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THE LQ/KYP PROBLEM FOR INFINITE-DIMENSIONAL SYSTEMS

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Abstract. Our aim is to present a solution to a general linear-quadratic (LQ) problem as well as to a Kalman-Yacubovich-Popov (KYP) problem for infinite-dimensional systems with bounded operators. The results are then applied, via the reciprocal system approach, to the question of solvability of some Lur'e resolving equations arising in the stability theory of infinite-dimensional systems in factor form with unbounded control and observation operators. To be more precise the Lur'e resolving equations determine a Lyapunov functional candidate for some closed-loop feedback systems on the base of some properties of an uncontrolled (open-loop) system. Our results are illustrated in details by an example of a temperature of a rod stabilization automatic control system.

Keywords: control of infinite-dimensional systems, semigroups, infinite-time LQ-control problem, Lur'e feedback systems.

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1. INTRODUCTION

Consider a linear matrix control system $\dot{x}(t) = Ax(t) + Bu(t)$, $A \in \mathbf{L}(\mathbb{R}^n)$, $B \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^n)$, $x(0) = x_0$ with a nonlinear feedback $u = -F(x)$, where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^r$ is a locally Lipschitz mapping, $F(0) = 0$. The following construction of a Lyapunov functional for the closed–loop Lur'e control system $\dot{x}(t) = Ax(t) - BF[x(t)]$ is known.

Assume that **(i)** there exists $Q \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^n)$ such that QF is a gradient-type (potential) operator, i.e., there exists a functional Φ such that $\nabla\Phi(x) = QF(x)$. Then, under the normalizing condition $\Phi(0) = 0$; Φ can be recovered from QF using

$$\Phi(x) = \int_0^1 x^T QF(sx)ds \underbrace{=}_{z=sx} \int_0^x dz^T QF(z).$$

Let **(ii)** there exist matrices $M = M^T \in \mathbf{L}(\mathbb{R}^n)$, $L \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^n)$ and $N = N^T \in \mathbf{L}(\mathbb{R}^r)$ such that

$$\begin{bmatrix} x \\ F(x) \end{bmatrix}^T \begin{bmatrix} -M & L \\ L^T & -N \end{bmatrix} \begin{bmatrix} x \\ F(x) \end{bmatrix} \geq 0, \quad x \in \mathbb{R}^n.$$

Finally assume that **(iii)** the *Lur'e system of resolving equations*

$$\begin{cases} A^T H + HA - M = -GG^T, \\ -HB + \frac{1}{2} A^T Q + L = -GV, \\ N + \frac{1}{2}(Q^T B + B^T Q) = V^T V \end{cases} \tag{1.1}$$

has a solution (H, G, V), $H = H^T \in \mathbf{L}(\mathbb{R}^n)$, $G \in \mathbf{L}(\mathbb{R}^r, \mathbb{R}^n)$, $V \in \mathbf{L}(\mathbb{R}^r)$. Then

$$x^T H x + \int_0^x dz^T Q F(z)$$

is a Lyapunov functional for the *closed-loop* system.

It was proved in [18] that if the pair (A, B) is *controllable* then a necessary and sufficient condition for solvability of (1.1) is

$$\begin{aligned} \Pi(j\omega) :=& N + L^T (j\omega I - A)^{-1} B + \left[L^T (j\omega I - A)^{-1} B\right]^* \\ &+ \frac{j\omega}{2} Q^T (j\omega I - A)^{-1} B + \left[\frac{j\omega}{2} Q^T (j\omega I - A)^{-1} B\right]^* \\ &+ \left[(j\omega I - A)^{-1} B\right]^* M (j\omega I - A)^{-1} B \geq 0, \quad \omega \in \mathbb{R}, j\omega \notin \sigma(A). \end{aligned}$$

Assume, in addition, that $N + \frac{1}{2}(Q^T B + B^T Q) = \Pi(j\infty) > 0$. Then V is nonsingular and by the second equation of (1.1) we have

$$\Big(HB - \frac{1}{2} A^T Q - L\Big) V^{-1} = G,$$

and from the first equation of (1.1) we conclude that H is a solution to the *matrix Riccati equation*

$$A^T H + HA - M + (HB - \frac{1}{2} A^T Q - L)(V^T V)^{-1} \Big(HB - \frac{1}{2} A^T Q - L\Big)^T = 0. \tag{1.2}$$

On the other side, if for some $\varepsilon > 0$

$$\Pi(j\omega) := R + 2 \operatorname{Re} \tilde{N}^* G(j\omega) + [G(j\omega)]^* \tilde{Q} G(j\omega) \geq \varepsilon I, \quad \omega \in \mathbb{R},$$

where $G(s) := C(sI - A)^{-1} B$, $C \in \mathbf{L}(\mathbb{R}^n, \mathbb{R}^r)$, then the quadratic performance index

$$J(x_0, u) = \int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{N} \\ \tilde{N}^* & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \quad y = Cx$$

achieves its minimum over $u \in \mathrm{L}^2(0,\infty;\mathbb{R}^r)$ (here x_0 is an arbitrary fixed initial condition). The minimal value of J is $x_0^T X x_0$ where X solves the matrix Riccati equation

$$XA + A^T X + C^T \tilde{Q} C - (XB + C^T \tilde{N}) R^{-1} (B^T X + \tilde{N}^T C) = 0$$

and it is given by

$$u = -R^{-1}(B^T X + \tilde{N}^T C)x;$$

the closed – loop system state matrix $A - BR^{-1}(B^T X + \tilde{N}^T C)$ need not be Hurwitz. Choosing $\tilde{Q}$, C and $\tilde{N}$ such that $M = C^T \tilde{Q} C$, $C^T \tilde{N} = \frac{1}{2} A^T Q + L$ and defining $R := N + \frac{1}{2}(Q^T B + B^T Q)$, $H = -X$ we can deduce solvability of (1.1) or (1.2) from the result characterizing optimality of J.

In the present paper, similar or related results will be formulated and solved for Hilbert spaces and operators acting on them in both bounded and unbounded cases. In particular, the last result for the matrix case is proved for bounded operators and completed with a result concerning the spectral factorization (Section 2, especially Theorem 2.4).

In next Section 3 we abbreviated the theory of boundary controlled systems in factor form. All theirs operators: state, observation and control are unbounded.

It is shown in Section 4 that the Hankel transformation jointly with the so-called reciprocality approach enables us to reduce the linear-quadratic (LQ) problem for a system of boundary control in factor form to that of Section 2; the main result of this section Theorem 4.5 is derived from Theorem 2.4. This means that the whole analysis reduces to the case of bounded operators only. Theorem 4.5 is a version of the result obtained in [21], see also [6] and the references therein. The reciprocality approach has been developed in [4, especially Theorem 4.5, p. 1695]. Therein it has been proved that the LQ problem (in the present paper denoted as (3.1)–(4.4)) is equivalent to a control problem for its reciprocal system (further denoted as (4.1)–(4.5)); the results of [21] were applied to get a new simplified Riccati characterization of the optimal control/controller in terms of bounded operators only. In the present paper a more natural and didactically simpler reversed approach is forced: we derive a solution to the LQ problem for infinite-dimensional systems in factor form using the results of Section 2.

In Section 5 we present an application of our results to the question of solvability of some Lur'e resolving systems and to its Riccati counterpart. The results are illustrated by an example of a rod temperature stabilization feedback control system (Section 5.2). Here the basic tool is the spectral analysis of the state operator. It enables us to reduce the solvability of the Lur'e resolving system or/and the Riccati operator equation to effective numerical procedures available in MATLAB/CONTROL TOOLBOX.

The paper ends with a discussion Section 6 of some problems which arose during derivation of our theory.

2. LQ PROBLEM WITH BOUNDED OPERATORS

2.1. FORMULATION OF THE PROBLEM

Let H, U and Y be Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{H}}$, $\langle\cdot,\cdot\rangle_{\mathrm{U}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$, respectively. Consider the system

$$\left\{\begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{array}\right\}, \quad t \geq 0. \tag{2.1}$$

where $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a triple of bounded operators $\mathbf{A} \in \mathbf{L}(\mathrm{H})$, $\mathbf{B} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ and $\mathbf{C} \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$.

Consider the infinite-time horizon LQ problem of finding the optimal control minimizing the quadratic performance index

$$\mathbf{J}(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix}^* \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^* & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{bmatrix} \mathrm{d}t, \tag{2.2}$$

over output trajectories of the system (2.1); here $\mathbf{Q} = \mathbf{Q}^* \in \mathbf{L}(\mathrm{Y})$, $\mathbf{N} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ and $\mathbf{R} = \mathbf{R}^* \in \mathbf{L}(\mathrm{U})$.

2.2. SOME AUXILIARY RESULTS

In what follows, a version of the Paley-Wiener theorem will be needed [2, Theorem 1.8.3, p. 48]; it does not require the *separability* of a Hilbert space.

Theorem 2.1 (Paley-Wiener). *Let* X *be a Hilbert space. Then the map* $f \longmapsto \hat{f}\big|_{\mathbb{C}^+}$ *is isometric isomorphism of* $\mathrm{L}^2(0,\infty;\mathrm{X})$ *onto* $\mathrm{H}^2(\mathbb{C}^+,\mathrm{X})$. *Moreover, for* $f \in \mathrm{L}^2(0,\infty;\mathrm{X})$,

$$\hat{f}(\sigma + j\omega) = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{(\mathcal{F}f)(r)}{\sigma^2 + (\omega - r)^2} \mathrm{d}r, \quad j^2 = -1.$$

As $\sigma \searrow 0$, $\|\hat{f}(\sigma + j\omega) - (\mathcal{F}f)(\omega)\|_{\mathrm{X}} \longrightarrow 0$ (ω)*-almost everywhere, and*

$$\int_{-\infty}^{\infty} \|\hat{f}(\sigma + j\omega) - (\mathcal{F}f)(\omega)\|_{\mathrm{X}}^2 \mathrm{d}\omega \longrightarrow 0.$$

Here $\hat{f}$ *is the Laplace transform of* f *and* $\mathcal{F}f$ *is its Fourier transform, whilst* $\mathrm{H}^2(\mathbb{C}^+,\mathrm{X})$ *stands for the space of holomorphic functions* $g : \mathbb{C}^+ \ni s \longmapsto g(s) \in \mathrm{X}$ *such that*

$$\|g\|^2_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{X})} = \sup_{\alpha>0} \int_{-\infty}^{\infty} \|g(\sigma + j\omega)\|_{\mathrm{X}}^2 \, \mathrm{d}\omega < \infty.$$

Actually the isomorphism established by the Laplace transform is a unitary map and the scalar product in $\mathrm{H}^2(\mathbb{C}^+, \mathrm{X})$ is given by the Plancharel theorem

$$\langle f_1, f_2\rangle_{\mathrm{L}^2(0,\infty;\mathrm{X})} = \int_0^\infty \langle f_1(t), f_2(t)\rangle_{\mathrm{X}} \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \hat{f}_1(j\omega), \hat{f}_2(j\omega)\rangle_{\mathrm{X}} \mathrm{d}\omega := \langle \hat{f}_1, \hat{f}_2\rangle_{\mathrm{H}^2(\mathbb{C}^+;\mathrm{X})}.$$

Later we shall use the semigroups of left-shifts on $\mathrm{L}^2(0,\infty;\mathrm{X})$, X is as in Theorem 2.1, which will be denoted as $\{T_{\mathrm{X}}(t)\}_{t\geq 0}$, $(T_{\mathrm{X}}(t)f)(\tau) := f(t+\tau)$ for almost all $t, \tau \geq 0$. It is generated by

$$\mathcal{L}f = f', \quad D(\mathcal{L}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{X}),$$

$$\mathrm{W}^{1,2}([0,\infty);\mathrm{X}) := \left\{f \in \mathrm{L}^2(0,\infty;\mathrm{X}) : f' \in \mathrm{L}^2(0,\infty;\mathrm{X})\right\} \subset \mathrm{C}([0,\infty);\mathrm{X}).$$

The adjoint of $T_{\mathrm{X}}(t)$,

$$(T^*_{\mathrm{X}}(t)f)(\tau) := \begin{cases} f(\tau - t) & \text{if } \tau \geq t, \\ 0 & \text{if } 0 \leq \tau < t, \end{cases}$$

is the right-shift operator on $\mathrm{L}^2(0,\infty;\mathrm{X})$ and it is clearly generated by $\mathcal{L}^* := \mathcal{R}$,

$$\mathcal{R}f = -f', \quad D(\mathcal{R}) = \mathrm{W}^{1,2}_0([0,\infty);\mathrm{X}),$$

$$\mathrm{W}^{1,2}_0([0,\infty);\mathrm{X}) := \left\{f \in \mathrm{W}^{1,2}([0,\infty);\mathrm{X}) : f(0) = 0\right\}.$$

2.3. SYSTEM-THEORETIC OPERATORS

Clearly, $\mathbf{A}$ generates an analytic group $\{e^{t\mathbf{A}}\}_{t\in\mathbb{R}}$,

$$e^{t\mathbf{A}} := \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k, \quad t \geq 0,$$

where the series uniformly converges on compact sets of $\mathbb{R}$, $\left[e^{t\mathbf{A}}\right]^* = e^{t\mathbf{A}^*}$. The resolvent of $\mathbf{A}$ is the Laplace transform of $e^{t\mathbf{A}}$ and it satisfies the estimate following from the geometric series expansion

$$\|(sI - \mathbf{A})^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq \frac{1}{|s| - \|\mathbf{A}\|_{\mathbf{L}(\mathrm{H})}}, \quad |s| \geq \|\mathbf{A}\|_{\mathbf{L}(\mathrm{H})}. \tag{2.3}$$

For every $\mathbf{x}_0 \in \mathrm{H}$ and $\mathbf{u} \in \mathrm{L}^2(0,\infty;\mathrm{U})$ the variation-of-constants formula

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau, \quad t \geq 0,$$

defines the unique *strong* (absolutely continuous) solution of the state equation (2.1) [17, Theorem 2.9, p. 109], and the output is given by

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau, \quad t \geq 0. \tag{2.4}$$

In order to obtain conditions under which the performance index (2.2) is well-defined we are interested in having: $y \in \mathrm{L}^2(0,\infty;\mathrm{Y})$ for any $\mathbf{x}_0 \in \mathrm{H}$ and any $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$.

Definition 2.2. The output (observation) operator $\mathbf{C}$ is said to be (infinite-time) *admissible* if the function $[0,\infty) \ni t \longmapsto \mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0$ belongs to $\mathrm{L}^2(0,\infty;\mathrm{Y})$ for every $\mathbf{x}_0 \in \mathrm{H}$.

By Theorem 2.1, $\mathbf{C}$ is admissible iff the function $s \longmapsto \mathbf{C}(sI-\mathbf{A})^{-1}\mathbf{x}_0$ is in $\mathrm{H}^2(\mathbb{C}^+;\mathrm{Y})$ for every $\mathbf{x}_0 \in \mathrm{H}$.

If $\mathbf{C}$ is admissible then, the *observability map*

$$\mathrm{H} \ni x_0 \longmapsto \mathbf{\Psi}\mathbf{x}_0 \in \mathrm{L}^2(0,\infty;\mathrm{Y}), \quad (\mathbf{\Psi}\mathbf{x}_0)(t) := \mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0$$

is also closed, whence by the closed graph theorem, $\mathbf{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$. Consequently, $\mathbf{\Psi}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{Y}),\mathrm{H})$,

$$\mathbf{\Psi}^* v = \int_0^\infty e^{t\mathbf{A}^*}\mathbf{C}^* \; v(t)\mathrm{d}t \;\;,$$

and $\mathbf{\Psi}^*\mathbf{\Psi}$ is called the *system observability gramian.* Moreover,

$$\|\mathbf{C}(sI-\mathbf{A})^{-1}\|_{\mathbf{L}(\mathrm{H},\mathrm{Y})} \leq \frac{1}{\sqrt{2\,\mathrm{Re}\,s}}\|\mathbf{\Psi}\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))}, \quad s \in \mathbb{C}^+, \tag{2.5}$$

because for $s \in \mathbb{C}^+$ there holds

$$\|\mathbf{C}(sI-\mathbf{A})^{-1}\mathbf{x}_0\|_{\mathrm{Y}} \leq \int_0^\infty \|e^{-st}\mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0\|_{\mathrm{Y}}\mathrm{d}t \leq \frac{1}{\sqrt{2\,\mathrm{Re}\,s}}\,\|\mathbf{\Psi}\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))}\,\|\mathbf{x}_0\|_{\mathrm{H}}\,.$$

It is also clear that

$$T_{\mathrm{Y}}(t)\mathbf{\Psi}\mathbf{x}_0 = \mathbf{\Psi}e^{t\mathbf{A}}\mathbf{x}_0, \quad \mathbf{x}_0 \in \mathrm{H}, t \geq 0. \tag{2.6}$$

The second component in the right hand side (RHS) of (2.4) naturally defines the *input-output operator* of convolution

$$(\mathbb{F}\mathbf{u})(t) := \int_0^t \mathbf{C}e^{\tau\mathbf{A}}\mathbf{B}\mathbf{u}(t-\tau)\mathrm{d}\tau, \quad u \in \mathrm{L}^2(0,\infty;\mathrm{U}),$$

but if $\mathbf{C}$ is admissible then, it reads as

$$(\mathbb{F}\mathbf{u})(t) = \int_0^t \mathbf{\Psi}\left[\mathbf{B}\mathbf{u}(t-\tau)\right](\tau)\mathrm{d}\tau = \int_0^\infty \mathbf{\Psi}\left[\mathbf{B}\left(R_t\mathbf{u}\right)(\tau)\right]\mathrm{d}\tau,$$

where R_t stands for the *reflection operator* at $t > 0$: $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty);\mathrm{U})$,

$$(R_t f)(\tau) := \left\{ \begin{array}{cc} f(t-\tau), & \tau \in [0,t) \\ 0, & \tau \geq t \end{array} \right\}, \quad R_t = R_t^*, \quad \|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty);\mathrm{U})} \leq 1.$$

The last representation is especially useful to find the Laplace transform of $\mathbb{F}\mathbf{u}$. Indeed, let us notice that the Laplace transform of the function $t \longmapsto (R_t\mathbf{u})(\tau)$ is $e^{-s\tau}\hat{\mathbf{u}}(s)$, whence

$$(\widehat{\mathbb{F}\mathbf{u}})(s) = \int_0^\infty \mathbf{\Psi}\left[\mathbf{B}e^{-s\tau}\hat{\mathbf{u}}(s)\right](\tau)\mathrm{d}\tau = \mathbf{\Psi}\widehat{[\mathbf{B}\hat{\mathbf{u}}(s)]} = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s),$$

which shows that $\mathbb{F}$ is under the Laplace transform similar to the operator of multiplication by an operator-valued *system transfer function* $\widehat{G}$

$$(\widehat{\mathbb{F}\mathbf{u}})(s) = \widehat{G}(s)\hat{\mathbf{u}}(s), \quad \widehat{G}(s) := \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}, \quad s \in \mathbb{C}^+.$$

However the estimate

$$\|\widehat{G}(s)\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \leq \frac{1}{\sqrt{2\,\mathrm{Re}\,s}}\,\|\mathbf{\Psi}\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))}\,\|\mathbf{B}\|_{\mathbf{L}(\mathrm{H},\mathrm{U})}\,, \quad s \in \mathbb{C}^+,$$

following from (2.5) is not enough to ensure the boundedness of the above operator of multiplication. Nevertheless, assuming that

$$\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$$

and making use of Theorem 2.1 we conclude that then $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y}))$.

Recall that $\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathrm{Z})$, $\mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}\,s > 0\}$, for some Banach space Z, if $\widehat{G} : \mathbb{C}^+ \ni s \longmapsto \widehat{G}(s) \in \mathrm{Z}$ is holomorphic and $\|\widehat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Z})} = \sup_{s\in\mathbb{C}^+} \|\widehat{G}(s)\|_{\mathrm{Z}} < \infty$; this definition applies to $\mathrm{Z} = \mathbf{L}(\mathrm{U},\mathrm{Y})$ as it is a Banach space.

If the latter holds then the adjoint operator $\mathbb{F}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{Y}), \mathrm{L}^2(0,\infty;\mathrm{U}))$ takes the form following from Tonelli's theorem:

$$\begin{aligned}(\mathbb{F}^*v)(t) &:= \int_t^\infty \mathbf{B}^* e^{(\tau-t)\mathbf{A}^*}\mathbf{C}^*\; v(\tau)\mathrm{d}\tau = \mathbf{B}^* \int_0^\infty e^{\xi\mathbf{A}^*}\mathbf{C}^* v(t+\xi)\mathrm{d}\xi \\ &= \mathbf{B}^*\mathbf{\Psi}^* T_{\mathrm{Y}}(t)v, \quad v \in \mathrm{L}^2(0,\infty;\mathrm{Y}).\end{aligned}$$

It follows from the last representation and the asymptotic stability of $\{T_{\mathrm{Y}}(t)\}_{t\geq 0}$ in strong topology, i.e., $\|T_{\mathrm{Y}}(t)v\|_{\mathrm{Y}} \longrightarrow 0$ for every $v \in \mathrm{Y}$ as $t \to \infty$, that the function $t \longmapsto (\mathbb{F}^*v)(t) \in \mathrm{U}$ is strongly continuous and strongly tends to 0 as $t \to \infty$.

Lemma 2.3. *The following two identities hold:*

$$T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u} = \mathbb{F}T_{\mathrm{U}}(t)\mathbf{u} + \mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau, \mathbf{u} \in \mathrm{L}^2(0,\infty;\mathrm{U}); \tag{2.7}$$

$$\mathbb{F}T_{\mathrm{U}}^*(t) = T_{\mathrm{Y}}^*(t)\mathbb{F} \quad \left(\iff T_{\mathrm{U}}(t)\mathbb{F}^* = \mathbb{F}^*T_{\mathrm{Y}}(t)\right). \tag{2.8}$$

(2.8) *means that* $\mathbb{F}$ *is shift-invariant.*

Proof. Indeed, with $\xi \geq 0$ and $\mathbf{u} \in \mathrm{L}^2(0,\infty;\mathrm{U})$,

$$\begin{aligned}
&\left(\mathbb{F}T_{\mathrm{U}}(t)\mathbf{u}\right)(\xi) + \left(\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau\right)(\xi)\\
&= \int_0^\xi \mathbf{C}e^{(\xi-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(t+\tau)\mathrm{d}\tau + \mathbf{C}e^{\xi\mathbf{A}}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau\\
&= \int_t^{t+\xi} \mathbf{C}e^{(\xi-r+t)\mathbf{A}}\mathbf{B}\mathbf{u}(r)\mathrm{d}r + \int_0^t \mathbf{C}e^{(\xi+t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau\\
&= \int_0^{t+\xi} \mathbf{C}e^{(\xi+t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau = \left(T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u}\right)(\xi),
\end{aligned}$$

giving (2.7), whilst (2.8) holds because

$$\begin{aligned}
\left(\mathbb{F}T_{\mathrm{U}}^*(t)\mathbf{u}\right)(\xi) &= \int_0^\xi \mathbf{C}e^{(\xi-\tau)\mathbf{A}}\mathbf{B}\left\{\begin{array}{cc} \mathbf{u}(\tau-t), & \tau \geq t\\ 0, & 0 \leq \tau < t\end{array}\right\}\mathrm{d}\tau\\
&= \int_{-t}^{\xi-t} \mathbf{C}e^{(\xi-r-t)\mathbf{A}}\mathbf{B}\left\{\begin{array}{cc} \mathbf{u}(r), & r \geq 0\\ 0, & -t \leq r < 0\end{array}\right\}\mathrm{d}r\\
&= \left\{\begin{array}{cc} \displaystyle\int_0^{\xi-t} \mathbf{C}e^{(\xi-r-t)\mathbf{A}}\mathbf{B}\mathbf{u}(r)\mathrm{d}r & \text{if} \quad \xi - t \geq 0\\ 0 & \text{if} \quad \xi - t < 0\end{array}\right\} = \left(T_{\mathrm{Y}}^*(t)\mathbb{F}\mathbf{u}\right)(\xi).
\end{aligned}$$

□

2.4. MAIN RESULTS

For the sake of simplicity the operators **Q**, **N** and **R** will be identified, if necessary, with the operator of multiplication induced by these operators – both in time and frequency domains. In particular, this enables us to notice that the following elementary commutative relationships hold:

$$\begin{aligned} T_{\mathrm{Y}}(t)\mathbf{Q} &= \mathbf{Q}T_{\mathrm{Y}}(t), \quad & T_{\mathrm{U}}(t)\mathbf{R} &= \mathbf{R}T_{\mathrm{U}}(t), \\ T_{\mathrm{U}}(t)\mathbf{N}^* &= \mathbf{N}^*T_{\mathrm{Y}}(t), \quad & \mathbf{N}T_{\mathrm{U}}(t) &= T_{\mathrm{Y}}(t)\mathbf{N}. \end{aligned} \tag{2.9}$$

Now, we are in position to proof our main result.

Theorem 2.4. *If* $\mathbf{C}$ *is admissible,* $\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+; \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ *and the Popov spectral function*

$$\Pi(j\omega) := \mathbf{R} + 2\,\mathrm{Re}\left[\mathbf{N}^*\widehat{G}(j\omega)\right] + \left[\widehat{G}(j\omega)\right]^* \mathbf{Q}\widehat{G}(j\omega) \tag{2.10}$$

is coercive, i.e., there exists $\varepsilon > 0$ *such that* $\Pi(j\omega) \geq \varepsilon I$ *for almost all* $\omega \in \mathbb{R}$*, then:*

(i) *the LQ problem has a unique solution,* $\mathbf{R}$ *is boundedly invertible and the optimal control* $\mathbf{u}^c \in \mathrm{L}^2(0, \infty; \mathrm{U})$ *can be realized in the* linear feedback form

$$\mathbf{u}^c(t) = -\mathbf{R}^{-1}\left[\mathbf{B}^*\mathcal{H}^c + \mathbf{N}^*\mathbf{C}\right]\mathbf{x}^c(t), \tag{2.11}$$

where $\mathcal{H}^c \in \mathbf{L}(\mathrm{H})$*,* $\mathcal{H}^c = (\mathcal{H}^c)^*$ *stands for the minimal cost operator given by* $\mathbf{J}(\mathbf{x}_0, \mathbf{u}^c) = \langle \mathbf{x}_0, \mathcal{H}^c\mathbf{x}_0\rangle_{\mathrm{H}}$*, and* $\mathcal{H}^c$ *satisfies the operator Riccati equation*

$$\mathbf{A}^*\mathcal{H}^c + \mathcal{H}^c\mathbf{A} + \mathbf{C}^*\mathbf{Q}\mathbf{C} = (\mathcal{H}^c\mathbf{B} + \mathbf{C}^*\mathbf{N})\mathbf{R}^{-1}(\mathbf{B}^*\mathcal{H}^c + \mathbf{N}^*\mathbf{C}). \tag{2.12}$$

Moreover, the pair $(\mathbf{H}, \mathbf{G})$

$$\mathbf{H} := -\mathcal{H}^c, \quad \mathbf{G} := -(\mathcal{H}^c\mathbf{B} + \mathbf{C}^*\mathbf{N})\mathbf{R}^{-1/2} \tag{2.13}$$

is a solution to the Lur'e system of resolving equations

$$\left\{ \begin{array}{rl} \mathbf{A}^*\mathbf{H} + \mathbf{H}\mathbf{A} - \mathbf{C}^*\mathbf{Q}\mathbf{C} &= -\mathbf{G}\mathbf{G}^* \\ -\mathbf{H}\mathbf{B} + \mathbf{C}^*\mathbf{N} &= -\mathbf{G}\mathbf{R}^{1/2} \end{array} \right\}, \tag{2.14}$$

satisfying $\mathbf{H} \in \mathbf{L}(\mathrm{H})$*,* $\mathbf{H} = \mathbf{H}^*$ *and* $\mathbf{G} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$*.*

(ii) *If, in addition:*

$$\{e^{t\mathbf{A}}\}_{t\geq 0} \text{ is uniformly bounded} \iff \exists M \geq 1 \forall t \geq 0 : \|e^{t\mathbf{A}}\|_{\mathbf{L}(\mathrm{H})} \leq M \tag{2.15}$$

and

$$\sigma(\mathbf{A}) \cap j\mathbb{R} \subset \{0\}, \tag{2.16}$$

then $\mathbf{G}^*$ *is an admissible observation operator and* $\Theta \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$*, where*

$$\Theta(s) := \mathbf{R}^{1/2} - \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}; \tag{2.17}$$

Θ *is a spectral factor of* Π*, i.e.,* $[\Theta(j\omega)]^*\,\Theta(j\omega) = \Pi(j\omega)$ *for almost all* $\omega \in \mathbb{R}$ *and*

$$[\Theta(s)]^{-1} = \mathbf{R}^{-1/2} + \mathbf{R}^{-1/2}\mathbf{G}^*(sI - \mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}, \tag{2.18}$$

$\Theta^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$*, where* $\mathbf{A}^c := \mathbf{A} + \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*$ *is the closed-loop state operator.*

Proof. Part (i). *Step 1.* Since $\mathbf{C}$ be admissible and $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y}))$ the output equation can be written as

$$\mathbf{y} = \mathbf{\Psi}\mathbf{x}_0 + \mathbb{F}\mathbf{u}, \quad \mathbf{x}_0 \in \mathrm{H}, \mathbf{u} \in \mathrm{L}^2(0,\infty;\mathrm{U}),$$

which enables us to eliminate $\mathbf{y}$ from $\mathbf{J}$:

$$\begin{aligned} \mathbf{J}(\mathbf{x}_0, \mathbf{u}) &= \langle \mathbf{Q}\mathbf{y} + \mathbf{N}\mathbf{u}, \mathbf{y}\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \langle \mathbf{N}^*\mathbf{y} + \mathbf{R}\mathbf{u}, \mathbf{u}\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} \\ &= \mathbf{x}_0^*\mathbf{\Psi}^*\mathbf{Q}\mathbf{\Psi}\mathbf{x}_0 + \mathbf{x}_0^*\mathbf{\Psi}^*(\mathbf{Q}\mathbb{F} + \mathbf{N})\mathbf{u} + \mathbf{u}^*(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0 + \mathbf{u}^*\mathcal{R}\mathbf{u}, \end{aligned} \tag{2.19}$$

where

$$\mathcal{R} := \mathbf{R} + \mathbb{F}^*\mathbf{N} + \mathbf{N}^*\mathbb{F} + \mathbb{F}^*\mathbf{Q}\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U})).$$

Assume that $\mathcal{R}$ is *coercive*, i.e., there exists $\varepsilon > 0$ such that $\mathcal{R} \geq \varepsilon I$. This assumption may turn to be difficult to check directly. In those instances one may try to verify its frequency-domain counterpart we can get using Theorem 2.1,

$$\langle \mathbf{u}, \mathcal{R}\mathbf{u}\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \langle \hat{\mathbf{u}}, \widehat{\mathcal{R}\mathbf{u}}\rangle_{\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\hat{\mathbf{u}}(j\omega)]^*\Pi(j\omega)\hat{\mathbf{u}}(j\omega)\mathrm{d}\omega,$$

and therefore $\mathcal{R}$ is coercive because $\Pi(j\omega)$ is coercive.

Since $\mathcal{R}$ is coercive the performance index $\mathbf{J}$ has a unique minimum over $\mathrm{L}^2(0,\infty;\mathrm{U})$ which satisfies $J'(u) = 0$, where $J'(u)$ denotes the Fréchet derivative of $\mathbf{J}$ with respect to $\mathbf{u}$. This necessary and sufficient condition of optimality yields the control

$$\mathbf{u}^c := -\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0, \tag{2.20}$$

which minimizes the functional (2.19) with respect to $\mathbf{u}$ and with $\mathbf{x}_0$ fixed arbitrarily.

Inserting this control back into (2.19) one obtains the minimal cost $\mathbf{J}(\mathbf{x}_0, \mathbf{u}^c) = \mathbf{x}_0^*\mathcal{H}^c\mathbf{x}_0$, where

$$\mathcal{H}^c := \mathbf{\Psi}^*\left[\mathbf{Q} - (\mathbf{Q}\mathbb{F} + \mathbf{N})\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\right]\mathbf{\Psi} \tag{2.21}$$

defines a bounded self-adjoint operator $\mathcal{H}^c \in \mathbf{L}(\mathrm{H})$.

Step 2. Next objective is to show that the optimal control (2.20) can be expressed as a linear feedback control law. For that we firstly show that for each $\mathbf{u} \in \mathrm{L}^2(0,\infty;\mathrm{U})$:

$$T_{\mathrm{U}}(t)\mathcal{R}\mathbf{u} = \mathcal{R}T_{\mathrm{U}}(t)\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau. \tag{2.22}$$

Indeed, by (2.9) and (2.8), we have

$$\begin{aligned} T_{\mathrm{U}}(t)\mathcal{R}\mathbf{u} &= T_{\mathrm{U}}(t)\mathbf{R}\mathbf{u} + T_{\mathrm{U}}(t)\mathbb{F}^*\mathbf{N}\mathbf{u} + T_{\mathrm{U}}(t)\mathbf{N}^*\mathbb{F}\mathbf{u} + T_{\mathrm{U}}(t)\mathbb{F}^*\mathbf{Q}\mathbb{F}\mathbf{u} \\ &= \mathbf{R}T_{\mathrm{U}}(t)\mathbf{u} + \mathbb{F}^*T_{\mathrm{Y}}(t)\mathbf{N}\mathbf{u} + \mathbf{N}^*T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u} + \mathbb{F}^*T_{\mathrm{Y}}(t)\mathbf{Q}\mathbb{F}\mathbf{u} \\ &= \mathbf{R}T_{\mathrm{U}}(t)\mathbf{u} + \mathbb{F}^*\mathbf{N}T_{\mathrm{U}}(t)\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u}. \end{aligned}$$

Applying (2.7) we get

$$T_{\mathrm{U}}(t)\mathcal{R}\mathbf{u} = (\mathbf{R} + \mathbb{F}^*\mathbf{N})T_{\mathrm{U}}(t)\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbb{F}T_{\mathrm{U}}(t)\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau = \mathcal{R}T_{\mathrm{U}}(t)\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau,$$

whence (2.22) holds.

Secondly, define the *dual state* variable

$$\begin{aligned}\lambda(t) &:= \int_t^\infty e^{(\tau-t)\mathbf{A}^*}\mathbf{C}^*\left[\mathbf{QCx}(\tau) + \mathbf{Nu}(\tau)\right]\mathrm{d}\tau \\ &= \int_0^\infty e^{\tau\mathbf{A}^*}\mathbf{C}^*\left[\mathbf{Qy}(t+\tau) + \mathbf{Nu}(t+\tau)\right]\mathrm{d}\tau \\ &= \mathbf{\Psi}^*\mathbf{Q}T_{\mathrm{Y}}(t)\mathbf{y} + \mathbf{\Psi}^*\mathbf{N}T_{\mathrm{U}}(t)\mathbf{u} = \mathbf{\Psi}^*\mathbf{Q}T_{\mathrm{Y}}(t)\mathbf{\Psi}\mathbf{x}_0 + \mathbf{\Psi}^*\mathbf{Q}T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u} + \mathbf{\Psi}^*\mathbf{N}T_{\mathrm{U}}(t)\mathbf{u} \\ &\underbrace{=}_{(2.6)\text{ and }(2.7)} \mathbf{\Psi}^*\mathbf{Q}\mathbf{\Psi}\mathbf{x}(t) + \mathbf{\Psi}^*(\mathbf{Q}\mathbb{F} + \mathbf{N})T_{\mathrm{U}}(t)\mathbf{u}.\end{aligned}$$

Now we claim that

$$\mathbf{Ru} + \mathbf{N}^*\mathbf{Cx} + \mathbf{B}^*\lambda = \mathcal{R}\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0. \tag{2.23}$$

Indeed, applying definitions of λ, $\mathbb{F}^*$, $\mathcal{R}$ and (2.9), we get

$$\begin{aligned}&\mathbf{Ru} + \mathbf{N}^*\mathbf{Cx} + \mathbf{B}^*\lambda = \mathbf{Ru} + \mathbf{N}^*\mathbf{y} + \mathbf{B}^*\lambda \\ &= \mathbf{Ru} + \mathbf{N}^*\mathbf{\Psi}\mathbf{x}_0 + \mathbf{N}^*\mathbb{F}\mathbf{u} + \mathbf{B}^*\mathbf{\Psi}^*\mathbf{Q}T_{\mathrm{Y}}(t)\mathbf{\Psi}\mathbf{x}_0 + \mathbf{B}^*\mathbf{\Psi}^*\mathbf{Q}T_{\mathrm{Y}}(t)\mathbb{F}\mathbf{u} + \mathbf{B}^*\mathbf{\Psi}^*\mathbf{N}T_{\mathrm{U}}(t)\mathbf{u} \\ &= (\mathbf{R} + \mathbf{N}^*\mathbb{F})\mathbf{u} + \mathbf{N}^*\mathbf{\Psi}\mathbf{x}_0 + \mathbf{B}^*\mathbf{\Psi}^*T_{\mathrm{Y}}(t)\mathbf{Q}\mathbf{\Psi}\mathbf{x}_0 + \mathbf{B}^*\mathbf{\Psi}^*T_{\mathrm{Y}}(t)\mathbf{Q}\mathbb{F}\mathbf{u} + \mathbf{B}^*\mathbf{\Psi}^*T_{\mathrm{Y}}(t)\mathbf{Nu} \\ &= (\mathbf{R} + \mathbf{N}^*\mathbb{F} + \mathbb{F}^*\mathbf{Q}\mathbb{F} + \mathbb{F}^*\mathbf{N})\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0 = \mathcal{R}\mathbf{u} + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0.\end{aligned}$$

In particular, for $\mathbf{u} = \mathbf{u}^c$, one gets applying subsequently (2.22), (2.20), (2.8), (2.9) and (2.6)

$$\begin{aligned}T_{\mathrm{U}}(t)\mathbf{u}^c &= \mathcal{R}^{-1}\left[T_{\mathrm{U}}(t)\mathcal{R}\mathbf{u}^c - (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}^c(\tau)\mathrm{d}\tau\right] \\ &= -\mathcal{R}^{-1}\left[T_{\mathrm{U}}(t)(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\mathbf{x}_0 + (\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*)\mathbf{\Psi}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}^c(\tau)\mathrm{d}\tau\right]\end{aligned}$$

$$= -\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q}+\mathbf{N}^*)\left[T_{\mathrm{Y}}(t)\mathbf{\Psi}\mathbf{x}_0+\mathbf{\Psi}\int\limits_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}^c(\tau)\mathrm{d}\tau\right]$$
$$= -\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q}+\mathbf{N}^*)\mathbf{\Psi}\left[e^{t\mathbf{A}}\mathbf{x}_0+\int\limits_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}^c(\tau)\mathrm{d}\tau\right]$$
$$= -\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q}+\mathbf{N}^*)\mathbf{\Psi}\mathbf{x}^c(t),$$

where $\mathbf{x}^c$ denotes the state generated by the optimal control $\mathbf{u}^c$ (solution of (2.1) corresponding to optimal control $\mathbf{u}^c$). Hence, by the analogous definition of λ^c and (2.21),

$$\lambda^c(t) = \mathbf{\Psi}^*\mathbf{Q}\mathbf{\Psi}\mathbf{x}^c(t)+\mathbf{\Psi}^*(\mathbb{F}\mathbf{Q}+\mathbf{N})T_{\mathrm{U}}(t)\mathbf{u}^c$$
$$= \mathbf{\Psi}^*\mathbf{Q}\mathbf{\Psi}\mathbf{x}^c(t)-\mathbf{\Psi}^*(\mathbb{F}\mathbf{Q}+\mathbf{N})\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q}+\mathbf{N}^*)\mathbf{\Psi}\mathbf{x}^c(t)=\mathcal{H}^c\mathbf{x}^c(t)$$

and consequently (apply directly the definition of λ to calculate $\dot{\lambda}^c$),

$$\begin{aligned} 0 &\equiv \dot{\lambda}^c(t)-\mathcal{H}^c\dot{\mathbf{x}}^c(t)\\ &= -\mathbf{A}^*\lambda^c(t)-\mathbf{C}^*\mathbf{Q}\mathbf{C}\mathbf{x}^c(t)-\mathbf{C}^*\mathbf{N}\mathbf{u}^c(t)-\mathcal{H}^c\left[\mathbf{A}\mathbf{x}^c(t)+\mathbf{B}\mathbf{u}^c(t)\right]\\ &= \left[-\mathbf{A}^*\mathcal{H}^c-\mathcal{H}^c\mathbf{A}-\mathbf{C}^*\mathbf{Q}\mathbf{C}\right]\mathbf{x}^c(t)+\left[-\mathbf{C}^*\mathbf{N}-\mathcal{H}^c\mathbf{B}\right]\mathbf{u}^c(t). \end{aligned} \tag{2.24}$$

For $\mathbf{u}=\mathbf{u}^c$ we get from (2.23)

$$\mathbf{R}\mathbf{u}^c+\mathbf{N}^*\mathbf{C}\mathbf{x}^c+\mathbf{B}^*\lambda^c=0,$$

whence

$$\mathbf{R}\mathbf{u}^c(t)+\mathbf{N}^*\mathbf{C}\mathbf{x}^c(t)+\mathbf{B}^*\mathcal{H}^c\mathbf{x}^c(t)\equiv 0.$$

It follows from (2.3) that $\widehat{G}(\infty)=0$ and therefore $\Pi(j\infty)=\mathbf{R}$. As $\Pi(j\omega)$ is coercive, the operator $\mathbf{R}$ is invertible with $\mathbf{R}^{-1}\in\mathbf{L}(\mathrm{U})$, and the optimal control is realizable in a feedback form (2.11).

Inserting (2.11) into (2.24) we get

$$\left[\mathbf{A}^*\mathcal{H}^c+\mathcal{H}^c\mathbf{A}+\mathbf{C}^*\mathbf{Q}\mathbf{C}-(\mathcal{H}^c\mathbf{B}+\mathbf{C}^*\mathbf{N})\mathbf{R}^{-1}(\mathbf{B}^*\mathcal{H}^c+\mathbf{N}^*\mathbf{C})\right]\mathbf{x}^c(t)\equiv 0$$

which, by taking $t=0$, implies that $\mathcal{H}^c$ satisfies the *Riccati operator equation* (2.12).

Step 3. The minimal cost operator $\mathcal{H}^c$ solves (2.12), whence the pair $(\mathbf{H},\mathbf{G})$ given by (2.13) is a solution to (2.14) and the optimal controller (2.11) reads as

$$\mathbf{u}^c(t)=\mathbf{R}^{-1/2}\mathbf{G}^*\mathbf{x}^c(t).$$

Part (ii). *Step 1.* Premultiplying the Lyapunov equation in (2.14) by $\mathbf{x}_0 e^{t\mathbf{A}^*}$ and postmultiplying it by $e^{t\mathbf{A}}\mathbf{x}_0$ we get:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{x}_0^*e^{t\mathbf{A}^*}\mathbf{H}e^{t\mathbf{A}}\mathbf{x}_0\right] &= \mathbf{x}_0^*e^{t\mathbf{A}^*}\left[\mathbf{A}^*\mathbf{H}+\mathbf{H}\mathbf{A}\right]e^{t\mathbf{A}}\mathbf{x}_0\\ &= \mathbf{x}_0^*e^{t\mathbf{A}^*}\mathbf{C}^*\mathbf{Q}\mathbf{C}e^{t\mathbf{A}}\mathbf{x}_0-\mathbf{x}_0^*e^{t\mathbf{A}^*}\mathbf{G}\mathbf{G}^*e^{t\mathbf{A}}\mathbf{x}_0, \end{aligned}$$

whence, integrating both sides from 0 to t one obtains (here R_t denotes the reflection operator on $\mathrm{L}^2(0,\infty;\mathrm{Y})$)

$$-\mathbf{x}_0^* e^{t\mathbf{A}^*}\mathbf{H}e^{t\mathbf{A}}\mathbf{x}_0 + \mathbf{x}_0^*\mathbf{H}\mathbf{x}_0 + \int_0^t \mathbf{x}_0^* e^{\tau\mathbf{A}^*}\mathbf{C}^*\mathbf{Q}\mathbf{C}e^{\tau\mathbf{A}}\mathbf{x}_0 \mathrm{d}\tau = \int_0^t \mathbf{x}_0^* e^{\tau\mathbf{A}^*}\mathbf{G}\mathbf{G}^* e^{\tau\mathbf{A}}\mathbf{x}_0 \mathrm{d}\tau$$

$$\iff -\mathbf{x}_0^* e^{t\mathbf{A}^*}\mathbf{H}e^{t\mathbf{A}}\mathbf{x}_0 + \mathbf{x}_0^*\mathbf{H}\mathbf{x}_0 + \langle R_t\mathbf{\Psi}\mathbf{x}_0, \mathbf{Q}R_t\mathbf{\Psi}\mathbf{x}_0\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} = \int_0^t \left\|\mathbf{G}^* e^{\tau\mathbf{A}}\mathbf{x}_0\right\|_{\mathrm{U}}^2 \mathrm{d}\tau.$$

Suppose that, in addition, the semigroup $\{e^{t\mathbf{A}}\}_{t\geq 0}$ is *uniformly bounded*, then we have

$$\begin{aligned} &(M^2+1)\|\mathbf{H}\|_{\mathbf{L}(\mathrm{H})}\|\mathbf{x}_0\|_{\mathrm{H}}^2 + \|\mathbf{Q}\|_{\mathbf{L}(\mathrm{Y})}\|\mathbf{\Psi}\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))}^2\|\mathbf{x}_0\|_{\mathrm{H}}^2 \\ &\geq \int_0^t \left\|\mathbf{G}^* e^{\tau\mathbf{A}}\mathbf{x}_0\right\|_{\mathrm{U}}^2 \mathrm{d}\tau, \quad t\geq 0, \end{aligned}$$

and $\mathbf{G}^*$ is an *admissible observation operator* with respect to $\{e^{t\mathbf{A}}\}_{t\geq 0}$. From the second equation of (2.14) in its adjoint form: $-\mathbf{B}^*\mathbf{H} = -\mathbf{R}^{1/2}\mathbf{G}^* - \mathbf{N}^*\mathbf{C}$ we conclude that $-\mathbf{B}^*\mathbf{H}$ is admissible with respect to $\{e^{t\mathbf{A}}\}_{t\geq 0}$.

Step 2. If, in addition (2.16) holds, then $(j\omega I - \mathbf{A})^{-1} \in \mathbf{L}(\mathrm{H})$, $(-j\omega I - \mathbf{A}^*)^{-1} \in \mathbf{L}(\mathrm{H})$ for $\omega \neq 0$, and consequently $(j\omega I - \mathbf{A})^{-1}\mathbf{B} \in \mathbf{L}(\mathrm{U},\mathrm{H})$, $\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{U})$. Premultiplying the Lyapunov equation in (2.14) by $\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}$ and postmultiplying it by $(j\omega I - \mathbf{A})^{-1}\mathbf{B}$ we get:

$$\begin{aligned} &\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{A}^*\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B} + \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{H}\mathbf{A}(j\omega I - \mathbf{A})^{-1}\mathbf{B} \\ &= -j\omega\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B} - \mathbf{B}^*\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B} \\ &\quad + j\omega\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B} - \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{H}\mathbf{B} \\ &= -2\,\mathrm{Re}\left[\mathbf{B}^*\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B}\right] \\ &= \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{C}^*\mathbf{Q}\mathbf{C}(j\omega I - \mathbf{A})^{-1}\mathbf{B} - \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{G}\mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B} \\ &= \left[\hat{G}(j\omega)\right]^*\mathbf{Q}\hat{G}(j\omega) - \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{G}\mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B}. \end{aligned}$$

From the second equation of (2.14)

$$\begin{aligned} -\mathbf{B}^*\mathbf{H}(j\omega I - \mathbf{A})^{-1}\mathbf{B} &= -\mathbf{R}^{1/2}\mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B} - \mathbf{N}^*\hat{G}(j\omega), \\ -\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{H}\mathbf{B} &= -\mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{G}\mathbf{R}^{1/2} - \left[\hat{G}(j\omega)\right]^*\mathbf{N} \end{aligned}$$

and thus

$$\begin{aligned} &\mathbf{R} - \mathbf{R}^{1/2}\mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B} - \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{G}\mathbf{R}^{1/2} \\ &+ \mathbf{B}^*(-j\omega I - \mathbf{A}^*)^{-1}\mathbf{G}\mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B} \\ &= \mathbf{R} + \mathbf{N}^*\hat{G}(j\omega) + \left[\hat{G}(j\omega)\right]^*\mathbf{N} + \left[\hat{G}(j\omega)\right]^*\mathbf{Q}\hat{G}(j\omega) \\ &\iff \Pi(j\omega) = \left[\mathbf{R}^{1/2} - \mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B}\right]^*\left[\mathbf{R}^{1/2} - \mathbf{G}^*(j\omega I - \mathbf{A})^{-1}\mathbf{B}\right]. \end{aligned}$$

Since $\mathbf{G}^*$ is admissible, we have that

$$\mathbb{C}^+ \ni s \mapsto \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \in \mathrm{H}^2(\mathbb{C}^+; \mathrm{Y})$$

for all $\mathbf{u} \in \mathrm{U}$. Thus the function

$$\Theta(s) = \mathbf{R}^{1/2} - \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}$$

is holomorphic on $\overline{\mathbb{C}^+} \setminus \{0\}$ and has boundary value

$$j\mathbb{R} \setminus \{0\} \ni j\omega \longmapsto \Theta(j\omega) \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$$

which, by $[\Theta(j\omega)]^* \, \Theta(j\omega) = \Pi(j\omega)$ belongs to $\mathrm{L}^\infty(j\mathbb{R}; \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ with $\|\Pi(j\omega)\|_{\mathbf{L}(\mathrm{U})} = \|\Theta(j\omega)\|^2_{\mathbf{L}(\mathrm{U},\mathrm{Y})}$. Applying [15, Lemma 3, p. 956] we conclude that $\Theta \in \mathrm{H}^\infty(\mathbb{C}^+; \mathbf{L}(\mathrm{U}, \mathrm{Y}))$.

Notice that $\mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}$ and $\Theta(s)$ are transfer functions of the system (2.1) with the output equation replaced by: $\mathbf{y}(t) = \mathbf{G}^*\mathbf{x}(t)$ and $\mathbf{y}(t) = -\mathbf{G}^*\mathbf{x}(t) + \mathbf{R}^{1/2}\mathbf{u}(t)$, respectively.

Step 3. By the optimality of $\mathbf{u}^c$, $\mathbf{G}^*$ is an admissible observation operator with respect to the closed-loop semigroup $\{e^{t\mathbf{A}^c}\}_{t\geq 0}$ generated by $\mathbf{A}^c$, equivalently,

$$s \longmapsto \mathbf{G}^*(sI - \mathbf{A}^c)^{-1}\mathbf{x}_0 \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U}).$$

Since $\widehat{\mathbf{u}^c} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$, $\widehat{\mathbf{u}^c}(s) = \mathbf{R}^{-1/2}\mathbf{G}^*(sI - \mathbf{A}^c)^{-1}\mathbf{x}_0$, we have

$$(sI - \mathbf{A}^c)^{-1}\mathbf{x}_0 = \widehat{\mathbf{x}^c}(s) = (sI - \mathbf{A})^{-1}\mathbf{x}_0 + (sI - \mathbf{A})^{-1}\mathbf{B}\widehat{\mathbf{u}^c}(s), \tag{2.25}$$

whence the resolvent of $\mathbf{A}^c$ is analytic on $\mathbb{C}^+$ and has boundary values almost everywhere on $j\mathbb{R}$. Moreover,

$$\begin{aligned} I - (sI - \mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^* &= (sI - \mathbf{A})^{-1}(sI - \mathbf{A} - \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*) \\ &= (sI - \mathbf{A})^{-1}(sI - \mathbf{A}^c) \end{aligned} \tag{2.26}$$

is boundedly invertible for every $s \in \mathbb{C}^+$ with the inverse

$$\Big[I - (sI - \mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*\Big]^{-1} = (sI - \mathbf{A}^c)^{-1}(sI - \mathbf{A})$$

having boundary values almost everywhere on $j\mathbb{R}$.

Recalling that $sI - \mathbf{A}^c = sI - \mathbf{A} - \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*$, we get

$$I - \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*(sI - \mathbf{A})^{-1} = (sI - \mathbf{A}^c)(sI - \mathbf{A})^{-1} \tag{2.27}$$

whence $I - \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*(sI - \mathbf{A})^{-1}$ is boundedly invertible for every $s \in \mathbb{C}^+$ with the inverse

$$\Big[I - \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*(sI - \mathbf{A})^{-1}\Big]^{-1} = (sI - \mathbf{A})(sI - \mathbf{A}^c)^{-1}$$

having boundary values almost everywhere on $j\mathbb{R}$. From (2.26) and (2.27) we obtain

$$\begin{aligned}&\left[I-(sI-\mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*\right](sI-\mathbf{A}^c)^{-1}=(sI-\mathbf{A})^{-1}\\&=(sI-\mathbf{A}^c)^{-1}\left[I-\mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^*(sI-\mathbf{A})^{-1}\right].\end{aligned}$$

Premultiplying both sides by $\mathbf{G}^*$ and postmultiplying them by $\mathbf{B}\mathbf{R}^{-1/2}$ we get the commutativity relation

$$\begin{aligned}&\left[\Theta(s)\mathbf{R}^{-1/2}\right]\left[\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}\right]=\mathbf{G}^*(sI-\mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1/2}\\&=\left[\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}\right]\left[\Theta(s)\mathbf{R}^{-1/2}\right].\end{aligned}\tag{2.28}$$

Now, by (2.28) and (2.17),

$$\begin{aligned}&\Theta(s)\left[\mathbf{R}^{-1/2}+\mathbf{R}^{-1/2}\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}\right]\\&=\left[\Theta(s)+\mathbf{G}^*(sI-\mathbf{A})^{-1}\mathbf{B}\right]\mathbf{R}^{-1/2}=I\end{aligned}$$

and

$$\begin{aligned}&\left[\mathbf{R}^{-1/2}+\mathbf{R}^{-1/2}\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}\right]\Theta(s)\\&=\mathbf{R}^{-1/2}\left[\Theta(s)+\mathbf{G}^*(sI-\mathbf{A})^{-1}\mathbf{B}\right]=I.\end{aligned}$$

This means that $\Theta(s)$ is boundedly invertible at any $s\in\mathbb{C}^+$ as well as almost everywhere on $j\mathbb{R}$ with the inverse given by (2.18), and $\Theta^{-1}(s)$ is the transfer function of system with the state operator $\mathbf{A}^c$, control operator $\mathbf{B}\mathbf{R}^{-1/2}$, observation operator $\mathbf{R}^{-1/2}\mathbf{G}^*$ and feedthrough operator $\mathbf{R}^{-1/2}$; see Figure 1 for an additional piece of information. We have $\Theta^{-1}\in\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{U})$ iff

$$s\mapsto\mathbf{R}^{-1/2}\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}$$

is in $\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{U})$. Since

$$\|\Pi(j\omega)\|_{\mathbf{L}(\mathrm{U})}\|\mathbf{u}\|^2_{\mathrm{U}}=\|\Theta(j\omega)\mathbf{u}\|^2_{\mathrm{U}}\geq\varepsilon\|\mathbf{u}\|^2_{\mathrm{U}}$$

then substituting $\mathbf{v}=[\Theta(j\omega)]^{-1}\mathbf{u}$ we get $\|[\Theta(j\omega)]^{-1}\|_{\mathbf{L}(\mathrm{U})}\leq\varepsilon^{-1/2}$ for almost all $\omega\in\mathbb{R}$, whence $\|\Theta^{-1}\|_{\mathrm{L}^\infty(j\mathbb{R},\mathbf{L}(\mathrm{U}))}\leq\varepsilon^{-1/2}$. Consequently, the boundary value of

$$s\longmapsto\mathbf{R}^{-1/2}\mathbf{G}^*(sI-\mathbf{A}^c)^{-1}\mathbf{B}\mathbf{R}^{-1/2}$$

at $j\mathbb{R}$ is in $\mathrm{L}^\infty(j\mathbb{R},\mathbf{L}(\mathrm{U}))$. Applying [15, Lemma 3, p. 956] once more we have $\Theta^{-1}\in\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{U})$. □

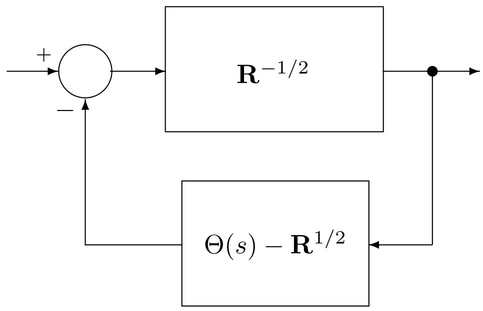

Fig. 1. Feedback realization of $[\Theta(s)]^{-1}$

3. AN OVERVIEW OF CONTROL SYSTEMS IN FACTOR FORM

Consider a class of controlled systems with observation governed by the model in factor form

$$\begin{cases} \dot{x}(t) = \mathcal{A}\left[x(t) + \mathcal{D}u(t)\right], \\ y(t) = \mathcal{C}x(t), \end{cases} \tag{3.1}$$

where the *state operator* $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$. A family $\{S(t)\}_{t\geq 0} \subset \mathbf{L}(\mathrm{H})$ is a C_0-*semigroup* on H if (**i**) $S(0) = I$, $S(t+\tau) = S(t)S(\tau)$ for $t, \tau \geq 0$ and (**ii**) $S(t)x_0 \to x_0$ as $t \to 0$ for every $x_0 \in \mathrm{H}$. $\{S(t)\}_{t\geq 0}$ is *exponentially stable* (**EXS**) if there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \leq Me^{-\alpha t}\|x_0\|_{\mathrm{H}}, \quad t \geq 0, x_0 \in \mathrm{H}. \tag{3.2}$$

We say that $\mathcal{A}$ *generates* $\{S(t)\}_{t\geq 0}$ if

$$\mathcal{A}x_0 = \lim_{h\to 0}\frac{1}{h}[S(h)x_0 - x_0], \quad D(\mathcal{A}) = \Big\{x_0 \in \mathrm{H}: \text{ there exists } \lim_{h\to 0}\frac{1}{h}[S(h)x_0 - x_0]\Big\}.$$

Since $s \mapsto (sI - \mathcal{A})^{-1}x_0$ is the *Laplace transform* of $t \mapsto S(t)x_0$ then, by (3.2), the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\alpha\}$ is contained in $\rho(\mathcal{A})$ – the resolvent set of $\mathcal{A}$ which, in particular, implies that $\mathcal{A}$ is invertible with bounded and everywhere defined inverse, $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

Next, $\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$, $\mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$, $\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{H})$ with range $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and Y and U are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, respectively.

Let us introduce $H := (\mathcal{C}\mathcal{A}^{-1})^* \iff H^* = \mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$ to simplify future notation.

Proofs of all results appearing in this Section are given in [7, Section 2].

3.1. ADMISSIBLE OBSERVATION AND CONTROL OPERATORS

Define $\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$,

$$\left(\mathcal{Z}x_0\right)(t) := H^* S(t) x_0 \qquad \left[\Longleftrightarrow \ \mathcal{Z}^* f = \int\limits_0^\infty S^*(t) H f(t) \mathrm{d}t \right].$$

The operator, called the *observability map*,

$$\mathbf{\Psi} := \mathcal{L}\mathcal{Z}, \quad D(\mathbf{\Psi}) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L})\},$$

is closed and densely defined, with $\mathbf{\Psi}|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Psi}^* = \mathcal{A}^*\mathcal{Z}^*, \quad D(\mathbf{\Psi}^*) = \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathcal{Z}^* y \in D(\mathcal{A}^*)\},$$

and $\mathbf{\Psi}^*|_{D(\mathcal{R})} = \mathcal{Z}^*\mathcal{R}$.

Definition 3.1. $\mathcal{C}$ is an admissible *observation (output) operator* if $\mathbf{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$ (or, by the closed graph theorem, $R(\mathcal{Z}) \subset D(\mathcal{L})$ or $\mathbf{\Psi}$ is bounded).

Lemma 3.2. *If $\mathcal{C}$ is admissible then $\mathbf{\Psi}$ is also a linear densely defined and bounded operator from* H *into* $\mathrm{L}^1(0,\infty;\mathrm{Y})$.

Next, we define $\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$,

$$\mathcal{W}f := \int\limits_0^\infty S(t)\mathcal{D}f(t)\mathrm{d}t \quad \left[\Longleftrightarrow \ \left(\mathcal{W}^* x_0\right)(t) = \mathcal{D}^* S^*(t) x_0 \right].$$

The operator, called the *reachability map*,

$$\mathbf{\Phi} := \mathcal{A}\mathcal{W}, \quad D(\mathbf{\Phi}) = \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathcal{W}u \in D(\mathcal{A})\},$$

is closed and densely defined, with $\mathbf{\Phi}|_{D(\mathcal{R})} = \mathcal{W}\mathcal{R}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Phi}^* = \mathcal{L}\mathcal{W}^*, \quad D(\mathbf{\Phi}) = \{x \in \mathrm{H} : \ \mathcal{W}^* x \in D(\mathcal{L})\},$$

with $\mathbf{\Phi}^*|_{D(\mathcal{A}^*)} = \mathcal{W}^*\mathcal{A}^*$.

Definition 3.3. $\mathcal{D}$ is an admissible *factor control operator* if $\mathbf{\Phi} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$ (or, by the closed graph theorem, $R(\mathcal{W}) \subset D(\mathcal{A})$ or $\mathbf{\Phi}$ is bounded).

Using duality arguments, we can state the following result.

Lemma 3.4. *$\mathcal{D}$ is an admissible factor control operator iff $\mathcal{D}^*\mathcal{A}^*$ is an admissible observation operator with respect to the semigroup $\{S^*(t)\}_{t\geq 0}$.*

3.2. REPRESENTATION OF THE STATE

Definition 3.5. Let $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$. A continuous vector valued function $t \mapsto x(t) \in \mathrm{H}$ is called a *weak solution* of (3.1) if $x(0) = x_0$ and x satisfies (3.1) in a *weak sense*, i.e., the function $t \mapsto \langle x(t), w\rangle_{\mathrm{H}}$ is absolutely continuous and for almost all $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = \langle x(t), \mathcal{A}^* w\rangle_{\mathrm{H}} + \langle \mathcal{D}u(t), \mathcal{A}^* w\rangle_{\mathrm{H}}, \quad w \in D(\mathcal{A}^*).$$

In what follows, $\mathrm{BUC}([0,\infty);\mathrm{Z})$ will denote the Banach space of *bounded, uniformly continuous* functions defined on $[0,\infty)$ and taking values in a Hilbert space Z, equipped with standard norm

$$\|f\|_{\mathrm{BUC}([0,\infty);\mathrm{Z})} := \sup_{t\geq 0}\|f(t)\|_{\mathrm{Z}}, \quad f \in \mathrm{BUC}([0,\infty);\mathrm{Z});$$

$\mathrm{BUC}_0([0,\infty);\mathrm{Z})$ will stand for its closed subspace consisting of functions that have zero limit at infinity.

Theorem 3.6. *If $\mathcal{A}$ generates an* ***EXS*** C_0*-semigroup $\{S(t)\}_{t\geq 0}$ and $\mathcal{D}$ is an admissible factor control operator, then for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$*

$$x(t) := S(t)x_0 + \mathbf{\Phi} R_t u, \tag{3.3}$$

we have $x \in \mathrm{BUC}_0([0,\infty),\mathrm{H})$ and x is a unique weak solution of (3.1)*. Furthermore, for every $z \in \mathrm{H}$ the function $t \longmapsto \langle x(t), z\rangle_{\mathrm{H}}$ is in $\mathrm{L}^2(0,\infty)$.*

3.3. REPRESENTATION OF THE OUTPUT

Now we pass to the *construction of the system output* in operator form. For that we assume that the system *transfer function*

$$\hat{g}(s) := s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D} = s^2 H^*(sI-\mathcal{A})^{-1}\mathcal{D} - sH^*\mathcal{D} - \mathcal{C}\mathcal{D} \tag{3.4}$$

(thus $\hat{g}$ is well-defined for $\operatorname{Re} s > -\alpha$) satisfies

$$\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y})) \tag{3.5}$$

Let us remark that $\hat{g}$ is analytic on a set containing $\overline{\mathbb{C}^+}$ and (3.4) implies that $\hat{g}$ grows no faster than quadratically on $\mathbb{C}^+$, whence by the *Phragmén-Lindelöf theorem* (3.5) is met if $\hat{g}$ is bounded on $j\mathbb{R}$. Moreover, (3.5) yields

$$\|\hat{g}(j\omega)\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \leq \|\hat{g}\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}, \quad \omega \in \mathbb{R}.$$

Theorem 3.7. *Let $\mathcal{A}$ generates an* ***EXS*** C_0*-semigroup $\{S(t)\}_{t\geq 0}$, $\mathcal{C}$ and $\mathcal{D}$ be admissible and* (3.5) *holds. Then, for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$:*

$$y = \Psi x_0 + \mathbb{F}u,$$

where $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{L}^2(0,\infty;\mathrm{Y}))$ *is the input-output operator,*

$$(\mathbb{F}u)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi[\mathcal{D}u(\tau)])(t-\tau)\mathrm{d}\tau - (\mathcal{C}\mathcal{D})u(t).$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}[H^* x(t)] = y(t) + \mathcal{C}\mathcal{D}u(t) = \mathcal{C}x(t) + \mathcal{C}\mathcal{D}u(t).$$

3.4. SYSTEMS WITH A LINEAR FEEDBACK

A basic result characterizing the well-posedness of systems with a linear feedback is the following version of the Weiss-Staffans perturbation theorem.

Theorem 3.8. *Let* $\mathcal{A}$ *generates an* ***EXS*** C_0*-semigroup* $\{S(t)\}_{t\geq 0}$ *on* H*,* $\mathcal{C}$ *and* $\mathcal{D}$ *be admissible operators, and* $\mathcal{D}^*\mathcal{A}^*$ *extends from* $D(\mathcal{A}^*)$ *to* $\mathcal{D}^{\#}$ *with domain* $D(\mathcal{D}^{\#})$ *such that* $R(H) \subset D(\mathcal{D}^{\#})$ *and* $\mathcal{D}^{\#}H = (\mathcal{C}\mathcal{D})^*$*,* $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$*. Assume that* (3.5) *is met and for some* $\mathcal{K} \in \mathbf{L}(\mathrm{Y},\mathrm{U})$ *there holds*

$$s \mapsto [I + \hat{g}(s)\mathcal{K}]^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{Y})) \iff s \mapsto [I + \mathcal{K}\hat{g}(s)]^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U})).$$

Then, the closed-loop operator $\mathcal{A}_c$ *arising by applying linear feedback control law* $u = -\mathcal{K}y = -\mathcal{K}\mathcal{C}x$ *to* (3.1)*, i.e.,*

$$\mathcal{A}_c x = \mathcal{A}(x - \mathcal{D}\mathcal{K}\mathcal{C}x), \quad D(\mathcal{A}_c) = \{x \in D(\mathcal{C}) :\ x - \mathcal{D}\mathcal{K}\mathcal{C}x \in D(\mathcal{A})\}, \tag{3.6}$$

generates an ***EXS*** C_0*-semigroup* $\{S_c(t)\}_{t\geq 0}$ *on* H.

4. APPLICATION TO LQ PROBLEM FOR SYSTEMS IN FACTOR FORM

4.1. RECIPROCAL SYSTEM

4.1.1. Preliminaria

The following result will be paramount [10, p. 134, Problem 3 and Corollary, p. 147].

Lemma 4.1. *Let* X *be a Hilbert space. Then, the operator*

$$\mathrm{H}^2(\mathbb{C}^+,\mathrm{X}) \ni \varphi \longmapsto s^{-1}\varphi(s^{-1}) \in \mathrm{H}^2(\mathbb{C}^+,\mathrm{X}),$$

is a unitary map, whilst the mapping

$$\mathcal{I} : \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y})) \ni \varphi(s) \longmapsto \varphi(s^{-1}) \in \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))$$

is an isometry.

The *Hankel transformation of order* 0 is defined as

$$(\mathfrak{H}_0 f)(t) := \int_0^\infty J_0(2\sqrt{t\tau}) f(\tau) \mathrm{d}\tau,$$

where J_0 stands for the *Bessel function* of the first kind and zero order (in early 1990s Gamage K. Watugala wrongly renamed the Hankel transformation to *Sumudu transform* and this error became persistent in Asian literature). It is straightforward to see (apply the Lebesgue dominated convergence theorem) that

$$\mathfrak{H}_0 \in \mathbf{L}(\mathrm{L}^1(0,\infty;\mathrm{Z}), \mathrm{L}^\infty(0,\infty;\mathrm{Z})); \quad (\mathfrak{H}_0 f) \in \mathrm{C}([0,\infty);\mathrm{Z}), \quad \lim_{t\to\infty} (\mathfrak{H}_0 f)(t) = 0.$$

Efros' theorem [1, pp. 12-13] states that under the assumption that a change of order of integration is possible while computing the Laplace transform, one has that the Laplace transform of $\int_0^\infty g(t,\tau) f(\tau)\mathrm{d}\tau$ equals $\hat{f}[q(s)]G(s)$, provided that the Laplace transform of $g(\cdot,\tau)$ is $G(s)e^{-\tau q(s)}$. Since $\widehat{J_0(2\sqrt{\cdot\tau})} = \frac{1}{s}e^{-\frac{\tau}{s}}$ [3, p. 185, Formula (25)] (here $G(s) = q(s) = \frac{1}{s}$), then by Fubini's and Efros' theorems [3, p. 132, Formula (32), $\nu = 0$]

$$\widehat{\mathfrak{H}_0 f}(s) = \frac{1}{s}\hat{f}\left(\frac{1}{s}\right), \quad f \in \mathrm{L}^1(0,\infty;\mathrm{Z}).$$

Since $\mathrm{L}^1(0,\infty;\mathrm{Z}) \cap \mathrm{L}^2(0,\infty;\mathrm{Z})$ is dense in $\mathrm{L}^2(0,\infty;\mathrm{Z})$ (here Z is a Hilbert space), then using Lemma 4.1, we conclude that the Hankel transform $\mathfrak{H}_0$ is a time-domain version of

$$(\widehat{\mathfrak{H}_0}\varphi)(s) := (1/s)\varphi(1/s),$$

so it maps unitarily $\mathrm{L}^2(0,\infty;\mathrm{Z})$ onto itself; confirmed by [3, p. 132, Formula (25), $\nu = 0$].

Lemma 4.2. *The admissibility of $\mathcal{C}$ with respect to the **EXS** semigroup $\{S(t)\}_{t\geq 0}$, is equivalent to the admissibility of H^*, with respect to the semigroup $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$. The admissibility of $\mathcal{D}$ with respect to the semigroup $\{S(t)\}_{t\geq 0}$, is equivalent to the admissibility of $\mathcal{D}^*$ with respect to the semigroup $\{e^{t\mathbf{A}^*}\}_{t\geq 0}$, $e^{t\mathbf{A}^*} = e^{t\mathcal{A}^{-*}}$.*

Proof. The first fact is an immediate consequence of

$$(\widehat{\Psi x_0})(s) = \mathcal{C}(sI - \mathcal{A})^{-1}x_0 = -\frac{1}{s}H^*\left(\frac{1}{s}I - \mathcal{A}^{-1}\right)^{-1} x_0,$$

where the RHS has to be identified with the Hankel transform of $\widehat{\mathbf{\Psi}_{H^*} x_0}$ – the Laplace transforms of the observability map associated with the output operator $(-H^*)$ with respect to $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$.

For the second statement notice that, by duality Lemma 3.4, $\mathcal{D}$ is admissible with respect to the semigroup $\{S(t)\}_{t\geq 0}$ iff $\mathcal{D}^*\mathcal{A}^*$ is admissible with respect to the semigroup $\{S^*(t)\}_{t\geq 0}$, but the latter holds iff $\mathcal{D}^*$ is admissible with respect to the semigroup $\{e^{t\mathbf{A}^*}\}_{t\geq 0}$, $e^{t\mathbf{A}^*} = e^{t\mathcal{A}^{-*}}$. □

4.1.2. Reciprocal system: case where both $\mathcal{C}$ and $\mathcal{D}$ are admissible

In this section we shall assume that $\mathcal{A}$ generates an **EXS** C_0-semigroup, both $\mathcal{C}$, $\mathcal{D}$ are admissible, and (3.5) holds.

By Theorem 3.6, especially (3.3),

$$x(t) = S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, \quad x \in \mathrm{BUC}_0([0,\infty);\mathrm{H})$$

is a unique weak solution of (3.1) with the Laplace transform

$$\hat{x}(s) = (sI-\mathcal{A})^{-1}x_0 + \mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s), \quad s \in \mathbb{C}^+.$$

The inverse $s \mapsto \dfrac{1}{s}$ conformally maps $\mathbb{C}^+$ onto itself and therefore

$$\begin{aligned}\frac{1}{s}\hat{x}\left(\frac{1}{s}\right) &= \frac{1}{s}\left(\frac{1}{s}I-\mathcal{A}\right)^{-1}x_0 + \mathcal{A}\left(\frac{1}{s}I-\mathcal{A}\right)^{-1}\mathcal{D}\left[\frac{1}{s}\hat{u}\left(\frac{1}{s}\right)\right] \\ &= -s\left(sI-\mathcal{A}^{-1}\right)^{-1}x_0 + x_0 - s\left(sI-\mathcal{A}^{-1}\right)^{-1}\mathcal{D}\left[\frac{1}{s}\hat{u}\left(\frac{1}{s}\right)\right],\end{aligned}$$

whence

$$\frac{1}{s}\hat{x}\left(\frac{1}{s}\right) = -\widehat{\dot{x}_{\mathrm{a}}}(s),$$

i.e., it is the Laplace transform of $\dot{x}_{\mathrm{a}} = -\mathfrak{H}_0 x$, where x_{a} is the unique solution of the *reciprocal system*

$$\left\{\begin{array}{l}\dot{x}_{\mathrm{a}}(t) = \mathcal{A}^{-1}x_{\mathrm{a}}(t) + \mathcal{D}u_{\mathrm{a}}(t)\\ x_{\mathrm{a}}(0) = x_0\\ y_{\mathrm{a}}(t) := \mathcal{C}\mathcal{A}^{-1}x_{\mathrm{a}}(t) = H^*x_{\mathrm{a}}(t)\end{array}\right\}, \quad u_{\mathrm{a}} := \mathfrak{H}_0 u \Leftrightarrow \widehat{u_{\mathrm{a}}} := \widehat{\mathfrak{H}_0 u}, \tag{4.1}$$

which contains bounded operators exclusively.

Conversely, since

$$\frac{1}{s}\hat{x}\left(\frac{1}{s}\right) = -\left[\mathcal{A}^{-1}\widehat{x_{\mathrm{a}}}(s) + \mathcal{D}\widehat{u_{\mathrm{a}}}(s)\right],$$

we have

$$\hat{x}(s) + \mathcal{D}\hat{u}(s) = -\mathcal{A}^{-1}\frac{1}{s}\widehat{x_{\mathrm{a}}}\left(\frac{1}{s}\right) \iff x(t) + \mathcal{D}u(t) = -\mathcal{A}^{-1}\mathfrak{H}_0 x_{\mathrm{a}}(t).$$

Hence

$$\hat{x}(s) + \mathcal{D}\hat{u}(s) \in D(\mathcal{A}), \quad \frac{1}{s}\widehat{x_{\mathrm{a}}}\left(\frac{1}{s}\right) = -\mathcal{A}\left[\hat{x}(s) + \mathcal{D}\hat{u}(s)\right]. \tag{4.2}$$

Next, the Laplace transform of the output of (3.1) reads as

$$\hat{y}(s) = \mathcal{C}(sI-\mathcal{A})^{-1}x_0 + \hat{g}(s)\hat{u}(s),$$

whence

$$\begin{aligned}(\widehat{\mathfrak{H}_0 y})(s) &= \frac{1}{s}\hat{y}\left(\frac{1}{s}\right) = \frac{1}{s}\mathcal{C}\left(\frac{1}{s}I-\mathcal{A}\right)^{-1}x_0 + \hat{g}\left(\frac{1}{s}\right)\left[\frac{1}{s}\hat{u}\left(\frac{1}{s}\right)\right] \\ &= -(\mathcal{C}\mathcal{A}^{-1})\left(sI-\mathcal{A}^{-1}\right)^{-1}x_0 - \widehat{G_{\rm a}}(s)\widehat{u_{\rm a}}(s) - (\mathcal{C}\mathcal{D})\widehat{u_{\rm a}}(s) \\ &= -\widehat{y_{\rm a}}(s) - (\mathcal{C}\mathcal{D})\widehat{u_{\rm a}}(s),\end{aligned}$$

where $\widehat{G_{\rm a}}(s) := H^*(sI-\mathcal{A}^{-1})^{-1}\mathcal{D}$ is the reciprocal system transfer function. Since

$$s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} = -\mathcal{C}\mathcal{A}^{-1}\left[\frac{1}{s}I-\mathcal{A}^{-1}\right]^{-1}\mathcal{D} = -\mathcal{I}\widehat{G_{\rm a}}(s), \tag{4.3}$$

we have the following results.

Lemma 4.3. $\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$ *iff* $\widehat{G_{\rm a}}(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$.

4.2. SOLUTION OF LQ PROBLEM

Let us consider the infinite-time horizon LQ problem of finding the optimal control or/and optimal controller, minimizing the quadratic performance index

$$\begin{aligned}J(x_0,u) &= \int_0^\infty \begin{bmatrix} y(t) \\ u(t)\end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R\end{bmatrix}\begin{bmatrix} y(t) \\ u(t)\end{bmatrix}\mathrm{d}t \\ &= \langle Qy+Nu, y\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \langle N^*y+Ru, u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})}\end{aligned} \tag{4.4}$$

over output trajectories of (3.1); $Q = Q^* \in \mathbf{L}(\mathrm{Y})$, $N \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $R = R^* \in \mathbf{L}(\mathrm{U})$.

Observe that

$$\begin{aligned}&\langle Qy+Nu, y\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \langle N^*y+Ru, u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} \\ &= \langle \mathfrak{H}_0 Qy+\mathfrak{H}_0 Nu, \mathfrak{H}_0 y\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \langle \mathfrak{H}_0 N^*y+\mathfrak{H}_0 Ru, \mathfrak{H}_0 u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} \\ &= \langle Q\mathfrak{H}_0 y+N\mathfrak{H}_0 u, \mathfrak{H}_0 y\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} + \langle N^*\mathfrak{H}_0 y+R\mathfrak{H}_0 u, \mathfrak{H}_0 u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} \\ &= \langle Q[-y_{\rm a}-\mathcal{C}\mathcal{D}u_{\rm a}]+Nu_{\rm a}, -y_{\rm a}-\mathcal{C}\mathcal{D}u_{\rm a}\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \\ &\quad + \langle N^*[-y_{\rm a}-\mathcal{C}\mathcal{D}u_{\rm a}]+Ru_{\rm a}, u_{\rm a}\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} \\ &= \int_0^\infty \begin{bmatrix} y_{\rm a}(t) \\ u_{\rm a}(t)\end{bmatrix}^* \begin{bmatrix} Q & -N_- \\ -N_-^* & R_-\end{bmatrix}\begin{bmatrix} y_{\rm a}(t) \\ u_{\rm a}(t)\end{bmatrix}\mathrm{d}t := J_{\rm a}(x_0,u_{\rm a}),\end{aligned}$$

where

$$N_- := N - Q(\mathcal{C}\mathcal{D}), \quad R_- := R - (\mathcal{C}\mathcal{D})^*N - N^*(\mathcal{C}\mathcal{D}) + (\mathcal{C}\mathcal{D})^*Q(\mathcal{C}\mathcal{D}) = R_-^*, \tag{4.5}$$

and the last performance index is being optimized over output trajectories of the reciprocal system (4.1).

However, the problem of minimization of $J_{\rm a}$ over trajectories of (4.1) is a particular case of minimization of $\mathbf{J}$, given by (2.2), over output trajectories of (2.1), to which Theorem 2.4 applies. To see this the following identifications are proper:

$$\begin{array}{r|r|r|c} \mathbf{x} = x_{\rm a}, & \mathbf{A} = \mathcal{A}^{-1}, & \mathbf{Q} = \ \ Q, & \\ \mathbf{u} = u_{\rm a}, & \mathbf{B} = \mathcal{D}, & \mathbf{N} = -N_-, & \hat{G} = \widehat{G_{\rm a}}, \ \mathbf{J} = J_{\rm a}, \\ \mathbf{y} = y_{\rm a}, & \mathbf{C} = H^*, & \mathbf{R} = \ \ R_-, & \end{array} \tag{4.6}$$

whence

$$\begin{aligned} \Pi(j\omega) &= \mathbf{R} + 2\,\mathrm{Re}\left[\mathbf{N}^*\hat{G}(j\omega)\right] + \left[\hat{G}(j\omega)\right]^* \mathbf{Q}\hat{G}(j\omega) \\ &= R_- - 2\,\mathrm{Re}\left[N_-^*\widehat{G_{\rm a}}(j\omega)\right] + \left[\widehat{G_{\rm a}}(j\omega)\right]^* Q\widehat{G_{\rm a}}(j\omega). \end{aligned}$$

Using (4.5), (3.4) and (4.3) we get the following result.

Lemma 4.4. Π *is coercive iff the Popov spectral function for* (3.1),

$$\pi(j\omega) := R + 2\,\mathrm{Re}\left[N^*\hat{g}(j\omega)\right] + \left[\hat{g}(j\omega)\right]^* Q\hat{g}(j\omega)$$

is coercive.

Theorem 4.5. *Let $\mathcal{A}$ generates an* ***EXS*** C_0*-semigroup* $\{S(t)\}_{t\geq 0}$*, $\mathcal{C}$ and $\mathcal{D}$ are admissible,* $\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$*, and π is coercive then both LQ problems* (3.1)*,* (4.4) *and* (4.1)*,* (4.4) *as well as* (2.14) *are solvable.*
If, in addition, the semigroup $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$ *is uniformly bounded then* $\mathcal{G}$*,*

$$\mathcal{G} := -\mathcal{D}^*\mathcal{H}^c\mathcal{A} + N_-\mathcal{C}, \quad \mathcal{G}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$$

is admissible with respect to the semigroup $\{S(t)\}_{t\geq 0}$ *and π has a spectral factorization*

$$\pi(j\omega) = [\phi(j\omega)]^*\phi(j\omega), \quad \phi(s) = R_-^{1/2} + sR_-^{-1/2}\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D},$$

ϕ*,* $\phi^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$*.*
Next, if moreover, $\mathcal{G}$ extends from $D(\mathcal{A})$ to $\mathcal{G}_\Lambda$ with domain $D(\mathcal{G}_\Lambda)$ such that $R(\mathcal{D}) \subset D(\mathcal{G}_\Lambda)$, $\mathcal{G}_\Lambda\mathcal{D} \in \mathbf{L}(\mathrm{U})$ and $R_- + \mathcal{G}_\Lambda\mathcal{D}$ is boundedly invertible then the closed-loop operator in optimally driven reciprocal system: $\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$ has a generally unbounded inverse:

$$\begin{aligned} \mathcal{A}^c v &= \mathcal{A}\left[v - \mathcal{D}\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda v\right], \\ D(\mathcal{A}^c) &= \left\{v \in D(\mathcal{G}_\Lambda) : v - \mathcal{D}\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda v \in D(\mathcal{A})\right\} \end{aligned} \tag{4.7}$$

and

$$u^c = -\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda x^c. \tag{4.8}$$

Finally, if in addition, $\mathcal{D}^*\mathcal{A}^*$ extends from $D(\mathcal{A}^*)$ to $\mathcal{D}^\#$ with domain $D(\mathcal{D}^\#)$ such that $R(\mathbf{G}) \subset D(\mathcal{D}^\#)$ and $\mathcal{D}^\#\mathbf{G}R_-^{1/2} = (\mathcal{G}_\Lambda\mathcal{D})^*$. Then the closed-loop state operator (4.7) *generates an* ***EXS*** C_0*-semigroup on* H.

Proof. By Lemmas 4.2, 4.3 and 4.4, and because of equivalences (4.6), the first claim immediately follows from Theorem 2.4/**(i)**.

By the additional assumption, the semigroup $\{e^{t\mathbf{A}}\}_{t\geq 0}$, $e^{t\mathbf{A}} = e^{t\mathcal{A}^{-1}}$ is uniformly bounded whilst **EXS** of $\{S(t)\}_{t\geq 0}$ clearly implies that $\sigma(\mathbf{A}) \cap j\mathbb{R} = \{0\}$, whence (2.16) holds; actually $0 \in \sigma_c(\mathbf{A})$, where $\sigma_c(\mathbf{A})$ denotes the continuous spectrum of $\mathbf{A}$. By Theorem 2.4 (ii) the operator $\mathbf{G}^*$ is admissible with respect to $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$. Since, by (4.6),

$$\mathcal{G} = -\mathcal{D}^*\mathcal{H}^c\mathcal{A} + N_-\mathcal{C} = (\mathbf{B}^*\mathbf{H} - \mathbf{N}\mathbf{C})\mathcal{A} = \mathbf{R}^{1/2}\mathbf{G}^*\mathcal{A}, \tag{4.9}$$

one can apply Lemma 4.2 with $\mathcal{C}$ and H replaced by $\mathbf{R}^{-1/2}\mathcal{G}$ and $\mathbf{G}$, respectively to conclude that $\mathcal{G}$ is admissible with respect to $\{S(t)\}_{t\geq 0}$.

Next,

$$\Theta(s) = \mathbf{R}^{1/2} - \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}$$

factorizes $\Pi(j\omega)$, but this holds iff

$$\begin{aligned}\phi(s) = \Theta(s^{-1}) &= \mathbf{R}^{1/2} - \mathbf{G}^*\left(s^{-1}I - \mathbf{A}\right)^{-1}\mathbf{B}\\ &= \mathbf{R}^{1/2} - \mathbf{G}^*\left[\left(-s^{-1}\mathbf{A}\right)\left(-\mathbf{A}^{-1} + sI\right)\right]^{-1}\mathbf{B}\\ &= R_-^{1/2} + s\mathbf{G}^*\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D} = R_-^{1/2} + sR_-^{-1/2}\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}\end{aligned}$$

factorizes $\pi(j\omega) = \Pi(-j\omega^{-1})$. Since Θ, $\Theta^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+; \mathbf{L}(\mathrm{U}))$, the same holds for ϕ and ϕ^{-1}.

It follows from Theorem 2.4 that the optimal feedback controller equation for the reciprocal system is given by (2.11) which here reads, again due to (4.6), as

$$u_{\mathrm{a}}^c = -R_-^{-1}\left[\mathcal{D}^*\mathcal{H}^c - N_-\mathcal{C}\mathcal{A}^{-1}\right]x_{\mathrm{a}}^c \Longrightarrow \widehat{u_{\mathrm{a}}}^c(s) = -R_-^{-1}\left[\mathcal{D}^*\mathcal{H}^c - N_-\mathcal{C}\mathcal{A}^{-1}\right]\widehat{x_{\mathrm{a}}}^c(s).$$

Thus applying the Hankel transform and (4.2) we conclude that the optimal controller for the system (3.1) is *implicitly defined* by

$$\hat{u}^c(s) = -R_-^{-1}\mathcal{G}\left[\hat{x}^c(s) + \mathcal{D}\hat{u}^c(s)\right]. \tag{4.10}$$

Since

$$s\mathcal{A}^{-1}\hat{x}^c(s) - \mathcal{A}^{-1}x_0 = \hat{x}^c(s) + \mathcal{D}\hat{u}^c(s), \tag{4.11}$$

then (4.10) can be written as

$$\hat{u}^c(s) = -sR_-^{-1}\mathcal{G}\mathcal{A}^{-1}\hat{x}^c(s) + R_-^{-1}\mathcal{G}\mathcal{A}^{-1}x_0.$$

Inserting this expression into (4.11) one obtains

$$s(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1})\hat{x}^c(s) - \hat{x}^c(s) = (\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1})x_0, \tag{4.12}$$

where $\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$ is the closed-loop state operator of the optimally driven reciprocal system. Indeed, making use of (4.6) and (4.9), we get

$$\mathbf{A}^c = \mathbf{A} + \mathbf{B}\mathbf{R}^{-1/2}\mathbf{G}^* = \mathcal{A}^{-1} + \mathcal{D}R_-^{-1}(R_-^{1/2}\mathbf{G}^*) = \mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}.$$

To find its inverse $\mathcal{A}^c = (\mathbf{A}^c)^{-1}$ we have to consider the equation

$$\mathcal{A}^{-1}x + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}x = v \in D(\mathcal{A}^c).$$

Since $\mathcal{G}$ extends from $D(\mathcal{A})$ to $\mathcal{G}_\Lambda$ with domain $D(\mathcal{G}_\Lambda)$ such that $R(\mathcal{D}) \subset D(\mathcal{G}_\Lambda)$, $\mathcal{G}_\Lambda\mathcal{D} \in \mathbf{L}(\mathrm{U})$, then $v \in D(\mathcal{G}_\Lambda)$ and

$$(R_- + \mathcal{G}_\Lambda\mathcal{D})\, R_-^{-1}\mathcal{G}\mathcal{A}^{-1}x = \mathcal{G}_\Lambda v.$$

But $R_- + \mathcal{G}_\Lambda\mathcal{D}$ is boundedly invertible and therefore we have

$$R_-^{-1}\mathcal{G}\mathcal{A}^{-1}x = (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda v.$$

Hence

$$\mathcal{A}^{-1}x = v - \mathcal{D}\,(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda v \in D(\mathcal{A})$$

and we come to (4.7).

Because $\mathcal{A}^c := \left(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}\right)^{-1}$, (4.12) shows that $\hat{x}^c$ satisfies the resolvent equation for $\mathcal{A}^c$, whence $\hat{x}^c$ is an analytic extension of the resolvent

$$(sI - \mathcal{A}^c)^{-1} = \frac{1}{s}(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1})\left[(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}) - \frac{1}{s}I\right]^{-1}$$

from the semicircle $|s| < \|\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}\|_{\mathbf{L}(\mathrm{H})}$, $s \in \mathbb{C}^+$ onto $\mathbb{C}^+$. Thus $\mathcal{A}^c$ is the unique candidate to be a state operator of the closed-loop (optimally controlled) system $\dot{x}^c(t) = \mathcal{A}^c x^c(t)$. Now comparing (4.7) with (3.1) we get (4.8).

To prove the last statement let us invoke Theorem 3.8 with (compare (4.7) with (3.1)):

$$\mathrm{Y} = \mathrm{U}, \quad \mathcal{K} = (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1} \in \mathbf{L}(\mathrm{U}), \quad \mathcal{C} = \mathcal{G}_\Lambda \Leftrightarrow H^* = R_-^{1/2}\mathbf{G}^*.$$

Thus $\hat{g}$ has to be replaced by $\hat{g}_\Lambda$,

$$\begin{aligned} \hat{g}_\Lambda(s) &:= s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D} - \mathcal{G}_\Lambda\mathcal{D} = R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D} - (R_- + \mathcal{G}_\Lambda\mathcal{D}) \\ &= R_-^{1/2}\phi(s) - (R_- + \mathcal{G}_\Lambda\mathcal{D}) \Longrightarrow \hat{g}_\Lambda \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U})), \end{aligned} \tag{4.13}$$

whence

$$\begin{aligned} I + \mathcal{K}\hat{g}_\Lambda(s) &= I + (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\left[s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D} - \mathcal{G}_\Lambda\mathcal{D}\right] \\ &= (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\left[R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}\right] \\ &= (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}R_-^{1/2}\phi(s) \Longrightarrow (I + \mathcal{K}\hat{g}_\Lambda)^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U})). \end{aligned}$$

By Theorem 3.8, the operator $\mathcal{A}_c$ defined by (3.6), here coinciding with the operator $\mathcal{A}^c$ defined by (4.7), generates an **EXS** semigroup. □

5. APPLICATION TO LUR'E EQUATIONS

5.1. SOLVABILITY OF LUR'E EQUATIONS: THE CASE OF DIRECT CONTROL

Consider an infinite-dimensional *direct* control system in factor form

$$\dot{x}(t) = \mathcal{A}\left\{x(t) - df\left[c^{\#}x(t)\right]\right\} \tag{5.1}$$

where $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates a linear **EXS** C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H, $d \in D(c^{\#})$ is a factor control vector, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a generally nonlinear function describing the static characteristic of a feedback controller and $c^{\#}$ is $\mathcal{A}$-bounded linear functional, $h \in \mathrm{H}$, $h^* := c^{\#}\mathcal{A}^{-1}$,

The directional derivative of a quadratic form $V(x) = x^*\mathbf{H}x$ dictated by $\mathbf{H} \in \mathbf{L}(\mathrm{H})$, $\mathbf{H} = \mathbf{H}^*$ at $x \in \mathrm{H}$ in the direction of the RHS of (5.1), i.e.,

$$\mathcal{A}_f x := \mathcal{A}[x - df(c^{\#}x)], \quad D(\mathcal{A}_f) = \{x \in D(c^{\#}) : x - df(c^{\#}x) \in D(\mathcal{A})\}$$

reads as

$$V'(x;\mathcal{A}_f x) = \langle \mathcal{A}(x - df), \mathbf{H}x\rangle_{\mathrm{H}} + \langle \mathbf{H}x, \mathcal{A}(x - df)\rangle_{\mathrm{H}}.$$

By definition of $\mathcal{A}_f$, $x \in D(c^{\#})$ and $x - df(c^{\#}x)$, must be of the form $x - df(c^{\#}x) = \mathcal{A}^{-1}w$ for some $w \in \mathrm{H}$. Thus

$$V'(x;\mathcal{A}_f x) = \langle w, \mathbf{H}(\mathcal{A}^{-1}w + df)\rangle_{\mathrm{H}} + \langle \mathbf{H}(\mathcal{A}^{-1}w + df), w\rangle_{\mathrm{H}}.$$

Assume that $f(0) = 0$ and f satisfies the sector condition

$$k_1 < \frac{f(y)}{y} < k_2 \iff [k_2 y - f(y)]\,[f(y) - k_1 y] > 0, \quad y \neq 0.$$

Since

$$\begin{aligned}[k_2 y - f(y)]\,[f(y) - k_1 y] &= \left[k_2 c^{\#}x - f(c^{\#}x)\right]\left[f(c^{\#}x) - k_1 c^{\#}x\right] \\ &= \left[k_2 h^* w - (1 - k_2 c^{\#}d)f\right]\left[(1 - k_1 c^{\#}d)f - k_1 h^* w\right],\end{aligned}$$

then adding and subtracting $[k_2 y - f(y)]\,[f(y) - k_1 y]$ yields

$$\begin{aligned}&V'(x;\mathcal{A}_f x) \\ &= \begin{bmatrix} w^* \\ f \end{bmatrix} \begin{bmatrix} \mathbf{H}\mathcal{A}^{-1} + (\mathcal{A}^{-1})^*\mathbf{H} - qhh^* & \mathbf{H}d - eh \\ d^*\mathbf{H} - eh^* & -\delta \end{bmatrix} \begin{bmatrix} w \\ f \end{bmatrix} \\ &\quad - [k_2 y - f(y)]\,[f(y) - k_1 y],\end{aligned}$$

where

$$\delta := (1 - k_1 c^{\#}d)(1 - k_2 c^{\#}d) \geq 0, \quad q := k_1 k_2, \quad e := -\frac{k_1 + k_2}{2} + k_1 k_2 c^{\#}d.$$

Problem 5.1 (Kalman-Yacubovich-Popov (KYP) problem I)**.** Find real constants k_1 and $k_2 > k_1$ such that with the *Lur'e system*

$$\left\{ \begin{array}{r} \left(\mathcal{A}^{-1}\right)^* \mathbf{H} + \mathbf{H}\mathcal{A}^{-1} - qhh^* = -\mathbf{G}\mathbf{G}^* \\ -\mathbf{H}d + eh = -\sqrt{\delta}\mathbf{G} \end{array} \right\} \tag{5.2}$$

has a solution $(\mathcal{H}, \mathbf{G})$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $\mathbf{G} \in \mathrm{H}$.

If the above infinite-dimensional version of the *Kalman-Yacubovich-Popov* (*KYP*) *problem* has a solution then

$$V'(x; \mathcal{A}_f x) = -\left[k_2 y - f(y)\right]\left[f(y) - k_1 y\right] - \left[\mathbf{G}^* w - \sqrt{\delta} f(c^{\#} x)\right]^2 \leq 0.$$

Thus if, in addition, for any initial condition $x(0) = x_0 \in D(\mathcal{A}_f)$ the abstract differential equation (5.1) has an absolutely continuous (strong) solution $x = x(t)$ for $t \geq 0$ then, $\frac{\mathrm{d}}{\mathrm{d}t} V[x(t)] = V'(x; \mathcal{A}_f x(t)) \leq 0$ for almost all $t \geq 0$ and V will be a Lyapunov functional for (5.1). If only existence of a weak solution is known one can approximate it by a sequence of strong solution using density arguments and continuity of V – see [9] for more details.

Putting: $\mathbf{A} = \mathcal{A}^{-1}, \mathbf{B} = d, \mathbf{C} = h^*; \mathbf{Q} = q, \mathbf{N} = e, \mathbf{R} = \delta$ in Theorem 2.4 we conclude that if:

(**L1**) $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$ is uniformly bounded,
(**L2**) h^* is admissible,
(**L3**) $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+)$, where $\hat{G}(s) = h^*\left(sI - \mathcal{A}^{-1}\right)^{-1} d$ and
(**L4**) for some $\eta > 0$ there holds

$$\begin{aligned} \Pi(jw) &:= \mathbf{R} + 2\,\mathrm{Re}[\mathbf{N}^* \hat{G}(jw)] + [\hat{G}(jw)]^* \mathbf{Q} \hat{G}(jw) \\ &= \delta + 2e\,\mathrm{Re}[\hat{G}(jw)] + q|\hat{G}(jw)|^2 \geq \eta, \quad w \neq 0, \end{aligned} \tag{5.3}$$

then, there exists a solution $(\mathbf{H}, \mathbf{G})$ to (5.2) and the Popov spectral function Π has a spectral factorization $\Pi(j\omega)| = |\Theta(j\omega)|^2$, $\Theta, \Theta^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+)$,

$$\begin{aligned} \Theta(s) &= \mathbf{R}^{1/2} - \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B} = \sqrt{\delta} - \mathbf{G}^*(sI - \mathcal{A}^{-1})^{-1} d \\ &= \sqrt{\delta} + s^{-1}\mathbf{G}^*\mathcal{A}(s^{-1}I - \mathcal{A})^{-1} d. \end{aligned}$$

Moreover, if $q \leq 0$ then, recalling (2.21), one obtains

$$\begin{aligned} \mathbf{H} &:= \Psi^* \left[(\mathbf{Q}\mathbb{F} + \mathbf{N})\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*) - \mathbf{Q}\right] \Psi \\ &= \Psi^* \left[(q\mathbb{F} + eI)\mathcal{R}^{-1}(q\mathbb{F}^* + eI) - qI\right] \Psi \geq 0. \end{aligned}$$

By **EXS** of $\{S(t)\}_{t\geq 0}$, there holds $\sigma(\mathbf{A}) \cap j\mathbb{R} = \{0\}$ and $0 \in \sigma_C(\mathbf{A})$ because $\mathbf{A}$ has an unbounded but densely defined inverse $\mathcal{A}$; thus (2.15) and (2.16) are met.

Conditions (**L2**)–(**L4**) can be expressed in terms of the original system (5.1), and the transformed conditions are usually easier to check:

(**L2**) holds iff $c^{\#}$ is admissible with respect to $\{S(t)\}_{t\geq 0}$ (Lemma 4.2).

(**L3**) holds iff $\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+)$, $\hat{g}(s) = sc^\#(sI - \mathcal{A})^{-1}d - c^\# d$ is the transfer function of the linear part of (5.1) (Lemma 4.3).

Finally (**L4**) holds as (5.3) reduces to frequency-domain inequality of the *circle criterion - type* (Lemma 4.4),

$$\begin{aligned} \Pi(jw) &= \delta + 2e\,\mathrm{Re}\left[\hat{G}(jw)\right] + q\left|\hat{G}(jw)\right|^2 \geq \eta \quad \forall w \neq 0 \\ &\overset{w=1/\omega}{\Longleftrightarrow} \pi(j\omega) := \delta - 2e\,\mathrm{Re}\left[\hat{g}(j\omega) + c^\# d\right] + q\left|\hat{g}(j\omega) + c^\# d\right|^2 \\ &= 1 + (k_1 + k_2)\,\mathrm{Re}\left[\hat{g}(j\omega)\right] + k_1 k_2 \left|\hat{g}(j\omega)\right|^2 \geq \eta \quad \forall \omega \in \mathbb{R}, \end{aligned} \tag{5.4}$$

which, in the case where $q < 0$, means geometrically that the *Nyquist plot* $\{\hat{g}(j\omega)\}_{\omega\in\mathbb{R}}$ is strictly inside the ball with center $-\frac{1}{2}(k_2^{-1} + k_1^{-1})$ and radius $\frac{1}{2}(k_2^{-1} - k_1^{-1})$.

5.2. EXAMPLE: HEATING OF A ROD

The dynamics of a plant, depicted in Figure 2 is governed by the controlled heat equation with observation

$$\left\{\begin{array}{ll} q_t(\theta,t) = aq_{\theta\theta}(\theta,t) - R_a q(\theta,t), & t \geq 0, \quad 0 \leq \theta \leq 1 \\ q_\theta(1,t) = 0, & t \geq 0, \\ q_\theta(0,t) = u(t), & t \geq 0, \\ y(t) = q(1,t), & t \geq 0, \end{array}\right\}$$

where $q(\theta,t)$ stands for the temperature at point θ and at time t, a is the *thermal diffusivity* and R_a is the *heat exchange coefficient* between the rod and its outside. Here the control servomechanism is a heater which steers the temperature gradient at the left end while the output is temperature measurement by a sensor located at the right end.

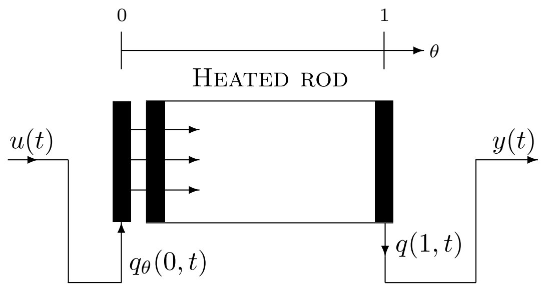

Fig. 2. The heating of a rod system

In the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product, the open-loop system dynamics can be written in the preliminary abstract form

$$\left\{\begin{array}{l} \dot{x} = \sigma x \\ \tau x = u \\ y = c^\# x \end{array}\right\} \tag{5.5}$$

with

$$\sigma x = ax'' - R_a x, \qquad D(\sigma) = \left\{x \in \mathrm{H}^2(0,1):\ x'(1) = 0\right\},$$
$$\tau x = x'(0), \qquad D(\tau) = \mathrm{C}^1[0,1] \supset D(\sigma)$$

and σ is a closed linear operator; the observation functional $c^{\#}$ is given by

$$c^{\#}x = x(1), \quad D(c^{\#}) = \mathrm{C}[0,1].$$

From the relationships: $d \in D(\sigma)$, $\sigma d = 0$, $\tau d = -1$ we find a factor control vector d,

$$\left\{\begin{array}{r} d''(\theta) - \mu d(\theta) = 0 \\ d'(1) = 0 \\ d'(0) = 1 \end{array}\right\} \iff d(\theta) = \frac{\cosh\sqrt{\mu}(1-\theta)}{\sqrt{\mu}\sinh\sqrt{\mu}}, \quad \theta \in [0,1]; \quad \mu := \frac{R_a}{a}.$$

Thanks to this

$$\tau[x(t) + du(t)] = \tau x(t) + \tau du(t) = \tau x(t) - u(t) = 0,$$

i.e., $x(t) = du(t) \in \ker\tau$. Next,

$$\dot{x}(t) = \sigma x(t) = \sigma x(t) + \sigma du(t) = \sigma[x(t) + du(t)] = \mathcal{A}[x(t) + du(t)],$$

provided that $\mathcal{A} := \sigma|_{\ker\tau}$, here given by

$$\mathcal{A}x = ax'' - R_a x, \quad D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1):\ x'(1) = 0,\ x'(0) = 0\}.$$

Consequently, $c^{\#}\big|_{D(\mathcal{A})} = h^*\mathcal{A}$, where $H = h$,

$$\left\{\begin{array}{r} h''(\theta) - \mu h(\theta) = 0 \\ h'(1) = -1/a \\ h'(0) = 0 \end{array}\right\} \iff h(\theta) = -\frac{\cosh\sqrt{\mu}\theta}{a\sqrt{\mu}\sinh\sqrt{\mu}}, \quad \theta \in [0,1],$$

and (5.5) is being reduced to (3.1) with $\mathcal{C} = c^{\#}$, $H = h$ and $\mathcal{D} = d$.

Since $\mathcal{A} = \mathcal{A}^* \leq -R_a I$ it generates an analytic **EXS** self-adjoint semigroup $\{S(t)\}_{t\geq 0}$ on H. $\mathcal{A}^{-1}$ is a compact operator, whence, by the discrete version of spectral theorem, $\mathcal{A}$ has countably many eigenvalues, namely:

$$\lambda_0 = -R_a, \quad \lambda_k = -ak^2\pi^2 - R_a, \quad k \in \mathbb{N}$$

and the corresponding sequence of eigenvectors

$$e_0 = 1, \quad e_k = \sqrt{2}\cos k\pi\theta, \quad k \in \mathbb{N}$$

forms an orthonormal basis (ONB) of H.

Passing to the question of solvability of Problem 5.1, we observe that (**L1**) holds as

$$\|e^{t\mathcal{A}^{-1}}x_0\|^2_{\mathrm{H}} = \sum_{n=0}^{\infty} e^{-2t/\lambda_n}|\langle x_0, e_k\rangle|^2_{\mathrm{H}} \leq \sum_{n=0}^{\infty} |\langle x_0, e_k\rangle|^2_{\mathrm{H}} = \|x_0\|^2_{\mathrm{H}}, \quad x_0 \in \mathrm{H}. \tag{5.6}$$

In fact we have more, by the Weierstrass test the first series in (5.6) uniformly converges to $\|e^{t\mathcal{A}^{-1}}x_0\|^2_{\mathrm{H}}$ on $[0,\infty)$ and because $\operatorname{Re}\lambda_n < 0$ $(n = 0,1,2,\ldots)$ its partial sums treated as function of t belong to $\mathrm{BUC}_0[0,\infty)$, whence $\|e^{(\cdot)\mathcal{A}^{-1}}x_0\|^2_{\mathrm{H}}$ is also in $\mathrm{BUC}_0[0,\infty)$.

Next,

$$c^{\#}e_0 = 1, \quad c^{\#}e_k = \sqrt{2}(-1)^k, \quad k \in \mathbb{N}$$

and because

$$f \in \mathrm{L}^2(0,\infty) \iff \hat{f} \in \mathrm{H}^2(\mathbb{C}^+) \Longrightarrow \sqrt{2\operatorname{Re}s}\;|\hat{f}(s)| \le \|f\|_{\mathrm{L}^2(0,\infty)},$$

for any $f \in \mathrm{L}^2(0,\infty)$ we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}|c^{\#}e_k|^2|\hat{f}(-\lambda_k)|^2 = |\hat{f}(-\lambda_0)|^2 + 2\sum_{k=1}^{\infty}\hat{f}(-\lambda_k)|^2\\
&\le \frac{1}{2R_a}\|f\|^2_{\mathrm{L}^2(0,\infty)} + \sum_{k=1}^{\infty}\frac{1}{ak^2\pi^2+R_a}\|f\|^2_{\mathrm{L}^2(0,\infty)}\\
&\le \frac{1}{2R_a}\|f\|^2_{\mathrm{L}^2(0,\infty)} + \frac{1}{a}\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}\|f\|^2_{\mathrm{L}^2(0,\infty)} \le \max\left\{\frac{1}{R_a},\frac{1}{3a}\right\}\|f\|^2_{\mathrm{L}^2(0,\infty)}.
\end{aligned}$$

Hence $c^{\#}$ is an admissible observation functional by the *spectral criterion* of admissibility [5] and the bibliography therein; (**L2**) is satisfied.

Similar arguments shows that $d^{\#}$,

$$d^{\#}x = -ax(0), \quad D(d^{\#}) = \mathrm{C}[0,1],$$

an extension of $d^*\mathcal{A}$ is an admissible observation functional, whence by Lemma 3.4, d is an admissible factor control vector, where now

$$d^{\#}e_0 = -a, \quad d^{\#}e_k = -a\sqrt{2}, \quad k \in \mathbb{N}.$$

Here, the transfer function equals

$$\hat{g}(s) = sc^{\#}(sI-\mathcal{A})^{-1}d - c^{\#}d = -\frac{1}{\sqrt{\dfrac{s+R_a}{a}}\sinh\sqrt{\dfrac{s+R_a}{a}}}.$$

This formula has been found using MAPLE-aided manipulations and can be easily confirmed by the fractional expansion of $z \mapsto \frac{\pi}{\sinh \pi z}$ [12, Problem 5.2.5 with $z = \frac{1}{\pi}\sqrt{\frac{s+R_a}{a}}$]:

$$\begin{aligned}
\frac{1}{\sqrt{\dfrac{s+R_a}{a}}\sinh\sqrt{\dfrac{s+R_a}{a}}} &= 2\sum_{k=1}^{\infty}\frac{(-1)^k a}{(a\pi^2k^2+R_a)+s} + \frac{a}{s+R_a}\\
&= 2\sum_{k=1}^{\infty}\frac{(-1)^k a}{s-\lambda_k} + \frac{a}{s-\lambda_0}\\
&= -\sum_{k=1}^{\infty}\frac{d^{\#}e_k c^{\#}e_k}{s-\lambda_k} - \frac{d^{\#}e_0 c^{\#}e_0}{s-\lambda_0}\\
&= -c^{\#}\mathcal{A}(sI-\mathcal{A})^{-1}d = -\hat{g}(s).
\end{aligned}$$

From this expansion and that of $z \mapsto \pi\coth\pi z$ [12, Problem 5.2.5, with $z = \frac{1}{\pi}\sqrt{\mu}$] we get

$$\begin{aligned}
|\hat{g}(s)| &\le \frac{a}{|s+R_a|} + \sum_{k=1}^{\infty}\frac{2a}{|(a\pi^2k^2+R_a)+s|}\\
&\le \frac{a}{R_a} + \sum_{k=1}^{\infty}\frac{2a}{a\pi^2k^2+R_a} = \frac{\coth\sqrt{\mu}}{\sqrt{\mu}}, \qquad s\in\mathbb{C}^+,
\end{aligned}$$

i.e., (3.5) is met, and (**L3**) holds true.

For a laboratory model the constants have been identified as

$$a = 0.00092054875946,\ R_a = 0.02719439502498 \Longrightarrow \mu = 29.54150417;$$

the corresponding circle-type frequency-domain inequality (5.4) is graphically verified in Figures 3, 4 which, with an aid of numerical computations, yields: (**L4**) is met for

$$k_1 = -897.56 < 0, \quad 0 < k_2 = 623.24 < -1/\hat{G}(0) = 623.2461689.$$

By the result of Section 5.1, Problem 5.1 with those k_1 and k_2, has a desired solution, i.e., there exists $(\mathbf{H},\mathbf{G})$, $\mathbf{H}\in\mathbf{L}(\mathrm{H})$, $\mathbf{H}=\mathbf{H}^*\ge 0$, $\mathbf{G}\in\mathrm{H}$ solving (5.2) in which

$$\mathbf{R} = \delta = 0.00002415247812 > 0, \quad q = -559395.2944 < 0, \quad e = -760.3911159.$$

We shall give a *modal approximation* of this solution.

Recall that the mapping, induced by ONB $\{e_k\}_{k\in\mathbb{Z}^*}$ of eigenvectors of $\mathcal{A}$,

$$\mathrm{H}\ni x \longmapsto x_\infty \in \ell^2(\mathbb{Z}^*), \quad x_\infty := [e_k^* x]_{k\in\mathbb{Z}^*}, \quad \mathbb{Z}^* := \{0\}\cup\mathbb{N},$$

is a unitary isomorphism of H onto $\ell^2(\mathbb{Z}^*)$. In particular,

$$h_\infty := [e_k^* h]_{k\in\mathbb{Z}^*} = \left\{\begin{array}{cl} \dfrac{1}{\lambda_0} & \text{if} \quad k=0 \\ \dfrac{(-1)^k\sqrt{2}}{\lambda_k} & \text{if} \quad k\in\mathbb{N} \end{array}\right\}.$$

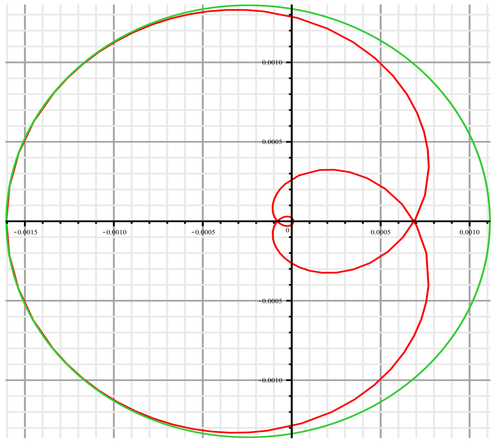

Fig. 3. Graphical verification of the circle criterion (5.4)

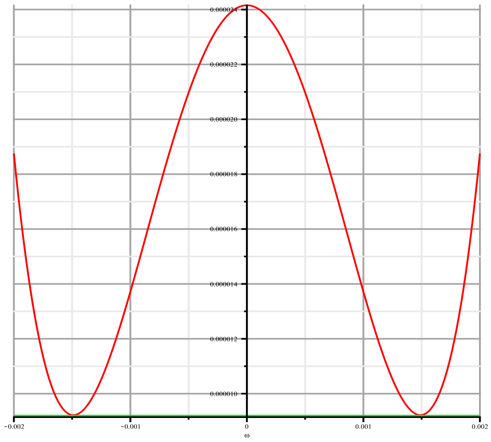

Fig. 4. Graphical verification of (5.4): $\pi(j\omega) > 0.0000092$

$$d_\infty := [e_k^* d]_{k\in\mathbb{Z}^*} = \left\{ \begin{array}{ll} \dfrac{-a}{\lambda_0} & \text{if} \quad k=0 \\ \dfrac{-a\sqrt{2}}{\lambda_k} & \text{if} \quad k\in\mathbb{N} \end{array} \right\}, \quad \mathbf{G}_\infty = [e_k^* \mathbf{G}]_{k\in\mathbb{Z}^*}\,.$$

Under the isomorphism above the infinite-matrix representations $\mathcal{A}_\infty^{-1}, \mathbf{H}_\infty \in \mathbf{L}(\ell^2(\mathbb{Z}^*))$ of operators $\mathcal{A}^{-1}$ and $\mathbf{H} \in \mathrm{L}(\mathrm{H})$ are, respectively:

$$\mathcal{A}_\infty^{-1} := \operatorname{diag}\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots\right\}, \quad \mathbf{H}_\infty = [e_k^* \mathbf{H} e_n]_{k,n\in\mathbb{Z}^*} = \mathbf{H}_\infty^*.$$

Premultiplying and postmultiplying the first equation of (5.2) by, respectively, e_k^* and e_n and then premultiplying the second equation of (5.2) by e_k^* and expanding d with respect to the basis $\{e_k\}_{k\in\mathbb{Z}^*}$ we get

$$\left\{ \begin{array}{c} \left(\mathcal{A}_\infty^{-1}\right)^* \mathbf{H}_\infty + \mathbf{H}_\infty \mathcal{A}_\infty^{-1} = q h_\infty h_\infty^* - \mathbf{G}_\infty \mathbf{G}_\infty^* \\ -\mathbf{H}_\infty d_\infty + e h_\infty = -\sqrt{\delta}\mathbf{G}_\infty \end{array} \right\}.$$

It is not difficult to see that truncations

$$\mathbf{H}_N := [e_k^* \mathbf{H} e_n]_{k,n=1,2,\dots N}\,, \quad \mathbf{G}_N := [e_k^* \mathbf{G}]_{k=1}^N$$

of $\mathbf{H}_\infty$ and $\mathbf{G}_\infty$ satisfy the matrix Lur'e system

$$\left\{ \begin{array}{c} \mathcal{A}_N^{-1} \mathbf{H}_N + \mathbf{H}_N \mathcal{A}_N^{-1} = q h_N h_N^* - \mathbf{G}_N \mathbf{G}_N^* \\ -\mathrm{H}_N d_N + e h_N = -\sqrt{\delta}\mathbf{G}_N \end{array} \right\}, \tag{5.7}$$

where $\mathcal{A}_N^{-1}$, d_N and h_N are the truncations of $\mathcal{A}_\infty^{-1}$, d_∞ and h_∞:

$$\mathcal{A}_N^{-1} := \operatorname{diag}\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_N}, \right\}, \quad d_N = [e_k^* d]_{k=1}^N\,, \quad h_N = [e_k^* h]_{k=1}^N\,.$$

(5.7) can be solved numerically using `care.m` procedure built in MATLAB/CONTROL TOOLBOX.

Theorem 5.2. *A solution* $(\mathbf{H}, \mathbf{G})$ *of the Lur'e system* (5.2)*, the existence of which is guaranteed by Theorem 2.4, has the following regularity properties:*

(i) $\mathbf{G} \in D(\mathcal{A}^{3/4-\varepsilon})$*, whence in particular the Fourier expansion*

$$\mathbf{G}(\theta) = \sum_{k=0}^{\infty} \mathbf{G}^* e_k \, e_k(\theta), \quad 0 \le \theta \le 1,$$

uniformly converges to $\mathbf{G}$*;* $\varepsilon \in [0, \frac{3}{4})$.

(ii) $\mathbf{H}$ *is a nuclear Hilbert-Schmidt operator given by*

$$(\mathbf{H}x)(\theta) = \int_0^1 k(\theta, \sigma) x(\sigma) d\sigma, \quad x \in \mathrm{H}, \tag{5.8}$$

with a symmetric kernel $k \in \mathrm{L}^2((0,1) \times (0,1))$.

Proof. Since $c^{\#}$ is admissible, then the *observability gramian* $\mathbf{H}_h = \mathbf{\Psi}^* \mathbf{\Psi} \in \mathbf{L}(\mathrm{H})$, $\mathbf{H}_h = \mathbf{H}_h^* \ge 0$, $\mathbf{H}_h$ satisfies the (equivalent) Lyapunov operator equations:

$$\begin{aligned} \langle \mathcal{A}x, \mathbf{H}_h x \rangle_{\mathrm{H}} + \langle x, \mathbf{H}_h \mathcal{A}x \rangle_{\mathrm{H}} &= -(c^{\#}x)^2, \quad x \in D(\mathcal{A}), \\ (\mathcal{A}^{-1})^* \mathbf{H}_h + \mathbf{H}_h (\mathcal{A}^{-1}) &= -hh^*. \end{aligned}$$

$\mathbf{H}$ can be represented as $\mathbf{H} = \mathbf{H}_{\mathbf{G}} - q\mathbf{H}_h$, where $\mathbf{H}_{\mathbf{G}} \in \mathbf{L}(\mathrm{H})$, $\mathbf{H}_{\mathbf{G}} = \mathbf{H}_{\mathbf{G}}^* \ge 0$, and the pair $(\mathbf{H}_{\mathbf{G}}, \mathbf{G})$ solves the system

$$\left\{ \begin{array}{c} \left(\mathcal{A}^{-1}\right)^* \mathbf{H}_{\mathbf{G}} + \mathbf{H}_{\mathbf{G}}(\mathcal{A}^{-1}) = -\mathbf{G}\mathbf{G}^* \\ -\mathbf{H}_{\mathbf{G}} d + (q\mathbf{H}_h d + eh) = -\sqrt{\delta}\mathbf{G} \end{array} \right\}. \tag{5.9}$$

Observe that

$$\begin{aligned} q\langle d, \mathbf{H}_h e_k \rangle_{\mathrm{H}} &= q\langle d, \mathbf{\Psi}^* \mathbf{\Psi} e_k \rangle_{\mathrm{H}} = q\langle \mathbf{\Psi} d, \mathbf{\Psi} e_k \rangle_{\mathrm{L}^2(0,\infty)} \\ &= q \left\langle \mathbf{\Psi} d, c^{\#} e_k e^{\lambda_k(\cdot)} \right\rangle_{\mathrm{L}^2(0,\infty)} = q \left\langle \mathbf{\Psi} d, h^* e_k \lambda_k e^{\lambda_k(\cdot)} \right\rangle_{\mathrm{L}^2(0,\infty)} \\ &= -qh^* e_k \ (-\lambda_k) \left(\widehat{\mathbf{\Psi} d}\right)(-\lambda_k) = -qh^* e_k \left[\widehat{g}(-\lambda_k) + c^{\#} d\right]. \end{aligned} \tag{5.10}$$

From Theorem 4.5 we know that $\mathbf{G}^* \mathcal{A}$ is admissible with respect to $\{S(t)\}_{t \ge 0}$. Therefore $\mathbf{H}_{\mathbf{G}} = \mathbf{\Psi}_{\mathbf{G}}^* \mathbf{\Psi}_{\mathbf{G}}$, where $\mathbf{\Psi}_{\mathbf{G}}$ denotes its observability map, $(\mathbf{\Psi}_{\mathbf{G}}|_{D(\mathcal{A})} \, x)(t) := \mathbf{G}^* \mathcal{A} S(t) x$. Now

$$\begin{aligned} -\langle d, \mathbf{H}_{\mathbf{G}} e_k \rangle_{\mathrm{H}} &= -\langle d, \mathbf{\Psi}_{\mathbf{G}}^* \mathbf{\Psi}_{\mathbf{G}} e_k \rangle_{\mathrm{H}} = -\left\langle \mathbf{\Psi}_{\mathbf{G}} d, \mathbf{G}^* e_k \lambda_k e^{\lambda_k(\cdot)} \right\rangle_{\mathrm{L}^2(0,\infty)} \\ &= \mathbf{G}^* e_k \ (-\lambda_k) (\widehat{\mathbf{\Psi}_{\mathbf{G}} d})(-\lambda_k) = \mathbf{G}^* e_k \left[\phi(-\lambda_k) - \sqrt{\delta}\right]. \end{aligned} \tag{5.11}$$

The last equality holds as, still by Theorem 4.5, $\phi(s) = \sqrt{\delta} + s\mathbf{G}^*\mathcal{A}(sI - \mathcal{A})^{-1}d$ is a spectral factor of the (coercive) Popov spectral function π satisfying ϕ, $\frac{1}{\phi} \in \mathrm{H}^\infty(\mathbb{C}^+)$. It follows from the second equation of (5.9), (5.10) and (5.11) that

$$\begin{aligned} -\sqrt{\delta}\mathbf{G}^* e_k &= -d^*\mathbf{H}_\mathbf{G} e_k + qd^*\mathbf{H}_h e_k + eh^* e_k \\ &= \mathbf{G}^* e_k \left[\phi(-\lambda_k) - \sqrt{\delta}\right] - qh^* e_k \left[\widehat{g}(-\lambda_k) + c^\# d\right] + eh^* e_k, \end{aligned}$$

whence by ϕ, $1/\phi \in \mathrm{H}^\infty$ and $qc^\# d - e = \dfrac{k_1 + k_2}{2} = -137.16$,

$$\mathbf{G}^* e_k = \frac{q\hat{g}(-\lambda_k) + [qc^\# d - e]}{\phi(-\lambda_k)}\, h^* e_k. \tag{5.12}$$

Hence (third inequality below holds as $\hat{g}$ increases for $s > 0$)

$$\begin{aligned} |\mathbf{G}^* e_k| &\le m\, |h^* e_k|\,, \quad m := \left\|\frac{1}{\phi}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+)} \left[|q|\, \|\hat{g}\|_{\mathrm{H}^\infty(\mathbb{C}^+)} + \left|\frac{k_1 + k_2}{2}\right|\right], \\ |h^* e_k| &\le |\mathbf{G}^* e_k|\, \frac{1}{|q|} \|\phi\|_{\mathrm{H}^\infty(\mathbb{C}^+)} \left|\hat{g}(-\lambda_k) + \frac{1}{2}\left[\frac{1}{k_1} + \frac{1}{k_2}\right]\right|^{-1}, \\ \frac{1}{2}\left[\frac{1}{k_1} + \frac{1}{k_2}\right] + \hat{g}(-\lambda_k) &\ge \frac{1}{2}\left[\frac{1}{k_1} + \frac{1}{k_2}\right] + \hat{g}(R_\mathrm{a}) > 0.0001 \end{aligned}$$

and the sequences $\{\mathbf{G}^* e_k\}_{k\in\mathbb{Z}^*}$, $\{h^* e_k\}_{k\in\mathbb{Z}^*}$ have the same asymptotic behaviour up to a nonzero constant. Thus $\mathbf{G} \in D[(-\mathcal{A})^{3/4-\epsilon}]$, because

$$h \in D[(-\mathcal{A})^{3/4-\epsilon}] \iff \sum_{k=0}^{\infty} |\lambda_k|^{3/2-2\epsilon}\, |h^* e_k|^2 < \infty,$$

where ϵ is an arbitrary small positive number.

To examine regularity of $\mathbf{H}$ we observe that

$$\sum_{n=0}^{\infty} \|\mathbf{\Psi} e_n\|^2_{\mathrm{L}^2(0,\infty)} = \sum_{n=0}^{\infty} \|e^{\lambda_n(\cdot)} c^\# e_n\|^2_{\mathrm{L}^2(0,\infty)} = \sum_{n=0}^{\infty} \frac{1}{-\lambda_n} < \infty,$$

and similarly, with aid of the estimate after (5.12),

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mathbf{\Psi}_\mathbf{G} e_n\|^2_{\mathrm{L}^2(0,\infty)} &= \sum_{n=0}^{\infty} \|e^{\lambda_n(\cdot)} \mathbf{G}^* \mathcal{A} e_n\|^2_{\mathrm{L}^2(0,\infty)} \\ &= m^2 \sum_{n=0}^{\infty} \|e^{\lambda_n(\cdot)}\|^2_{\mathrm{L}^2(0,\infty)}\, |\lambda_k h^* e_k|^2 \le m^2 \sum_{n=0}^{\infty} \frac{1}{-\lambda_n} < \infty. \end{aligned}$$

Hence $\mathbf{\Psi}, \mathbf{\Psi}_{\mathbf{G}}$ are *Hilbert-Schmidt* operators. Consequently $\mathbf{\Psi}^*, \mathbf{\Psi}^*_{\mathbf{G}} \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$ are Hilbert-Schmidt operators [20, Theorem 6.9, p. 136]. Next, applying [20, Corollary, p. 138] and [16, Proposition 1.10, p.11] we conclude that $\mathbf{H}_h = \mathbf{\Psi}^*\mathbf{\Psi}$ and $\mathbf{H}_{\mathbf{G}} = \mathbf{\Psi}^*_{\mathbf{G}}\mathbf{\Psi}_{\mathbf{G}}$ are *nuclear* (or *trace*) Hilbert-Schmidt operators. Finally $\mathbf{H}$ is a nuclear Hilbert-Schmidt operator as a linear combination of $\mathbf{H}_h$ and $\mathbf{H}_{\mathbf{G}}$. In particular this implies [20, Theorem 6.11, p. 139] that $\mathbf{H}$ is an *integral* operator (5.8) with a *symmetric kernel* $k \in \mathrm{L}^2((0,1)^2)$. □

By Theorem 5.2 (i), the linear functional

$$x \longmapsto \mathbf{R}^{1/2}\mathbf{G}^*\mathcal{A}x = \sqrt{\delta}\langle \mathcal{A}x, \mathbf{G}\rangle_{\mathrm{H}} = -\sqrt{\delta}\langle(-\mathcal{A})^{1/4+\varepsilon}x, (-\mathcal{A})^{3/4-\varepsilon}\mathbf{G}\rangle_{\mathrm{H}}$$

extends to a bounded linear functional $\mathcal{G}_\Lambda$ on $D[(-\mathcal{A})^{1/4+\varepsilon}]$, but $d \in D[(-\mathcal{A})^{3/4-\varepsilon}] \subset D[(-\mathcal{A})^{1/4+\varepsilon}]$; the inclusion holds for $\varepsilon \leq \frac{1}{4}$.

Similarly, the functional

$$x \longmapsto \langle \mathcal{A}x, d\rangle_{\mathrm{H}} = -\langle(-\mathcal{A})^{1/4+\varepsilon}x, (-\mathcal{A})^{3/4-\varepsilon}d\rangle_{\mathrm{H}}$$

extends to a bounded linear functional $d^{\#}$ on $D[(-\mathcal{A})^{1/4+\varepsilon}]$, but $\mathbf{G} \in D[(-\mathcal{A})^{3/4-\varepsilon}] \subset D[(-\mathcal{A})^{1/4+\varepsilon}]$; the inclusion holds for $\varepsilon \leq \frac{1}{4}$. Thus, taking $\varepsilon = \frac{1}{4}$ one obtains

$$\begin{aligned} d^{\#}\mathbf{G}\mathbf{R}^{1/2} &= \mathbf{G}\sqrt{\delta} \\ &= -\sqrt{\delta}\langle(-\mathcal{A})^{1/2}\mathbf{G}, (-\mathcal{A})^{1/2}d\rangle_{\mathrm{H}} \\ &= -\sqrt{\delta}\langle(-\mathcal{A})^{1/2}d, (-\mathcal{A})^{1/2}\mathbf{G}\rangle_{\mathrm{H}} = \mathcal{G}_\Lambda d. \end{aligned}$$

Hence the transfer function of optimal control system (4.13) reads as

$$\begin{aligned} \hat{g}_\Lambda(s) &= sR_-^{1/2}\mathbf{G}^*\mathcal{A}(sI-\mathcal{A})^{-1}d - \mathcal{G}_\Lambda d \\ &= \sqrt{\delta}\langle[I - s(sI-\mathcal{A})^{-1}](-\mathcal{A})^{1/2}d, (-\mathcal{A})^{1/2}\mathbf{G}\rangle_{\mathrm{H}}. \end{aligned}$$

$$s = Re^{\pm j\varphi} \Longrightarrow [s(sI-\mathcal{A})^{-1} - I](-\mathcal{A})^{1/2}d = [R(RI - e^{\mp j\varphi}\mathcal{A})^{-1} - I](-\mathcal{A})^{1/2}d \quad (5.13)$$

If $\varphi \in (0, \frac{\pi}{2}]$ then, by [2, Proposition 3.9.1 and Remark 3.9.3, pp. 171–172], $e^{\mp j\varphi}\mathcal{A}$ generate bounded C_0-semigroups, whence the RHS of (5.13) tends to 0 as $R \to \infty$ and consequently $\hat{g}_\Lambda(s) \longrightarrow 0$ as $|s| \to \infty$, $\operatorname{Re} s \geq 0$. Analogous arguments apply to $\hat{g}$ as well, giving $\hat{g}(s) \longrightarrow 0$ as $|s| \to \infty$, $\operatorname{Re} s \geq 0$ and $\pi(\mp j\infty) = |\phi(\mp j\infty)|^2 = 1$. Taking this into account in (4.13) we conclude that $R_- + \mathcal{G}_\Lambda d \neq 0$. All assumptions of Theorem 4.5 hold and by its assertion the semigroup generated by $\mathcal{A}^c$ is **EXS**.

5.3. NUMERICAL EXPERIMENT

For $k_1 = -897.56$ and $k_2 = 623.24$ a simple m-script has been written and executed under MATLAB to calculate $N = 100$-dimensional truncation $\mathbf{G}_{100}$ of $\mathbf{G}_\infty$. The `care.m` procedure from MATLAB/CONTROL TOOLBOX was used to find $\mathbf{G}_{100}$ and $\mathbf{H}_{100}$. Having an access to $\mathbf{G}_{100}$ we can approximate $\mathbf{G}$ using $\mathbf{G}_N^T e_N(\cdot)$.

The result of computations are depicted in Figures 5, 6 and 7: the Fourier expansions of h, d and $\mathbf{G}$ are uniformly convergent. $\mathbf{G}$ is satisfactory approximated. The spectra of the open-loop and closed-loop systems are depicted in Figure 8. The closed-loop (optimal) system has approximately 20 pairs of complex eigenvalues which means that the optimal process is of oscillatory type.

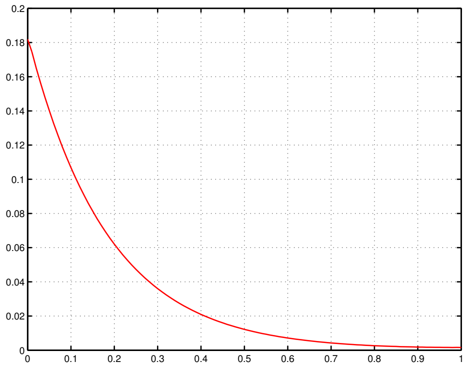

Fig. 5. Fourier series approximation of d

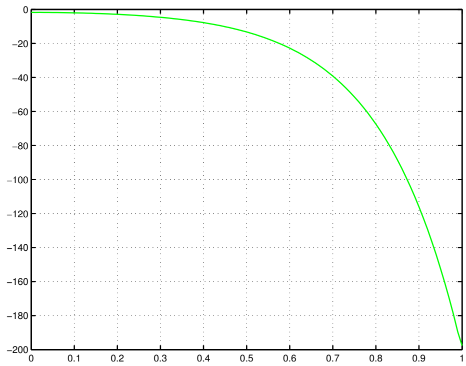

Fig. 6. Fourier series approximation of h

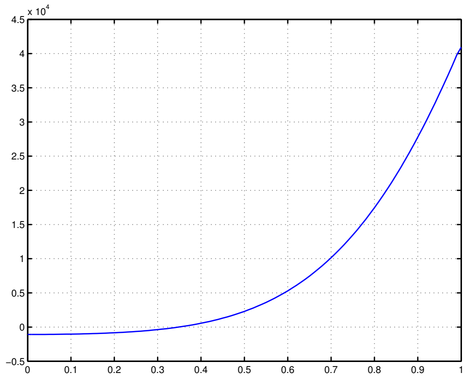

Fig. 7. Fourier series approximation of $\mathbf{G}$

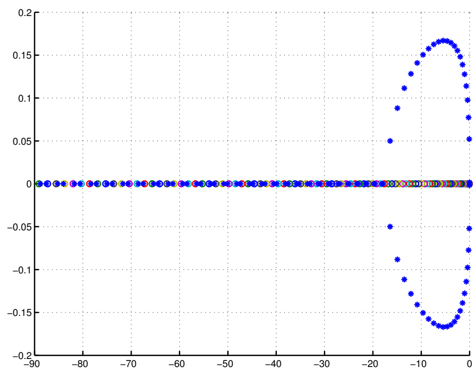

Fig. 8. Spectra of open ($\circ$)/closed ($*$)-loop systems

Since

$$x^*\mathbf{H}x = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^* e_m \mathbf{H}_\infty e_m^* x = x^* \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e_m \mathbf{H}_\infty e_m^* x,$$

then comparing the last term with (5.8) we obtain

$$k(\theta,\sigma) = \begin{bmatrix} e_1(\theta) \\ e_2(\theta) \\ \ldots \\ e_m(\theta) \\ \ldots \end{bmatrix}^T \mathbf{H}_\infty \begin{bmatrix} e_1(\sigma) \\ e_2(\sigma) \\ \ldots \\ e_n(\sigma) \\ \ldots \end{bmatrix} \tag{5.14}$$

in almost all points of the square $[0,1]^2$, but one cannot expect that k is bounded as confirmed by Figure 9.

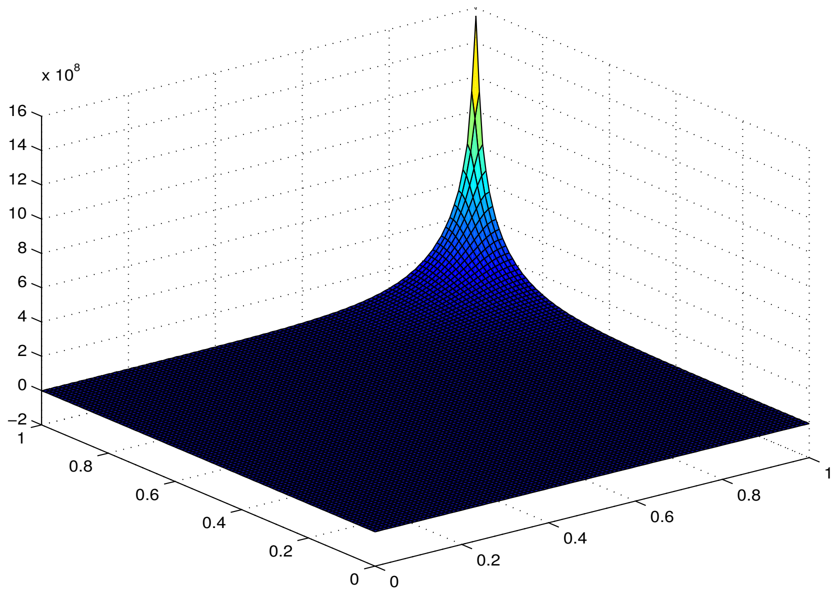

Fig. 9. Plot of the kernel k approximated with the aid of (5.14)

Numerical algorithms going beyond MATLAB are recently discussed in [13] and references therein.

5.4. SOLVABILITY OF LUR'E EQUATIONS: CASE OF THE INDIRECT CONTROL

Consider an infinite-dimensional *indirect* control system in factor form

$$\left\{ \begin{array}{l} \dot{x}(t) = \mathcal{A}\left\{x(t) + df\left[\sigma(t)\right]\right\} \\ \dot{\sigma}(t) = c^{\#}x(t) - \rho f\left[\sigma(t)\right] \end{array} \right\}, \tag{5.15}$$

where $\mathcal{A}$, d, f, $c^{\#}$ and h are as in Section 5.1, $\rho + c^{\#}d > 0$. Repeating, with appropriate modifications the analysis of Section 5.1, we can represent the directional derivative of a functional (compare with [11, p. 201] where the case $\mathrm{H} = \mathbb{R}^n$ was examined)

$$V(x,\sigma) = x^*\mathbf{H}x + \frac{1}{2(\rho + c^{\#}d)}\left[\sigma - h^*x\right]^2 + q\int_0^{\sigma} f(\xi)\mathrm{d}\xi$$

in the direction of the RHS of (5.15) as

$$\begin{aligned}
&V'\left(\begin{bmatrix} x \\ \sigma \end{bmatrix};\begin{bmatrix} \mathcal{A}(x+df) \\ c^{\#}x-\rho f \end{bmatrix}\right)\\
&= \langle w, \mathbf{H}(\mathcal{A}^{-1}w-df)\rangle_{\mathrm{H}} + \langle \mathbf{H}(\mathcal{A}^{-1}w-df), w\rangle_{\mathrm{H}}\\
&\quad -\sigma f + h^*(\mathcal{A}^{-1}w-df)f + qf\left[c^{\#}(\mathcal{A}^{-1}w-df)-\rho f\right].
\end{aligned}$$

Assume that $f(0)=0$ and f satisfies the sector condition

$$0<\frac{f(\sigma)}{\sigma}<k\le\infty \iff \left[\sigma-\frac{1}{k}f(\sigma)\right]f(\sigma)>0, \quad y\neq 0.$$

Adding and subtracting $\frac{1}{k}f^2(\sigma)$ we get

$$\begin{aligned}
&V'\left(\begin{bmatrix} x \\ \sigma \end{bmatrix};\begin{bmatrix} \mathcal{A}(x+df) \\ c^{\#}x-\rho f \end{bmatrix}\right)\\
&= -\left[\sigma-\frac{1}{k}f(\sigma)\right]f(\sigma)\\
&\quad+\begin{bmatrix} w^* \\ f \end{bmatrix}\begin{bmatrix} \mathbf{H}(\mathcal{A}^{-1})^*+\mathcal{A}^{-*}\mathbf{H} & -\mathbf{H}d+\frac{1}{2}[\mathcal{A}^{-1}+qI]^*h \\ -d^*\mathbf{H}+\frac{1}{2}h^*[\mathcal{A}^{-1}+qI] & -\delta \end{bmatrix}\begin{bmatrix} w \\ f \end{bmatrix},
\end{aligned}$$

where

$$\delta := \frac{1}{k}+h^*d+q(\rho+c^{\#}d)\ge 0.$$

Problem 5.3 (Kalman-Yacubovich-Popov (KYP) problem II)**.** Find real constants $k>0$ and $q\ge 0$ such that the *Lur'e system*

$$\left\{\begin{aligned} \mathbf{H}\mathcal{A}^{-1}+\left(\mathcal{A}^{-1}\right)^*\mathbf{H} &= -\mathbf{G}\mathbf{G}^* \\ -\mathbf{H}d+\tfrac{1}{2}[\mathcal{A}^{-1}+qI]^*h &= -\sqrt{\delta}\,\mathbf{G} \end{aligned}\right\} \tag{5.16}$$

has a solution $(\mathbf{H},\mathbf{G})$, $\mathbf{G}\in\mathrm{H}$, $\mathbf{H}\in\mathbf{L}(\mathrm{H})$, $\mathbf{H}=\mathbf{H}^*\ge 0$.

Putting: $\mathbf{A}=\mathcal{A}^{-1},\mathbf{B}=d,\mathbf{C}=h^*(\mathcal{A}^{-1}+qI);\mathbf{Q}=0,\mathbf{N}=\dfrac{1}{2},\mathbf{R}=\delta$ in Theorem 2.4, we conclude from its statement, that if:

– $s\mapsto h^*\left(sI-\mathcal{A}^{-1}\right)^{-1}x_0$ is in $\mathrm{H}^2(\mathbb{C}^+)$ for every $x_0\in\mathrm{H}$ or, equivalently $s\mapsto c^{\#}(sI-\mathcal{A})^{-1}x_0$ is in $\mathrm{H}^2(\mathbb{C}^+)$ for every $x_0\in\mathrm{H}$;

– $\hat{G}\in\mathrm{H}^\infty(\mathbb{C}^+)$, where

$$\begin{aligned}
\hat{G}(s) &= -q[\hat{g}(s^{-1})+c^{\#}d]-s^{-1}h^*(s^{-1}I-\mathcal{A})^{-1}d\\
&= -q[\hat{g}(s^{-1})+c^{\#}d]-[c^{\#}(s^{-1}I-\mathcal{A})^{-1}d+h^*d]\\
&= -(q+s)[\hat{g}(s^{-1})+c^{\#}d]-h^*d
\end{aligned}$$

or, equivalently $\hat{g}\in\mathrm{H}^\infty(\mathbb{C}^+)$, because **EXS** and $s\mapsto c^{\#}(sI-\mathcal{A})^{-1}x_0\in\mathrm{H}^2(\mathbb{C}^+)$ for every $x_0\in\mathrm{H}$ implies by Lemma 3.2 that $s\mapsto c^{\#}(sI-\mathcal{A})^{-1}x_0\in\mathrm{H}^\infty(\mathbb{C}^+)$ for every $x_0\in\mathrm{H}$;

– for some $\eta > 0$ there holds

$$\begin{aligned} \Pi(j\omega) &= \mathbf{R} + 2\,\mathrm{Re}\left[\mathbf{N}^*\hat{G}(j\omega)\right] + \left[\hat{G}(j\omega)\right]^* \mathbf{Q}\hat{G}(j\omega) \\ &= \delta + \mathrm{Re}\left[\hat{G}(j\omega)\right] \geq \eta, \quad \omega \neq 0, \end{aligned} \tag{5.17}$$

then, there exists a solution $(\mathbf{H}, \mathbf{G})$ to (5.2). Here (5.17) reduces to

$$\frac{1}{k} - \mathrm{Re}\left[(1 + j\omega q)\frac{\hat{g}(j\omega) - \rho}{j\omega}\right] \geq \eta > 0, \quad \omega \in \mathbb{R}. \tag{5.18}$$

Moreover, recalling (2.21), one obtains

$$\mathbf{H} := \Psi^* \left[(\mathbf{Q}\mathbb{F} + \mathbf{N})\mathcal{R}^{-1}(\mathbb{F}^*\mathbf{Q} + \mathbf{N}^*) - \mathbf{Q}\right] \Psi = \frac{1}{4}\ \Psi^*\mathcal{R}^{-1}\Psi \geq 0.$$

We can see that again conditions of solvability of the Lur'e system (5.16) can be formulated in the language of the original system (5.15).

Introducing $\mathcal{X}(j\omega) = \mathrm{Re}\,\hat{g}(j\omega)$ and $\mathcal{Y}(j\omega) := \dfrac{\mathrm{Im}\,\hat{g}(j\omega)}{\omega}$ we can see that (5.18) holds iff the parametric curve $\mathbb{R} \ni \omega \longmapsto (\mathcal{X}(j\omega), \mathcal{Y}(j\omega))$ is located strictly to the left of the straight line $\mathcal{Y} = \left(\dfrac{1}{k} + \rho\right) - q\mathcal{X}$.

6. DISCUSSION

REMARKS ON ASSUMPTIONS

As concerns (2.15) notice that $\mathcal{A}$ is a *dissipative operator* in an equivalent scalar product of H iff $\mathbf{A} = \mathcal{A}^{-1}$ is a *dissipative operator* in the same scalar product. Therefore $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$ is uniformly bounded if $\mathcal{A}$ is a dissipative operator in an equivalent scalar product of H.

If (2.15) is satisfied then (2.16) holds iff $\lim_{t\to\infty} \|\mathbf{A}e^{t\mathbf{A}}\|_{\mathbf{L}(\mathrm{H})} = 0$. This equivalence in is due to [2, Theorem 4.4.16, p. 284]. If this condition is satisfied then $\lim_{t\to\infty} \|e^{t\mathbf{A}}\mathbf{A}f\|_{\mathrm{H}} = 0$ for every $f \in \mathrm{H}$, i.e., the origin strongly attracts trajectories starting from the range $R(\mathbf{A})$ of $\mathbf{A}$. Thus, if $\overline{R(\mathbf{A})} = \mathrm{H}$ then the origin attracts trajectories starting from a dense subset of H which, jointly with (2.15) implies that the semigroup $\{e^{t\mathbf{A}}\}_{t\geq 0}$ is *strongly asymptotically stable*, i.e., $\lim_{t\to\infty} \|e^{t\mathbf{A}}f\|_{\mathrm{H}} = 0$ for every $f \in \mathrm{H}$.

In particular, this holds when $\mathbf{A} = \mathcal{A}^{-1}$ and $\mathcal{A}$ generates an **EXS** C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H.

The arguments above indicate that if $\{S(t)\}_{t\geq 0}$ is **EXS** then it is reasonable to assume that $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$ is strongly asymptotically stable. Such an assumption has been made in [15] and the results on the LQ problem derived therein are applicable to the reciprocal system as announced in [8] and fully applied in [9].

RECIPROCALITY APPROACH

The reciprocality approach has been developed in [4, especially Theorem 4.5, p. 1695]. It was noticed therein that the LQ problem (3.1)–(4.4) is equivalent to a control problem for its reciprocal system (4.1)–(4.5). Next, the general theory of the LQ problem presented in [21] (see also [6]) was applied to get a new simplified Riccati characterization of the optimal control/controller in terms of bounded operators only.

In the present paper we derived a solution to the LQ problem for infinite-dimensional systems in factor form using the results of Section 2.4, which is a *reversed approach* in comparison with the results of [4], i.e., we derived the general theory of LQ problem with unbounded operators starting from the LQ theory for bounded operators only.

WEISS' DEFINITION OF $\mathcal{G}_\Lambda$

A possible extension $\mathcal{G}_\Lambda$ of $\mathcal{G}$ has been proposed in [21]:

$$\mathcal{G}_\Lambda x := \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}x,$$
$$D(\mathcal{G}_\Lambda) = \{x \in \mathrm{H} : \text{ there exists } \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}x\}.$$

Since $s\mathcal{G}(sI-\mathcal{A})^{-1}x = s\mathbf{R}^{1/2}\mathbf{G}^*\mathcal{A}(sI-\mathcal{A})^{-1}x$, then by the well-known fact

$$\lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{A}(sI-\mathcal{A})^{-1}x = \mathcal{A}x, \quad x \in D(\mathcal{A}),$$

where $s\mathcal{A}(sI-\mathcal{A})^{-1} \in \mathbf{L}(\mathrm{H})$ is called the *Yosida approximation* of $\mathcal{A}$, there holds $\mathcal{G}_\Lambda x = \mathcal{G}x$ for $x \in D(\mathcal{A})$, i.e., $\mathcal{G}_\Lambda$ extends $\mathcal{G}$ from $D(\mathcal{A})$ onto $D(\mathcal{G}_\Lambda)$.

Next,

$$\begin{aligned}
&R(\mathcal{D}) \subset D(\mathcal{G}_\Lambda)\\
&\Longleftrightarrow \forall v \in \mathrm{U}: \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}v = \mathcal{G}_\Lambda\mathcal{D}v = \lim_{s\to\infty, s\in\mathbb{R}} R_-^{1/2}\phi(s)v - R_-v\\
&\Longleftrightarrow \lim_{s\to\infty, s\in\mathbb{R}} \phi(s)v = R_-^{-1/2}(R_- + \mathcal{G}_\Lambda\mathcal{D})v
\end{aligned}$$

and if the last limit exists then, by [20, Theorem 4.23(a), p. 75], one has: $R_- + \mathcal{G}_\Lambda\mathcal{D} \in \mathbf{L}(\mathrm{U})$. Now, if $R_- + \mathcal{G}_\Lambda\mathcal{D}$ is boundedly invertible then in the terminology of [21, Definition 12.1, p. 319]: ϕ is a *regular spectral factor*. It is known [21, Proposition 12.3, p. 319] that in this case $\phi^{-1}(s)f$ tends to $(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}R_-^{1/2}f$ as $s \to \infty$, $s \in \mathbb{R}$. To verify that our theory is consistent with that of [21] we shall prove that the latter is true. Indeed,

$$\begin{aligned}
R_-^{-1/2}\mathbf{G}^*\mathcal{A}^c x &= R_-^{-1}R_-^{1/2}\mathbf{G}^*\mathcal{A}[x - \mathcal{D}(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x]\\
&= R_-^{-1}\mathcal{G}[x - \mathcal{D}(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x],
\end{aligned}$$

whence $R_-^{-1/2}\mathbf{G}^*\mathcal{A}^c$ extends to

$$R_-^{-1}[I - \mathcal{G}_\Lambda\mathcal{D}(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}]\mathcal{G}_\Lambda x = (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x.$$

Now, it follows from (2.18) with appropriate substitutions that

$$\lim_{s\to 0, s\in\mathbb{R}} \Theta^{-1}(s)v = [I - (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda\mathcal{D}]R_-^{-1/2}v = (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}R_-^{1/2}v$$
$$= \lim_{s\to\infty, s\in\mathbb{R}} \phi(s)v := \phi(\infty)v, \quad v \in \mathrm{U}.$$

On the other side (2.17) yields

$$\lim_{s\to 0, s\in\mathbb{R}} \Theta(s)v = R_-^{1/2}v + R_-^{-1/2}\mathcal{G}_\Lambda\mathcal{D}vR_-^{-1/2}(R_- + \mathcal{G}_\Lambda\mathcal{D})v$$
$$= \lim_{s\to\infty, s\in\mathbb{R}} \phi^{-1}(s)v := \phi^{-1}(\infty), \quad v \in \mathrm{U}.$$

It should be emphasized that usually it is enough to define $\mathcal{G}_\Lambda$ on a certain domain intermediate between $D(\mathcal{A})$ and $D(\mathcal{G}_\Lambda)$ – see examples given in [6].

Introducing $F_\Lambda := R_-^{-1/2}\mathcal{G}_\Lambda$ we can represent the spectral factor ϕ and the optimal feedback controller u as:

$$\phi(s) = (R_-^{1/2} + F_\Lambda\mathcal{D}) + F_\Lambda\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}, \tag{6.1}$$

$$u^c = -\left[(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}R_-^{1/2}\right]F_\Lambda x^c = -\phi^{-1}(\infty)F_\Lambda x^c. \tag{6.2}$$

SPECTRAL FACTORIZATION METHOD

Substituting a general form of $F_\Lambda : \mathrm{H} \longrightarrow \mathrm{U}$ into (6.1) we can determine, if possible, its particular form using the identity $\phi(j\omega)^*\phi(j\omega) = \Pi(j\omega)$ and the condition $\phi^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$; the optimal controller is then uniquely given by (6.2). This method of finding the optimal controller could be recommended when $\dim \mathrm{U} < \infty$.

THE CASE OF **EXS** IN THEOREM 2.4

Notice that if $\{e^{t\mathbf{A}}\}_{t\geq 0}$ is **EXS** then in Theorem 2.4: $\mathbf{C}$ is admissible, $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+; \mathbf{L}(\mathrm{U}, \mathrm{Y}))$, $\{e^{t\mathbf{A}}\}_{t\geq 0}$ is uniformly bounded and (2.16) is satisfied. Consequently its assertions **(i)**, **(ii)** are met if (2.10) holds. Furthermore, $\{e^{t\mathbf{A}^c}\}_{t\geq 0}$ is **EXS** too. Indeed, in (2.25): $\hat{\mathbf{u}}^c \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$, $(sI - \mathbf{A})^{-1}\mathbf{x}_0 \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{H})$ for every $\mathbf{x}_0 \in \mathrm{H}$ and $(sI - \mathbf{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+; \mathbf{L}(\mathrm{H}))$, which yields $(sI - \mathbf{A}^c)^{-1}\mathbf{x}_0 \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{H})$. Now **EXS** follows from the Paley-Wiener and Datko theorems. The concept of stabilizability we shall introduce below enable us to reduce a general system (2.1) to this particularly simple case.

Definition 6.1. The pair $(\mathbf{A}, \mathbf{B})$ is *exponentially stabilizable* if there exists $\mathbf{F} \in \mathbf{L}(\mathrm{H}, \mathrm{U})$ such that the semigroup $e^{t\tilde{\mathbf{A}}}$, $\tilde{\mathbf{A}} := \mathbf{A} + \mathbf{BF}$ is **EXS**.

If $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable then for any $\mathbf{Z} = \mathbf{Z}^* \in \mathbf{L}(\mathrm{H})$ there exists a unique $\mathbf{W} = \mathbf{W}^* \in \mathbf{L}(\mathrm{H})$, $\mathbf{W} \geq 0$ namely $\mathbf{W} := \int_0^\infty e^{t\tilde{\mathbf{A}}^*}\mathbf{Z}e^{t\tilde{\mathbf{A}}}\mathrm{d}t$ such that

$$\tilde{\mathbf{A}}^*\mathbf{W} + \mathbf{W}\tilde{\mathbf{A}} = -\mathbf{Z}.$$

Consider $\mathbf{W}$ corresponding to $\mathbf{Z} := \mathbf{C}^*\mathbf{QC} + \mathbf{F}^*\mathbf{RF} + \mathbf{F}^*\mathbf{N}^*\mathbf{C} + \mathbf{C}^*\mathbf{NF}$ and let $\mathbf{L} := \mathbf{WB} + \mathbf{C}^*\mathbf{N} + \mathbf{F}^*\mathbf{R}$. Then *Popov's transformations*:

$$0 \leq \tilde{\mathbf{H}} = \mathbf{H} + \mathbf{W}, \quad \tilde{\mathbf{G}} = \mathbf{G} - \mathbf{F}^*\mathbf{R}^{1/2}$$

reduce (2.14) to its canonical form

$$\left\{ \begin{array}{ll} \tilde{\mathbf{A}}^*\tilde{\mathbf{H}} + \tilde{\mathbf{H}}\tilde{\mathbf{A}} & = -\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* \\ -\tilde{\mathbf{H}}\mathbf{B} + \mathbf{L} = -\tilde{\mathbf{G}}\mathbf{R}^{1/2} & \end{array} \right\}. \tag{6.3}$$

Replacing $(\mathbf{A}, \mathbf{H}, \mathbf{C}, \mathbf{Q}, \mathbf{G}, \mathbf{N})$ by $(\tilde{\mathbf{A}}, \tilde{\mathbf{H}}, I, 0, \tilde{\mathbf{G}}, \mathbf{L})$, we conclude from Theorem 2.4 that (6.3) has a solution $(\tilde{\mathbf{H}}, \tilde{\mathbf{G}})$, $\tilde{\mathbf{H}} \geq 0$, provided that

$$\Pi(j\omega) := \mathbf{R} + 2\,\mathrm{Re}[\mathbf{L}^*(j\omega I - \tilde{\mathbf{A}})^{-1}\mathbf{B}]$$

is coercive, i.e., $\frac{1}{2}\mathbf{R} + \mathbf{L}^*(sI - \tilde{\mathbf{A}})^{-1}\mathbf{B}$ is *strictly positive real.*

It should be stressed that the assumption that $\{e^{t\mathbf{A}}\}_{t\geq 0}$ is **EXS** is very restrictive and it mainly concerns the case of a finite dimensional spaces H, U and Y though in a recently published report [14] (collecting earlier author's results) this case is in a way regarded as a starting point to discuss the general systems with unbounded state, output and observation operators.

STANDARD LQ PROBLEM

This problem corresponds to $\mathbf{Q} \geq 0$ and $\mathbf{N} = 0$ in (2.2) and is the most classical lq problem. Let $\mathbf{C}$ be admissible and $\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ as required in Theorem 2.4, where $\Pi(j\omega) = \mathbf{R} + [\widehat{G}(j\omega)]^*\mathbf{Q}\widehat{G}(j\omega)$. Then we have $\mathcal{H}^c \geq 0$ because here the performance index $\mathbf{J}(\mathbf{x}_0, \mathbf{u}) \geq 0$ for all $\mathbf{x}_0 \in \mathrm{H}$ and $\mathbf{u} \in \mathrm{L}^2(0, \infty; \mathrm{U})$. The operator Riccati equation takes a simplified form:

$$\mathbf{A}^*\mathcal{H}^c + \mathcal{H}^c\mathbf{A} + \mathbf{C}^*\mathbf{QC} - \mathcal{H}^c\mathbf{BR}^{-1}\mathbf{B}^*\mathcal{H}^c = 0. \tag{6.4}$$

In some cases $\mathrm{U} = \mathrm{H}$ and $\mathbf{B}$ is boundedly invertible. Then (6.4) can be solved using *Shubert's idea* originally proposed for matrices [19]; (6.4) can equivalently be written as:

$$\mathbf{S}^*\mathbf{S} = \mathbf{A}^*\mathbf{B}^{-1}\mathbf{R}(\mathbf{B}^{-1})^*\mathbf{A} + \mathbf{C}^*\mathbf{QC} \geq 0,$$

and $\mathbf{S}$ is being selected in such a way that the LHS of

$$(\mathbf{B}^{-1})^*\mathbf{R}^{1/2}\mathbf{S} + (\mathbf{B}^{-1})^*\mathbf{R}(\mathbf{B}^{-1})^*\mathbf{A} = \mathcal{H}^c$$

is a self-adjoint and nonnegative operator.

LINEAR VIZ NONLINEAR THEORY OF SECTION 4

Stability sector (k_1, k_2) for a nonlinear feedback control obtained for the Example of Section 5.2 is significantly smaller than the *so-called Hurwitz sector* ensuring stability in the case of a linear feedback – see [7] for detailed presentation of the linear theory.

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