bound of summation by the natural number L, which is called the length of the practical implementation [7]. Thus, equation (4) can be written in the form

$$\begin{aligned} x((k+1)h) = {} & \left(h^{\alpha} A_0 + I\alpha\right) x(kh) + h^{\alpha} \sum_{r=1}^{q} A_r x((k-r)h) \\ & + \sum_{i=1}^{L} c_i(\alpha) x((k-i)h). \end{aligned} \tag{7}$$

Equation (7) is called the practical realization of fractional system (3).

The definition of practical stability for fractional discrete-time systems has been introduced in [7]. With respect to equations (3) and (7) we have the following definition.

Definition 2 *The fractional system (3) is called practically stable for given length* L *of practical implementation if system (7) is asymptotically stable.*

If system (7) is asymptotically stable for $L \to \infty$ *then fractional system (3) is called asymptotically stable.*

Theorem 1 *The fractional system (3) with given length* L *of practical implementation is practically stable if and only if all roots of characteristic equation*

$$\det\left\{Iz - (h^{\alpha}A_0 + I\alpha) - h^{\alpha}\sum_{r=1}^{q} A_r z^{-r} - \sum_{i=1}^{L} Ic_i(\alpha)z^{-i}\right\} = 0 \tag{8}$$

are strictly inside the unit circle.

Proof. Taking the Z-transform to both side of (4) for initial conditions $x(0) \neq 0$ and $x(-r) = 0$ for $r = 1, \ldots, q$, yields

$$zX(z) - zx(0) = (h^{\alpha}A_0 + I\alpha)\,X(z) + h^{\alpha}\sum_{r=1}^{q} A_r z^{-r}X(z) + \sum_{i=1}^{L} c_i(\alpha)z^{-i}X(z), \tag{9}$$

where $X(z) = Z\{x(kh)\}$. Solving equation (9) for $X(z)$ we obtain

$$X(z) = \left(Iz - (h^{\alpha}A_0 + I\alpha) - h^{\alpha}\sum_{r=1}^{q} A_r z^{-r} - \sum_{i=1}^{L} Ic_i(\alpha)z^{-i}\right)^{-1} zx(0), \tag{10}$$

which leads to the characteristic equation (8). □

The main aim of this paper is to give new necessary and sufficient conditions for practical stability and for asymptotic stability of fractional system (3) with Grünwald-Letnikov fractional difference (1).

3. Solution of the problem

3.1. Asymptotic stability

Firstly, we will analyse system (7) with $L \to \infty$ to formulate asymptotic stability conditions of system (3).

Theorem 2 *The fractional system (3) is asymptotically stable if and only if all roots of characteristic equation*

$$\det\left\{z(I-z^{-1})^{\alpha}-h^{\alpha}\sum_{r=0}^{q}A_r z^{-r}\right\}=0 \tag{11}$$

are strictly inside the unit circle.

Proof. Taking into account the following formula [4]

$$\sum_{i=1}^{\infty}c_i(\alpha)z^{-i}=z-\alpha-(z-1)^{\alpha}z^{1-\alpha} \tag{12}$$

in (8) for $L\to\infty$ we obtain (11). □

The asymptotic stability of system (3) in the scalar case has been considered in the papers [2, 17–19] for the case with pure delay, one delay, two delays and multiple delays, respectively. The necessary conditions for asymptotic stability have been given. The condition in Theorem 2 is the general condition (necessary and sufficient). However, the stability analysis using Theorem 2 will be presented on the example of a scalar system with two delays.

Example 1 Let us consider system (3) in the scalar case for $\alpha=0.5$, $q=2$, $h=1$ and $A_0=a_0$, $A_1=a_1$, $A_2=a_2$ described by the equation

$$\Delta^{0.5}x(k+1)=a_0x(k)+a_1x(k-1)+a_2x(k-2), \tag{13}$$

with initial conditions $x(0)=1$ and $x(-2)=x(-1)=0$.

In the example, the influence of the parameters values on the stability will be presented. Example values for three cases of system: asymptotically stable, limit of stability and unstable, are shown in Table 1. The roots of equation (11) are obtained by using Mathcad software. The solution of system (4) for values in Table 1 are also calculated. Plots with example values are shown in Figs. 1–3. This confirms that system (3) is asymptotically stable when all roots of characteristic equation (11) are strictly inside the unit circle, i.e. the condition $|z|<1$ holds.

Next, we will consider system (3) involving only one matrix A_q occurring in the state equation at a delayed moment, i.e. $A_r=0$ for $r\neq q$ and $A_r=A_q\neq 0$ for $r=q$. Then, equation (3) has the form

$$\Delta^{\alpha}((k+1)h)=A_q((k-q)h),\qquad k=0,1,\ldots,\qquad q=1,2,\ldots. \tag{14}$$

Note that if we put $q=0$ in (14) we obtain the system without delay.

Table 1: Stability analysis for system (13).

System	Coefficient values			Roots	Modulus of roots
	$A_0 = a_0$	$A_1 = a_1$	$A_2 = a_2$		
asymptotically stable	−0.5	−0.2	−0.4	$z = -0.6807$ $z = 0.3154 \pm j0.6625$	$\|z\| = 0.6807$ $\|z\| = 0.7337$
	−0.5	−0.3	−0.4	$z = -0.6352$ $z = 0.2945 \pm j0.7027$	$\|z\| = 0.6352$ $\|z\| = 0.7619$
	−0.5	−0.2	−0.8	$z = -0.8862$ $z = 0.4267 \pm j0.8197$	$\|z\| = 0.8862$ $\|z\| = 0.9241$
limit of stability	−1.21425	−0.2	−0.4	$z = -1$ $z = 0.1248 \pm j0.5988$	$\|z\| = 1$ $\|z\| = 0.6117$
	−0.5	−0.97305	−0.4	$z = -0.3784$ $z = 0.1738 \pm j0.9848$	$\|z\| = 0.3784$ $\|z\| = 1$
	−0.5	−0.2	−1.0118	$z = -0.9657$ $z = 0.469 \pm j0.8832$	$\|z\| = 0.9657$ $\|z\| = 1$
unstable	−1.5	−0.2	−0.4	$z = -1.19$ $z = 0.07863 \pm j0.5567$	$\|z\| = 1.19$ $\|z\| = 0.5622$
	−0.5	−1.5	−0.4	$z = -0.2631$ $z = 0.1194 \pm j1.2$	$\|z\| = 0.2631$ $\|z\| = 1.206$
	−0.5	−0.2	−1.1	$z = -0.9954$ $z = 0.4846 \pm j0.9072$	$\|z\| = 0.9954$ $\|z\| = 1.029$

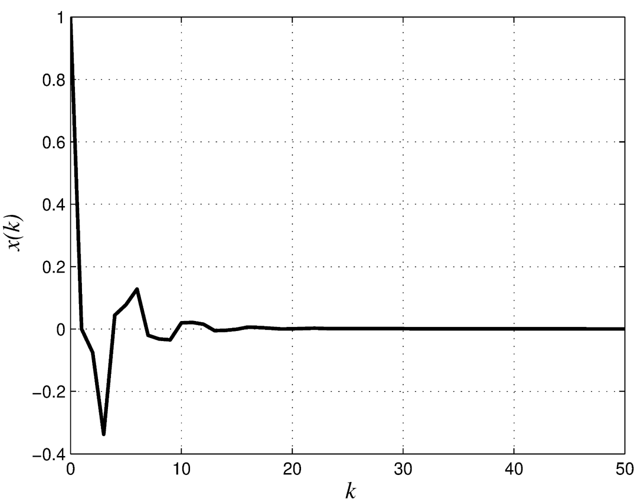

Figure 1: Solution of system (4) for $\alpha = 0.5$, $q = 2$, $h = 1$, $A_0 = a_0 = -0.5$, $A_1 = a_1 = -0.2$, $A_2 = a_2 = -0.4$ (asymptotically stable case).

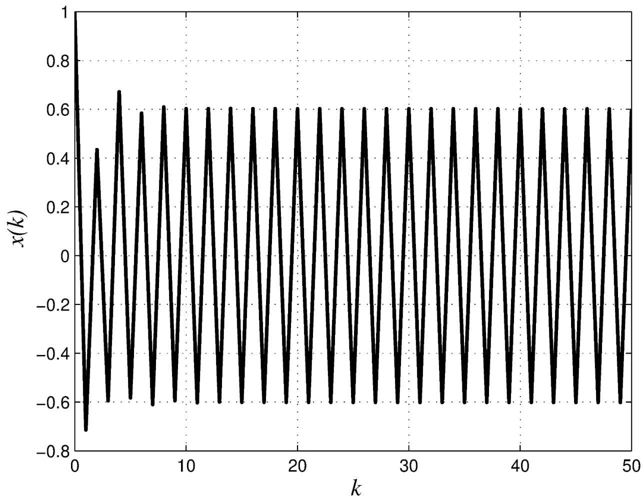

Figure 2: Solution of system (4) for $\alpha = 0.5$, $q = 2$, $h = 1$, $A_0 = a_0 = -1.21425$, $A_1 = a_1 = -0.2$, $A_2 = a_2 = -0.4$ (limit of stability case).

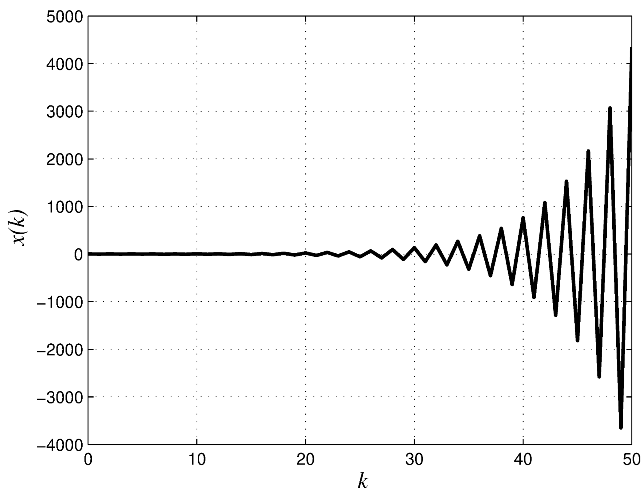

Figure 3: Solution of system (4) for $\alpha = 0.5$, $q = 2$, $h = 1$, $A_0 = a_0 = -1.5$, $A_1 = a_1 = -0.2$, $A_2 = a_2 = -0.4$ (unstable case).

Theorem 3 *The fractional system (14) is asymptotically stable if and only if all roots of characteristic equation*

$$\det\left\{\frac{z}{h^{\alpha}}\left(I - z^{-1}\right)^{\alpha} z^{q} - A_q\right\} = 0 \tag{15}$$

are strictly inside the unit circle.

Proof. The proof directly follows from Theorem 2 for the assumption $A_r = 0$ for $r \neq q$ and $A_r = A_q \neq 0$ for $r = q$. □

Theorem 4 *The fractional system (14) is asymptotically stable if and only if*

$$\arg \lambda_i \in \left[\alpha\frac{\pi}{2},\ 2\pi - \alpha\frac{\pi}{2}\right] \ \wedge\ |\lambda_i| < |w_i|, \quad i = 1, 2, \ldots, n, \tag{16}$$

where $\arg \lambda_i$ *and* $|\lambda_i|$ *are the main argument and modulus, respectively, of the i-th eigenvalue* λ_i *of the matrix* A_q *and*

$$|w_i| = \begin{cases} \left(\dfrac{2}{h}\left|\sin\dfrac{2\arg\lambda_i - \alpha\pi}{2(2q+2-\alpha)}\right|\right)^{\alpha} & \text{for } \arg\lambda_i \in [0, \pi], \\ \left(\dfrac{2}{h}\left|\sin\dfrac{2\arg\lambda_i - \alpha\pi + 4q\pi}{2(2q+2-\alpha)}\right|\right)^{\alpha} & \text{for } \arg\lambda_i \in (\pi, 2\pi). \end{cases} \tag{17}$$

Proof. We will present that if conditions (16) and (17) hold then all roots of the equation (15) are inside the unit circle. If $z \in C$ and $\alpha \in (0,\ 1)$ than we have

$$z^{\alpha} = |z|^{\alpha} e^{j\alpha\varphi + j2l\pi}, \tag{18}$$

where $|z|$ is modulus, $\varphi = \arg z \in [0,\ 2\pi]$ is the principal argument and $l \in Z$.

Let

$$w = \frac{z}{h^{\alpha}}\left(1 - z^{-1}\right)^{\alpha} z^{q}. \tag{19}$$

In the polar form for unit circle we have $|z| = 1$ and $z = e^{j\varphi}$. Then, for $(1 - z^{-1})^{\alpha}$ we obtain

$$\arg\left(1 - z^{-1}\right)^{\alpha} = \alpha \arctan\frac{\sin\varphi}{1 - \cos\varphi} = \alpha\arctan\left(\tan\left(\frac{\pi}{2} - \frac{\varphi}{2}\right)\right) = \alpha\frac{\pi - \varphi}{2}$$

and

$$\left|1 - z^{-1}\right|^{\alpha} = \left(\sqrt{(1 - \cos\varphi)^2 + \sin^2\varphi}\right)^{\alpha} = \left(\sqrt{2(1 - \cos\varphi)}\right)^{\alpha} = \left(2\left|\sin\frac{\varphi}{2}\right|\right)^{\alpha}.$$

Hence

$$w = |w|\, e^{j\arg w}, \tag{20}$$

where $|w| = \left(\dfrac{2}{h}\left|\sin\dfrac{\varphi}{2}\right|\right)^{\alpha}$ and $\arg w = \varphi + \alpha\dfrac{\pi - \varphi}{2} + q\varphi + 2l\pi$.

Note that since $0 \leqslant \varphi \leqslant 2\pi$ and

$$\arg w = \left(q + 1 - \frac{\alpha}{2}\right)\varphi + \alpha\frac{\pi}{2} + 2l\pi \tag{21}$$

we have

$$\alpha\frac{\pi}{2} \leqslant \left(q + 1 - \frac{\alpha}{2}\right)\varphi + \alpha\frac{\pi}{2} \leqslant 2\pi(q + 1) - \alpha\frac{\pi}{2}\,. \tag{22}$$

Then, for the principal argument of w we have

$$\alpha\frac{\pi}{2} \leqslant \arg w \leqslant 2\pi - \alpha\frac{\pi}{2}\,. \tag{23}$$

Since for all eigenvalues λ_i, $i = 1, \ldots, n$, the principal argument of λ_i should be equal to the principal argument of w_i then we obtain the first condition of (16).

From (21) we have

$$\frac{\varphi}{2} = \frac{2\arg w - \alpha\pi - 4l\pi}{2(2q + 2 - \alpha)}\,. \tag{24}$$

Since $\varphi \leqslant 2\pi$, we obtain $\arg w \leqslant 2(q + l)\pi + (1 - \alpha)\frac{\pi}{2}$. Then, for $\arg w < 2\pi$ it should be $l = -q$. Therefore, from (24) if $\arg w \in [0, \pi]$ we obtain

$$\frac{\varphi}{2} = \frac{2\arg w - \alpha\pi}{2(2q + 2 - \alpha)} \tag{25}$$

and if $\arg w \in (\pi, 2\pi)$

$$\frac{\varphi}{2} = \frac{2\arg w - \alpha\pi + 4q\pi}{2(2q + 2 - \alpha)}\,. \tag{26}$$

Taking (25) and (26) into formula $|w| = \left(\frac{2}{h}\left|\sin\frac{\varphi}{2}\right|\right)^{\alpha}$ for $\arg w_i = \arg\lambda_i$ we obtain (17). Note that all roots of the equation (15) are inside the unit circle if $|\lambda_i| < |w_i|$ and conditions of (16) hold. Accordingly Theorem 3 we have that system (14) is asymptotically stable and the proof is completed. □

Remark 1 *Note that if we put $q = 0$ in (17) we obtain the asymptotic stability condition for system (3) without delays similarly as presented in [22].*

Remark 2 *Theorem 4 for model (14) has a similar form as Proposition 8 in [12] for the model without a time shift in the difference. It differs only in the form of the formula of $|w_i|$. In addition, the method of determining the characteristic equation for the considered systems is different in this paper and [12].*

Theorem 5 *The fractional system (14) is asymptotically stable if and only if all eigenvalues $\lambda_i(A_q)$ $(i = 1, 2, \ldots, n)$ are strictly inside the stability region*

$$S = \left\{h^{-\alpha}e^{j\omega(1+q)}(1-e^{-j\omega})^{\alpha};\ \omega\in\left[0,\ \frac{\pi(2-\alpha)}{2q+2-\alpha}\right] \wedge \left(\frac{\pi(2-\alpha+4q)}{2q+2-\alpha},\ 2\pi\right]\right\}. \tag{27}$$

Proof. The proof follows immediately from Theorem 4. □

The asymptotic stability regions of system (14) on the plane of eigenvalues of A_q for some values of fractional order $\alpha \in (0, 1)$ and $q = 1$, $q = 10$ are shown in Fig. 4 and Fig. 5, respectively. Notice that for fixed delay q we obtain smaller stability region for bigger values of fractional order α. Fig. 6 shows the stability regions of system (14) for $\alpha = 0.5$ and different values of delay q. We can see that

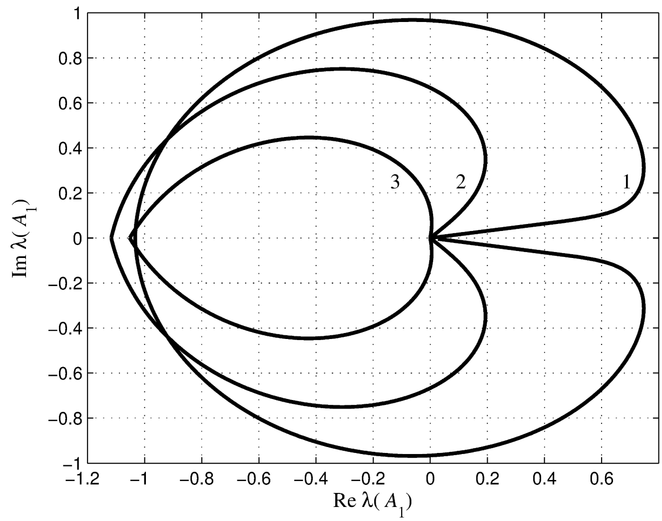

Figure 4: Stability boundaries of system (14) for $h = 1$, $q = 1$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3).

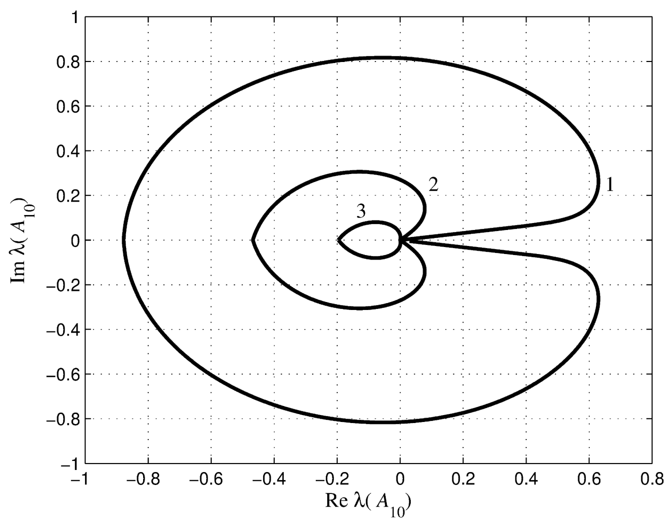

Figure 5: Stability boundaries of system (14) for $h = 1$, $q = 10$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3).

for fixed α, bigger values of q result in smaller stability regions. The influence of step h on stability region is shown in Fig. 7. Smaller values of step h result in bigger stability regions.

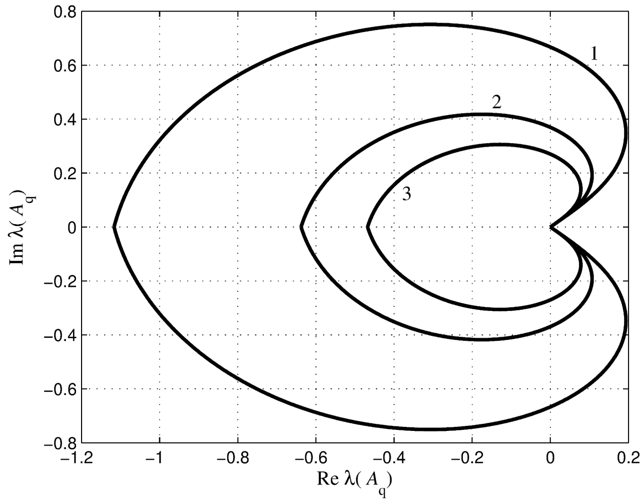

Figure 6: Stability boundaries of system (14) for $h = 1$, $\alpha = 0.5$ and $q = 1$ (boundary 1), $q = 5$ (boundary 2), $q = 10$ (boundary 3).

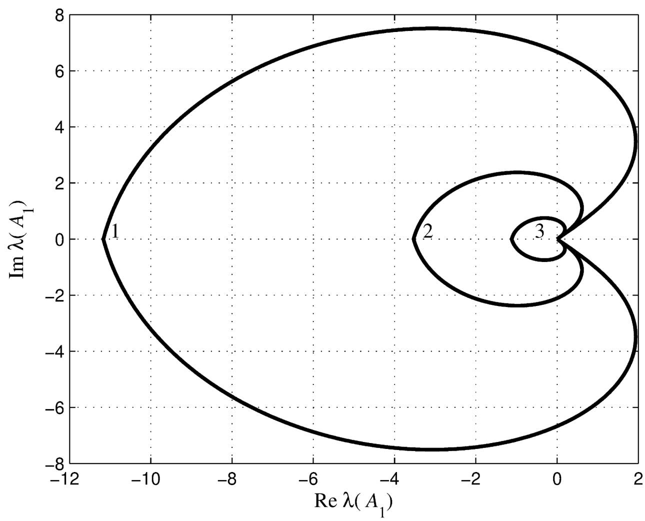

Figure 7: Stability boundaries of system (14) for $\alpha = 0.5$, $q = 1$ and $h = 0.01$ (boundary 1), $h = 0.1$ (boundary 2), $h = 1$ (boundary 3).

For the scalar system (14), i.e. $A_q = \lambda < 0$, we obtain the following asymptotic stability condition.

Lemma 1 *The scalar system (14) with* $\lambda < 0$ *is asymptotically stable if and only if*

$$|\lambda| < \left(\frac{2}{h}\sin\frac{2\pi - \alpha\pi}{2(2q+2-\alpha)}\right)^{\alpha}. \tag{28}$$

Proof. For the scalar system with $\lambda < 0$ we have $\arg\lambda = \pi$. Substituting $\arg\lambda = \pi$ in (17) we obtain (28). □

Lemma 2 *If all eigenvalues* $\lambda_i(A_q)$ *are real and* $\lambda_i(A_q) < 0$*, then the fractional system (14) is asymptotically stable if and only if*

$$|\lambda_i| < \left(\frac{2}{h}\sin\frac{2\pi - \alpha\pi}{2(2q+2-\alpha)}\right)^{\alpha}, \quad i = 1, 2, \ldots, n. \tag{29}$$

Proof. The proof directly follows from Lemma 1. □

Example 2 Check asymptotic stability of fractional system (14) with $q = 2$, $h = 1$ and the matrix

$$A_2 = \begin{bmatrix} -1.7 & -0.62 & 1.52 \\ 1.05 & 1.37 & -3.16 \\ -0.08 & 0.58 & -1.26 \end{bmatrix}. \tag{30}$$

The matrix A_2 has the following eigenvalues: $\lambda_1 = -0.907$, $\lambda_2 = -0.5511$ and $\lambda_3 = -0.1319$. Note that the considered system has only real negative eigenvalues. According to Lemma 2 this system is asymptotically stable for any

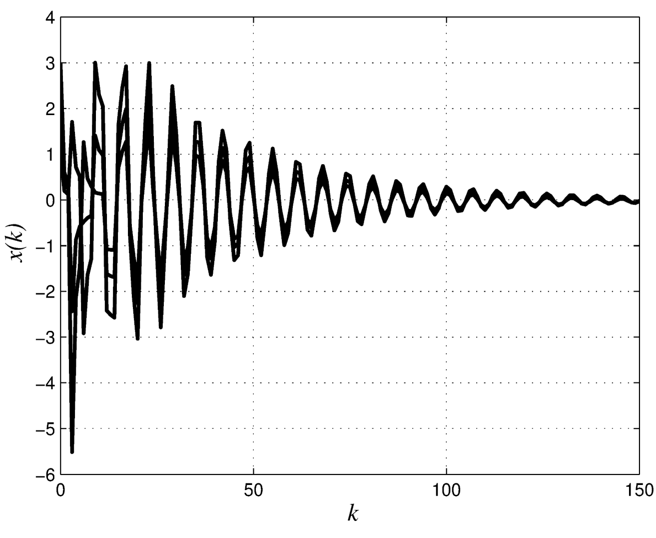

Figure 8: Solution of system (14) with matrix (30) for $h = 1$, $q = 2$, $\alpha = 0.2$ (asymptotically stable case).

$\alpha \in (0,\ 0.5117)$, because from (29) for $\alpha < 0.5117$ we have $|\lambda_i| < 0.907$. The solution of system (14) with matrix (30), $h = 1$, $q = 2$ for $\alpha = 0.2$ and $\alpha = 0.5117$ are shown in Figs. 8 and 9, respectively.

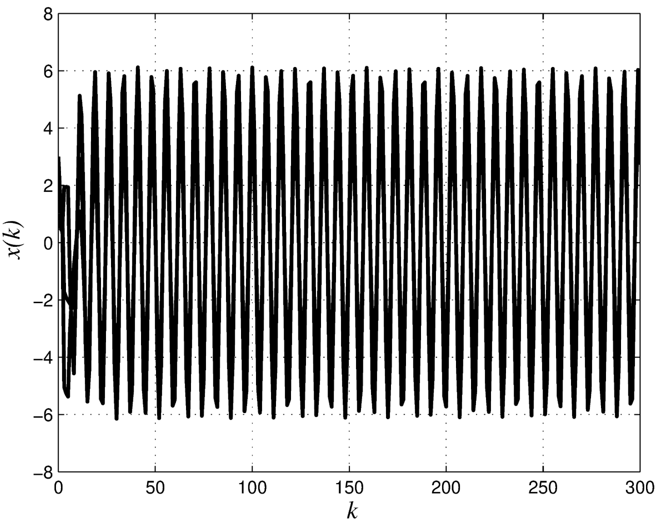

Figure 9: Solution of system (14) with matrix (30) for $h = 1$, $q = 2$, $\alpha = 0.5117$ (limit of stability case).

3.2. Practical stability

The practical stability of fractional system (3) is equivalent to asymptotic stability of discrete-time system (7) with given length L of practical implementation and fractional order α. The fractional system (3) is practically stable if the condition given in Theorem 1 holds.

For system (14) characteristic equation (8) has the form

$$\det\left\{(z-\alpha)I - \sum_{i=1}^{L} Ic_i(\alpha)z^{-i} - h^{\alpha}A_q z^{-q}\right\} = 0. \tag{31}$$

By multiplying both sides of equation (31) by $h^{-\alpha}z^q$ we obtain

$$\det\left\{h^{-\alpha}z^q(z-\alpha)I - h^{-\alpha}\sum_{i=1}^{L} Ic_i(\alpha)z^{q-i} - A_q\right\} = 0. \tag{32}$$

Lemma 3 *The fractional system (14) with given length L of practical implementation is practically stable if and only if all eigenvalues* $\lambda_i(A_q)$ $(i = 1, 2, \ldots, n)$

are strictly inside the stability region

$$S = \left\{ h^{-\alpha} e^{j\omega q} \left(e^{j\omega} - \alpha \right) - h^{-\alpha} \sum_{i=1}^{L} c_i(\alpha) e^{j\omega(q-i)};\ \omega \in [0,\ \omega_1] \wedge (\omega_2,\ 2\pi] \right\}, \quad (33)$$

where ω_1 and ω_2 depend on given values of fractional order α, length L and number of delay q.

Proof. The proof follows from Theorem 5 and the substitution $z = \exp(j\omega)$, $\omega \in [0,\ 2\pi]$, i.e. boundary of the unit circle in the complex z-plane, in $h^{-\alpha} z^q (z-\alpha)$ $-h^{-\alpha} \sum_{i=1}^{L} c_i(\alpha) z^{q-i}$. □

Figs. 10 and 11 show the boundary of practical stability regions of system (14) for $h = 1$, $\alpha = 0.5$, $L = 100$ and $q = 1$, $q = 2$, respectively. The boundaries are plotted for $\omega \in [0,\ 2\pi]$ and stability regions are marked. For plotting the boundaries of stability regions it is enough using $\omega \in [0,\ 1.3464] \wedge (4.9368,\ 2\pi]$ for the plot in Fig. 10 and $\omega \in [0,\ 0.8568] \wedge (5.4264,\ 2\pi]$ for the plot in Fig. 11. These ranges depend on values of fractional order α, length L and delay q.

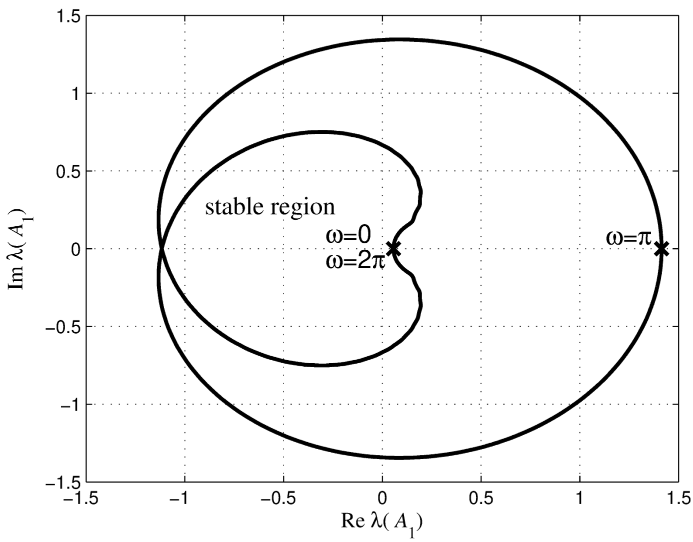

Figure 10: Practical stability boundaries of system (14) for $h = 1$, $L = 100$, $\alpha = 0.5$ and $q = 1$.

Figs. 12–14 show the boundary of practical stability regions of system (14) for different values of α, L and q.

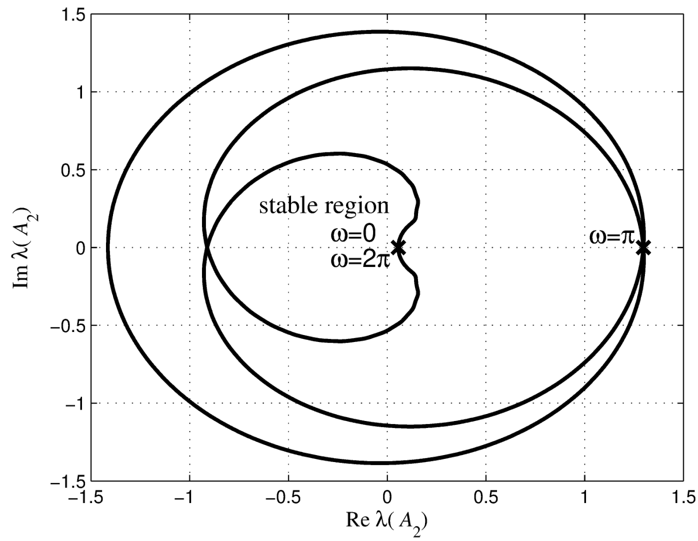

Figure 11: Practical stability boundaries of system (14) for $h = 1$, $L = 100$, $\alpha = 0.5$ and $q = 2$.

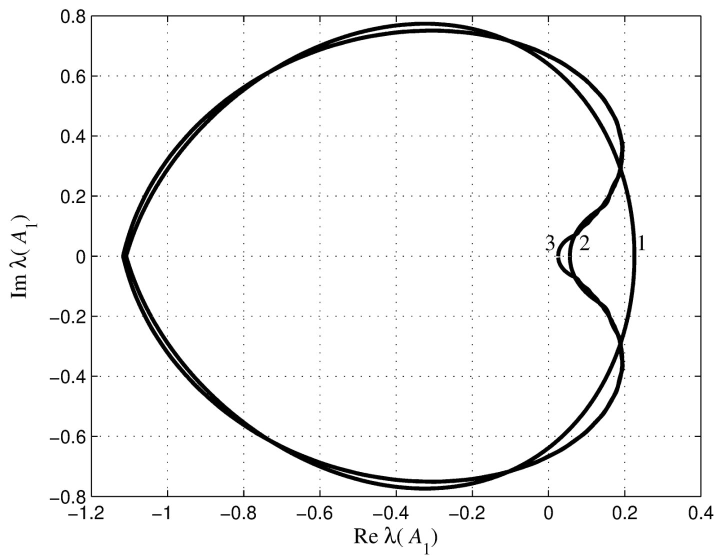

Figure 12: Practical stability boundaries of system (14) for $h = 1$, $\alpha = 0.5$, $q = 1$ and $L = 5$ (boundary 1), $L = 100$ (boundary 2), $L = 500$ (boundary 3).

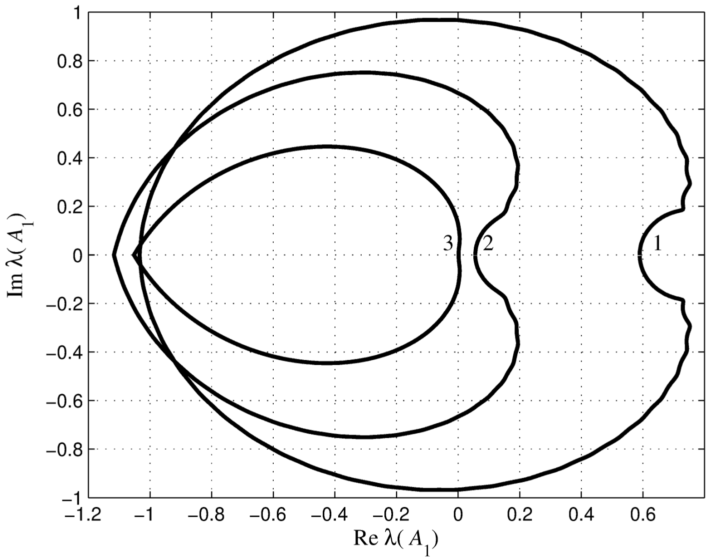

Figure 13: Practical stability boundaries of system (14) for $h = 1$, $L = 100$, $q = 1$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3).

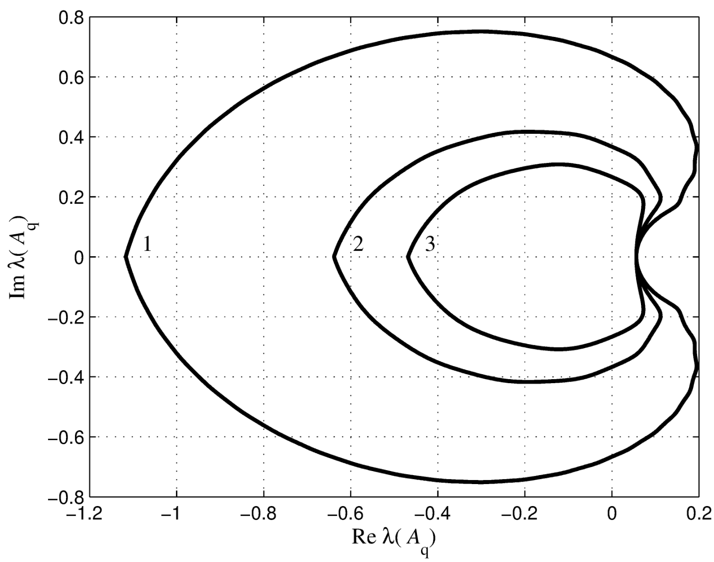

Figure 14: Practical stability boundaries of system (14) for $h = 1$, $L = 100$, $\alpha = 0.5$ and $q = 1$ (boundary 1), $q = 5$ (boundary 2), $q = 10$ (boundary 3).

4. Concluding remarks

The practical and asymptotic stabilities of the fractional order $\alpha \in (0, 1)$ discrete-time linear systems with delays have been analysed. The state model with a time shift in the difference and the Grünwald-Letnikov-type fractional-order difference have been used.

Necessary and sufficient conditions for practical stability and for asymptotic stability have been established (Theorem 1 and Theorem 2). The system with only one matrix occurring in the state equation at a delayed moment has been also considered. In this case the practical stability and asymptotic stability conditions have been given. Moreover analytical conditions for asymptotic stability have been established (Theorem 4). Parametric descriptions of the boundary of practical stability and asymptotic stability regions have been also presented. The considered system is practically (asymptotically) stable if all eigenvalues of the state matrix lie in stability region in the complex plane.

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