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GLOBAL EXISTENCE AND BLOW-UP PHENOMENON FOR A QUASILINEAR VISCOELASTIC EQUATION WITH STRONG DAMPING AND SOURCE TERMS

Huafei Di and Zefang Song

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Abstract. Considered herein is the global existence and non-global existence of the initial-boundary value problem for a quasilinear viscoelastic equation with strong damping and source terms. Firstly, we introduce a family of potential wells and give the invariance of some sets, which are essential to derive the main results. Secondly, we establish the existence of global weak solutions under the low initial energy and critical initial energy by the combination of the Galerkin approximation and improved potential well method involving with t. Thirdly, we obtain the finite time blow-up result for certain solutions with the non-positive initial energy and positive initial energy, and then give the upper bound for the blow-up time T^*. Especially, the threshold result between global existence and non-global existence is given under some certain conditions. Finally, a lower bound for the life span T^* is derived by the means of integro-differential inequality techniques.

Keywords: viscoelastic equation, strong damping and source, blow-up, upper and lower bounds, potential well, invariant set.

Mathematics Subject Classification: 35L35, 35L75, 35R15.

1. INTRODUCTION

In this article, we consider the following initial-boundary value problem of the quasilinear viscoelastic equation

$$\begin{cases} |u_t|^{\rho-1}u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t - \triangle u_{tt} = f(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The functions g, f represent the kernel of memory term and source term, respectively. The following conditions are the basic assumptions to obtain the main results.

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(A1) Assume that the nonlinear source term f satisfies:
  (i) $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f(0) = f'(0) = 0$,
  (ii) $f(s)$ is a monotone function for $s \in \mathbb{R}$, and is convex function for $s \in \mathbb{R}_+$, concave function for $s \in \mathbb{R}_-$,
  (iii) $|f(s)| \le \alpha |s|^p$ for $\alpha > 0$, and $(p+1)F(s) \le sf(s)$, where $F(s) = \int_0^s f(\tau)d\tau$, $1 < p < \infty$ when $n \le 2$, $1 < p \le \frac{n+2}{n-2}$ when $n \ge 3$.

(A2) The relaxation function g: $[0,\infty) \to (0,\infty)$ is a C^1 function, which meet

$$g'(t) \le 0, \quad l(t) = 1 - \int_0^t g(s)ds \ge 1 - \int_0^\infty g(s)ds = l > 0.$$

(A3) For the nonlinear term $|u_t|^{\rho-1}u_{tt}$, we further assume

$$1 < \rho < \infty \text{ when } n \le 2, \quad \text{and} \quad 1 < \rho \le \frac{n+2}{n-2} \text{ when } n \ge 3.$$

It is worth mentioning that the viscoelastic materials describe natural damping, mainly because that these materials have some special properties keeping memory of their past traces. In mathematics, these damping effects can be modeled by using integro-differential operators. Hence, the equation (1.1) has extensive physical background and also describes various important physical processes, for instance, heat conduction and viscoelastic flow in viscoelastic materials [1], vibration appearing in the viscoelastic rods [14], analysis of bidirectional shallow-water waves [23], velocity evolution of ion acoustic waves with ionic viscosity [21], etc. In addition, the another reason why this type of equation is so attractive is that it has rich theoretical connotation. In the last few years, much effort in mathematics have been devoted to the study of qualitative properties for the viscoelastic equations. For example, when the parameter $\rho = 1$, source term $f(u) = u|u|^{p-2}$ and absence of strong damping term $\triangle u_t$, dispersion term $\triangle u_{tt}$, the equation (1.1) is reduced to the following nonlinear viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds = u|u|^{p-2}, \quad x \in \Omega, t > 0. \tag{1.2}$$

Here, the source term $u|u|^{p-2}$ is the main factor that causes the finite time blow-up of the solutions with the negative initial energy (see [3,13]). In [4], Berrimi and Messaoudi proved the global existence of the solutions to equation (1.2) and then gave some energy decaying estimates in the exponential and polynomial forms under certain conditions on g and p.

In the absence of dispersion term $\triangle u_{tt}$ and parameter $\rho = 1$, source term $f(u) = u|u|^{p-2}$, the model (1.1) becomes

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t = u|u|^{p-2}, \quad x \in \Omega, t > 0. \tag{1.3}$$

Song and Zhong [24] considered the initial-boundary value problem of (1.3) and obtained a finite time blow-up result of the solutions with certain positive initial energy. Later, Song and Xue [25] improved the result of [24], in which the finite time blow-up is investigated under arbitrary high initial energy. Wang *et al.* [26, 29] changed the strong damping term $\triangle u_t$ into weak damping term u_t in equation (1.3). By using the potential well method, they established the global existence and exponential decay of energy [29] and then under appropriate conditions on g and initial data, discussed the finite time blow-up phenomenon of the solutions with arbitrary high initial energy [26].

In [30], Xu, Yang and Liu considered the following viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t - \triangle u_{tt} + u_t = u|u|^{p-2}, \quad x \in \Omega, t > 0.$$

By introducing a family of potential wells, the global/non-global existence of weak solutions were investigated at the low initial energy. Then, a finite time blow-up result was given at arbitrary positive initial energy. In [2], Anaya and Messaoudi discussed a class of fourth-order viscoelastic wave equation with a general weak damping term, and derived a general decay rate estimates under some certain restrictions for relaxation function g.

In the absence of source term $f(u)$, the model (1.1) is reduced to the nonlinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \gamma\triangle u_t - \triangle u_{tt} = 0, \quad x \in \Omega, t > 0. \tag{1.4}$$

Cavalcanti *et al.* [5] proved that there exists the global existence of the solutions for $\gamma \geq 0$ and exponential uniform decay estimate of the energy for $\gamma > 0$.

When the source term $f(u) = u|u|^{p-2}$ and lack of strong damping term $\triangle u_t$, the model (1.1) becomes the nonlinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_{tt} = bu|u|^{p-2}, \quad x \in \Omega, t > 0. \tag{1.5}$$

When the initial data of (1.5) are in the stable set, Messaoudi and Tatar [20] obtained the global existence and uniform decay of the solutions by applying the potential well method. Later, Liu [18] further discussed the energy decaying estimates by constructing a suitable Lyapunov function and using the perturbed energy method, and also gave a finite time blow-up result at the positive initial energy for certain relaxation function g and initial data in the unstable set.

Here, let us mention that the potential well theory has become one of most popular tools in the study of the qualitative properties of the solutions to the nonlinear evolution equations. Sattinger [22] firstly proposed this method used to investigate the global existence of the solutions for the hyperbolic equations. Especially, the potential well method also plays a very key role in deriving the threshold result between global existence and non-global existence of the solutions (see [1,7,8,15,28,30]). Therefore, it has been widely used and extended by many authors to study the qualitative theory for different kinds of evolution equations, such as heat equation, pseudo-parabolic equation, viscoelastic wave equation, Kirchhoff-type equation and so on, we refer the readers to see [6–12,15,16,18–20,22,26–32] and the papers cited therein.

Motivated by the above researches, in the present work we will study the initial-boundary value problem (1.1) for the viscoelastic equation. To our knowledge, there is little information on the global existence and non-global existence of the above problem. The global existence, blow-up and uniform decay rates of the energy of the solutions to equations (1.4) and (1.5) were discussed in references [5,18,20]. Compared with equations (1.4) and (1.5), we note that the nonlinear term $|u_t|^{\rho-1}u_{tt}$, strong damping term $\triangle u_t$, dispersion term $\triangle u_{tt}$ and nonlinear source term $f(u)$ appear simultaneously in the equation (1.1), which cause some difficulties in the way of the proof when we study the qualitative theory of the solutions. Especially, the interaction among the above terms in (1.1) makes that it requires a rather delicate analysis in proving the relative qualitative properties of the solutions. Another obvious difference between this paper and references [5,18,20] is the definition of total energy $E(t)$. In references [5,18,20], the function $l = 1 - \int_0^\infty g(s)ds$ appearing in $E(t)$ is independent of time t. However, in this paper the function l of $E(t)$ is replaced by $l(t)$, which is equal to $1 - \int_0^t g(s)ds$ involving the time t. Thus, we need to invent some new skills and methods to overcome the above difficulties in the study of the global existence and blow-up phenomenon for the problem (1.1).

In order to state our main results precisely, we first define the weak solutions for the problem (1.1) over the interval $\Omega \times [0,T)$. Here, it is to be understood that T is either infinity or the limit of the existence interval.

Definition 1.1. We say that $u(x,t)$ is called a weak solution of the problem (1.1) on the interval $\Omega \times [0,T)$. If $u \in L^\infty(0,T;H_0^1(\Omega))$, $u_t \in L^\infty(0,T;H_0^1(\Omega))$ and $u_{tt} \in L^\infty(0,T;H_0^1(\Omega))$ satisfy the following conditions:

(i) for any $v \in H_0^1(\Omega)$ and a.e. $t \in [0,T)$, we have

$$\begin{aligned}(|u_t|^{\rho-1}u_{tt}, v) + (\nabla u, \nabla v) - \Big(\int_0^t g(t-s)\nabla u(s)ds, \nabla v\Big)\\ + (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) = (f(u), v),\end{aligned} \tag{1.6}$$

(ii) $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0) = u_1(x)$ in $H_0^1(\Omega)$,

(iii) the energy inequality

$$E(t) \le E(0) \tag{1.7}$$

holds for any $0 \le t < T$, where the concrete expression of total energy $E(t)$ will be given later.

Remark 1.2. Since $1 < \rho < \infty$ when $n \leq 2$, and $1 < \rho \leq \frac{n+2}{n-2}$ when $n \geq 3$, according to Sobolev embedding theorem, we have

$$H_0^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega) \quad \text{and} \quad \|v\|_{\rho+1} \leq B_{\rho+1}\|\nabla v\|_2 \text{ for all } v \in H_0^1(\Omega),$$

where $B_{\rho+1}$ is the optimal embedding constant form Sobolev spaces $H_0^1(\Omega)$ to $L^{\rho+1}(\Omega)$. Noting that $\frac{\rho-1}{\rho+1} + \frac{1}{\rho+1} + \frac{1}{\rho+1} = 1$, by the Hölder inequality we can see that the nonlinear term $\int_\Omega |u_t|^{\rho-1}u_{tt}v dx$ makes sense.

Next, we will introduce some functionals which are all associated with the potential wells.

$$\begin{aligned} E(t) &= E(u, u_t) \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}l(t)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int\limits_\Omega F(u)dx \\ &\geq \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int\limits_\Omega F(u)dx, \end{aligned} \tag{1.8}$$

$$J(u) = \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int\limits_\Omega F(u)dx, \tag{1.9}$$

and

$$I(u) = l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \int\limits_\Omega f(u)udx, \tag{1.10}$$

where

$$(g \circ \nabla u)(t) = \int\limits_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds \quad \forall u \in H_0^1(\Omega).$$

The corresponding potential well sets are defined in the form

$$W = \left\{u \in H_0^1(\Omega) : I(u) > 0,\ E(t) < d\right\} \cup \{0\}, \tag{1.11}$$

$$V = \left\{u \in H_0^1(\Omega) : I(u) < 0,\ E(t) < d\right\}. \tag{1.12}$$

Here, the depth d of potential well set W is defined by

$$d = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \left\{\sup_{\lambda>0} J(\lambda u)\right\}. \tag{1.13}$$

In fact (see [16,20] for details), the potential well depth d is a positive constant equaled to

$$\inf_{I(u)=0, u\neq 0} J(u). \tag{1.14}$$

Furthermore, for $\delta > 0$, we introduce the functional

$$I_\delta(u) = \delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] - \int\limits_\Omega f(u)udx, \tag{1.15}$$

and give the modified potential well depth d_δ by the form

$$d_\delta = d(\delta) = \inf_{I_\delta(u)=0, u\neq 0} J(u). \tag{1.16}$$

Then, a series of potential well sets can be given as follows

$$W_\delta = \left\{u \in H_0^1(\Omega) : I_\delta(u) > 0,\ E(t) < d(\delta)\right\} \cup \{0\}, \tag{1.17}$$

$$V_\delta = \left\{u \in H_0^1(\Omega) : I_\delta(u) < 0,\ E(t) < d(\delta)\right\}. \tag{1.18}$$

To obtain the global existence and non-global existence of the problem (1.1), our first intention in this paper is to establish some important lemmas about the properties of the potential wells and the invariant sets, cf. Lemmas 2.1–2.10 in Section 2. On the basis of these lemmas, our second intention is to prove the existence of the global weak solutions for certain initial data in the stable sets W_δ and $\overline{W}_\delta$, respectively, by the combination of Galerkin approximation and improved potential well method, where $\overline{W}_\delta = W_\delta \cup \partial W_\delta$, $W_1 = W$ and $\overline{W}_1 = \overline{W}$. Finally, we will investigate the finite time blow-up phenomenon of the solutions with certain initial data in the unstable sets, and then discuss the upper and lower bounds for the finite blow-up time T^*.

Now, we are ready to state the main results of this paper.

Theorem 1.3 (Global existence at low initial energy level)**.** *Let the hypotheses* (A1)–(A3) *hold and* $u_0(x), u_1(x) \in H_0^1(\Omega)$. *Further assume that* $E(0) < d$ *and* $I(u_0) > 0$ *or* $\|u_0\|_{H^1} = 0$. *Then the problem* (1.1) *admits a global weak solution* $u \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_t \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_{tt} \in L^\infty(0,\infty; H_0^1(\Omega))$ *and* $u(t) \in W$ *for* $0 \le t < \infty$.

Theorem 1.4 (Global existence at critical initial energy level)**.** *Let the hypotheses* (A1)–(A3) *hold and* $u_0(x), u_1(x) \in H_0^1(\Omega)$. *Further, assume that* $E(0) = d$ *and* $I(u_0) \ge 0$. *Then the problem* (1.1) *admits a global weak solution* $u \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_t \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_{tt} \in L^\infty(0,\infty; H_0^1(\Omega))$ *and* $u(t) \in \overline{W} = W \cup \partial W$ *for* $0 \le t < \infty$.

Applying Lemma 2.7 (see Section 2), from $0 < E(0) < d$, $I_{\delta_2}(u_0) > 0$, it follows that $I_\delta(u_0) > 0$ for all $\delta \in (\delta_1, \delta_2)$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d_\delta(t) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Repeating the arguments of Theorem 1.3 for $\delta \in (\delta_1, \delta_2)$, then we can obtain the following conclusion.

Corollary 1.5. *If the hypothetical conditions "*$E(0) < d$, $I(u_0) > 0$*" in Theorem 1.3 are changed to "*$0 < E(0) < d$, $I_{\delta_2}(u_0) > 0$*", then the problem* (1.1) *admits a global weak solution* $u \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_t \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_{tt} \in L^\infty(0,\infty; H_0^1(\Omega))$ *and* $u(t) \in W_\delta$ *for* $0 \le t < \infty$.

Based on the conclusion of Corollary 1.5, using the similar arguments as Theorem 1.4, we immediately obtain the following global existence of the solutions. We omit the proof here for the sake of brevity.

Corollary 1.6. *Let the hypotheses* (A1)–(A3) *hold and* $u_0(x), u_1(x) \in H_0^1(\Omega)$*. Further, assume that* $E(0) = d$ *and* $I_\delta(u_0) \geq 0$*. Then the problem* (1.1) *admits a global weak solution* $u \in L^\infty(0, \infty; H_0^1(\Omega))$, $u_t \in L^\infty(0, \infty; H_0^1(\Omega))$, $u_{tt} \in L^\infty(0, \infty; H_0^1(\Omega))$ *and* $u(t) \in \overline{W}_\delta = W_\delta \cup \partial W_\delta$ *for* $0 \leq t < \infty$*.*

Theorem 1.7 (Blow-up and upper bound for blow-up time)**.** *Let the hypotheses* (A1)–(A3) *hold. For any fixed number* $\beta \in (0, 1)$*, we assume that* $u_0(x), u_1(x) \in H_0^1(\Omega)$ *satisfying*

$$I(u_0) < 0, \quad E(0) < \beta \tilde{d}, \tag{1.19}$$

where $\tilde{d}$ *is given by* (2.26) *in Section 2. We further choose* $\rho < p$ *and* g *satisfies*

$$\int_0^\infty g(s)ds < \frac{[(p-1)(1-\beta) - \vartheta]^2 + 2[(p-1)(1-\beta) - \vartheta]}{[(p-1)(1-\beta) - \vartheta + 2]^2 + 1}, \tag{1.20}$$

where $0 < \vartheta < (p - 1)(1 - \beta)$*. Then, the solutions of the problem* (1.1) *blow up in finite time, i.e. there exists a time* $T^* < +\infty$ *such that*

$$\lim_{t \to T^{*-}} \left(\|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u_t\|_2^2 + \|u\|_{p+1}^{p+1} \right) = +\infty.$$

Theorem 1.8 (Lower bound for blow-up time)**.** *Under the assumption of Theorem 1.7, let* $u_0(x), u_1(x) \in H_0^1(\Omega)$ *and* u *be a blow-up solution of the problem* (1.1)*. Then a lower bound for blow-up time* T^* *can be estimated by*

$$T^* \geq \int_{F(0)}^\infty \frac{d\eta}{\alpha 2^{p-1} B_{p+1}^{2p} l^{-p} \eta^p + \alpha B_{p+1}^2 \eta}, \tag{1.21}$$

where

$$F(0) = \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2,$$

B_{p+1} *is the Sobolev embedding constant satisfying the inequality*

$$\|v\|_{p+1} \leq B_{p+1}\|\nabla v\|_2$$

for all $v \in H_0^1(\Omega)$*.*

In view of Theorem 1.3 and Theorem 1.7, a threshold result between global existence and non-global existence of the solutions for the problem (1.1) can be obtained as follows.

Corollary 1.9. *Let the hypotheses* (A1)–(A3), (1.20) *hold and* $u_0(x), u_1(x) \in H_0^1(\Omega)$*. Further, assume that* $E(0) < \beta \tilde{d} \leq d$ *with* $\beta \in (0, 1)$*. Then the problem* (1.1) *has a unique global weak solution provided* $I(u_0) \geq 0$ *(including* $\|\nabla u_0\|_2^2 = 0$*). The problem* (1.1) *does not admit any global solution provided* $I(u_0) < 0$*.*

Remark 1.10. In view of conditions (A1)(iii) and (1.8)–(1.10), we have

$$\begin{aligned}&\frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1}+\frac{1}{2}\|\nabla u_1\|_2^2+\frac{p-1}{2(p+1)}l\|\nabla u_0\|_2^2+\frac{1}{p+1}I(u_0)\\&\leq\frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1}+\frac{1}{2}\|\nabla u_1\|_2^2+J(u_0)\leq E(0).\end{aligned}\tag{1.22}$$

It is easy to see that if the initial energy $E(0)<0$, then the inequality $I(u_0)\geq 0$ is impossible. When the initial energy $E(0)=0$, we see that either $I(u_0)>0$ or $I(u_0)=0$ with $\|\nabla u_0\|_2^2\neq 0$ are all impossible. If the initial energy $0<E(0)<\beta\tilde{d}\leq d$ (choose $\beta\in(0,1)$ is small such that $\beta\tilde{d}\leq d$), we discover that the equality $I(u_0)=0$ with $\|\nabla u_0\|_2^2\neq 0$ is impossible. Otherwise, by the definition of d we have that $J(u_0)\geq d$, which contradicts (1.22). Therefore, all possible scenarios have already been investigated in this paper.

2. FAMILY OF POTENTIAL WELLS AND INVARIANT SETS

In this section, we shall introduce some notations and important lemmas about the properties of the potential wells and the invariant sets, which are essential to derive the main results.

Lemma 2.1 (see [18,22])**.** *If $f(s)$, $F(s)$ satisfy the hypothesis* (A1)*, then:*

(1) *for $|s|\geq 1$ and some constant $B>0$, we have $F(s)\geq B|s|^{p+1}$,*
(2) *for $|s|\geq 1$, we have $sf(s)\geq(p+1)B|s|^{p+1}$,*
(3) *we have $s(sf'(s)-f(s))\geq 0$.*

Lemma 2.2. *If the hypotheses* (A1), (A2) *hold, then for any $u\in H_0^1(\Omega)$, $\|u\|_{H^1}\neq 0$, it follows that:*

(1) $\lim_{\lambda\to 0^+}J(\lambda u)=0$, $\lim_{\lambda\to+\infty}J(\lambda u)=-\infty$,
(2) $\varphi(\lambda)=\frac{1}{\lambda}\int_\Omega uf(\lambda u)dx$ *is an increasing function for $\lambda\in(0,\infty)$,*
(3) $\lim_{\lambda\to 0^+}\varphi(\lambda)=0$, $\lim_{\lambda\to+\infty}\varphi(\lambda)=+\infty$.

Proof. (1) From the hypothesis (A1) and (1.9), it follows that

$$\begin{aligned}J(\lambda u)&=\frac{1}{2}l\|\nabla\lambda u\|_2^2+\frac{1}{2}(g\circ\nabla\lambda u)(t)-\int_\Omega F(\lambda u)dx\\&=\frac{1}{2}l\lambda^2\|\nabla u\|_2^2+\frac{1}{2}\lambda^2(g\circ\nabla u)(t)-\int_\Omega F(\lambda u)dx,\end{aligned}\tag{2.1}$$

which implies that $\lim_{\lambda\to 0^+}J(\lambda u)=0$. In addition, we have from Lemma 2.1 (1) that

$$\int_\Omega F(\lambda u)dx\geq B\lambda^{p+1}\int_{\Omega_\lambda}|u|^{p+1}dx,$$

where

$$\Omega_\lambda = \Big\{x \in \Omega : |u| \geq \frac{1}{\lambda}\Big\}.$$

Furthermore, we note that

$$\lim_{\lambda\to+\infty} \int\limits_{\Omega_\lambda} |u|^{p+1} dx = \|u\|_{p+1}^{p+1}.$$

So there appears the relation

$$\int\limits_{\Omega} F(\lambda u) dx \geq B\lambda^{p+1}\|u\|_{p+1}^{p+1} \quad \text{as } \lambda \to +\infty. \tag{2.2}$$

By (1.9) and (2.2), we discover that

$$J(\lambda u) \leq \frac{1}{2} l\lambda^2 \|\nabla \lambda u\|_2^2 + \frac{1}{2}\lambda^2 (g \circ \nabla u)(t) - B\lambda^{p+1}\|u\|_{p+1}^{p+1}, \tag{2.3}$$

as $\lambda \to +\infty$. Therefore, we conclude that $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.

(2) Applying Lemma 2.1 (3), it follows that

$$\varphi'(\lambda) = \frac{1}{\lambda^2}\int\limits_{\Omega} [\lambda u^2 f'(\lambda u) - u f(\lambda u)] dx = \frac{1}{\lambda^3}\int\limits_{\Omega} \lambda u[\lambda u f'(\lambda u) - f(\lambda u)] dx > 0, \tag{2.4}$$

which implies that $\varphi(\lambda)$ is an increasing function for $\lambda \in (0, \infty)$.

(3) From the hypotheses (A1), there appears the relation

$$0 \leq \varphi(\lambda) = \frac{1}{\lambda^2}\int\limits_{\Omega} \lambda u f(\lambda u) dx \leq \frac{\alpha}{\lambda^2}\|\lambda u\|_{p+1}^{p+1} = \alpha\lambda^{p-1}\|u\|_{p+1}^{p+1}. \tag{2.5}$$

So we have that $\lim_{\lambda\to 0^+} \varphi(\lambda) = 0$. To go a step further, by Lemma 2.1 (2), one has

$$\begin{aligned} \varphi(\lambda) &= \frac{1}{\lambda^2}\int\limits_{\Omega} \lambda u f(\lambda u) dx \geq \frac{(p+1)B}{\lambda^2}\int\limits_{\Omega_\lambda} |\lambda u|^{p+1} dx \\ &= (p+1)B\lambda^{p-1}\int\limits_{\Omega_\lambda} |u|^{p+1} dx, \end{aligned} \tag{2.6}$$

where Ω_λ is shown as part (1) of this lemma. Applying

$$\lim_{\lambda\to+\infty} \int\limits_{\Omega_\lambda} |u|^{p+1} dx = \|u\|_{p+1}^{p+1}$$

again, we discover that

$$\varphi(\lambda) \geq (p+1)B\lambda^{p-1}\int\limits_{\Omega} |u|^{p+1} dx \quad \text{as } \lambda \to +\infty. \tag{2.7}$$

Therefore, we deduce that $\lim_{\lambda\to+\infty} \varphi(\lambda) = +\infty$. $\square$

Lemma 2.3. *If the hypotheses* (A1), (A2) *hold, then for any* $u \in H_0^1(\Omega)$ *with* $\|u\|_{H^1} \neq 0$*, we have:*

(1) *there exists a unique* $\lambda^* = \lambda^*(u) \in (0, \infty)$ *satisfying*

$$\frac{dJ(\lambda u)}{d\lambda}\Big|_{\lambda=\lambda^*} = 0,$$

(2) $J(\lambda u)$ *is an increasing function for* $\lambda \in [0, \lambda^*]$*, decreasing function for* $\lambda \in [\lambda^*, \infty)$*, and takes the maximum at* $\lambda = \lambda^*$*,*

(3) $I(\lambda u) > 0$ *for* $\lambda \in (0, \lambda^*)$*,* $I(\lambda u) < 0$ *for* $\lambda \in (\lambda^*, \infty)$*, and* $I(\lambda^* u) = 0$*.*

Proof. (1) We have from (1.9) that

$$\begin{aligned} \frac{dJ(\lambda u)}{d\lambda} &= \lambda l \|\nabla u\|_2^2 + \lambda (g \circ \nabla u)(t) - \int_\Omega f(\lambda u) u dx \\ &= \lambda \left[l \|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \varphi(\lambda) \right], \end{aligned} \tag{2.8}$$

where $\varphi(\lambda)$ has been defined in Lemma 2.2. It shows that $\varphi(\lambda) = \frac{1}{\lambda}\int_\Omega u f(\lambda u) dx$ is an increasing function for $\lambda \in [0, \infty)$, $\lim_{\lambda \to 0^+} \varphi(\lambda) = 0$ and $\lim_{\lambda \to +\infty} \varphi(\lambda) = +\infty$. Hence, it is easy to see that there exists a unique $\lambda^* = \lambda^*(u)$ satisfying

$$\varphi(\lambda^*) = l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \quad \text{and} \quad \frac{dJ(\lambda u)}{d\lambda}\Big|_{\lambda=\lambda^*} = 0.$$

(2) By the combination of $\varphi'(\lambda) > 0$ and result of part (1), it follows that

$$\frac{dJ(\lambda u)}{d\lambda} \geq 0 \text{ for } \lambda \in [0, \lambda^*], \quad \frac{dJ(\lambda u)}{d\lambda} \leq 0 \text{ for } \lambda \in [\lambda^*, \infty).$$

Thus, we get that $J(\lambda u)$ is an increasing function for $\lambda \in [0, \lambda^*]$, decreasing function for $\lambda \in [\lambda^*, \infty)$.

(3) Combining the proof process of part (2) and

$$I(\lambda u) = \lambda^2 l \|\nabla u\|_2^2 + \lambda^2 (g \circ \nabla u)(t) - \int_\Omega f(\lambda u) \lambda u dx = \lambda \frac{dJ(\lambda u)}{d\lambda},$$

it is easy to see that the conclusion of part (3) holds. □

Lemma 2.4. *Assume that the conditions* (A1), (A2) *hold. Furthermore,*

(1) *if* $I_\delta(u) < 0$*, then we have*

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) > r(\delta),$$

where

$$r(\delta) = \left(\frac{\delta l^{\frac{p+1}{2}}}{\alpha B_{p+1}^{p+1}} \right)^{\frac{2}{p-1}}$$

and the constant B_{p+1} *is given in Theorem 1.8; particularly, if* $I(u) < 0$*, then we have*

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) > r(1),$$

(2) *if* $I_\delta(u) = 0$, *then we have*

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq r(\delta) \quad or \quad \|u\|_{H^1} = 0;$$

particularly, if $I(u) = 0$, *then we have* $l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq r(1)$ *or* $\|u\|_{H^1} = 0$.

Proof. (1) By conditions (A1)(iii) and Sobolev inequality, a series of calculation gives

$$\begin{aligned}\int_\Omega f(u)udx &\leq \alpha\|u\|_{p+1}^{p+1} \leq \alpha B_{p+1}^{p+1}\|\nabla u\|_2^{p+1} \\ &\leq \frac{\alpha}{l^{\frac{p+1}{2}}} B_{p+1}^{p+1}[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)]^{\frac{p+1}{2}},\end{aligned} \tag{2.9}$$

where use has been made of the fact that the functional $(g \circ \nabla u)(t) > 0$ in the third inequality. Applying $I_\delta(u) < 0$, we know that $u \neq 0$ and

$$\int_\Omega f(u)udx > \delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)]. \tag{2.10}$$

In view of (2.9) and (2.10), it follows that

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) > \Big(\frac{\delta l^{\frac{p+1}{2}}}{\alpha B_{p+1}^{p+1}}\Big)^{\frac{2}{p-1}} = r(\delta).$$

(2) From the condition $I_\delta(u) = 0$, we can see that

$$\delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] = \int_\Omega f(u)udx, \quad \|u\|_{H^1} \neq 0 \quad \text{or} \quad \|u\|_{H^1} = 0.$$

When $\|u\|_{H^1} \neq 0$, a simple calculation gives

$$\delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] \leq \frac{\alpha}{l^{\frac{p+1}{2}}} B_{p+1}^{p+1}[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)]^{\frac{p+1}{2}},$$

which implies that

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq r(\delta) \quad \text{or} \quad \|u\|_{H^1} = 0.$$

Thus, the proof is completed. □

Lemma 2.5. *If the hypotheses* (A1), (A2) *hold, then*

(1) *for* $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$ $(\delta \in (0, \frac{p+1}{2}))$, *we have* $d(\delta) \geq a(\delta)r(\delta)$,
(2) $\lim_{\delta \to 0^+} d(\delta) = 0$, $\lim_{\delta \to +\infty} d(\delta) = -\infty$ *and there exists a constant* $\delta_0 \geq \frac{p+1}{2}$ *satisfying* $d(\delta_0) = 0$,
(3) $d(\delta)$ *is a strictly increasing function for* $\delta \in (0, 1]$, *strictly decreasing function for* $\delta \in [1, \delta_0)$ *and takes the maximum* $d(1) = d$.

Proof. (1) Considering $I_\delta(u) = 0$ ($\|u\|_{H^1} \neq 0$), we have from Lemma 2.4 (2) that

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq r(\delta).$$

Thus, a series of calculation gives

$$\begin{aligned} J(u) &= \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega F(u)dx \\ &\geq \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\int_\Omega f(u)udx \\ &= a(\delta)[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] + \frac{1}{p+1}I_\delta(u) \\ &\geq a(\delta)r(\delta), \end{aligned} \tag{2.11}$$

which together with $d(\delta) \geq J(u)$ yields that the conclusion holds.

(2) For any $u \in H_0^1(\Omega)$ ($\|u\|_{H^1} \neq 0$), we introduce $\lambda = \lambda(\delta)$ such that

$$\delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] = \frac{1}{\lambda}\int_\Omega f(\lambda u)udx. \tag{2.12}$$

Similar arguments as the proof process of Lemma 2.2, we also introduce functional

$$\varphi(\lambda) = \delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)]. \tag{2.13}$$

It is easy to see that there exists a unique

$$\lambda = \varphi^{-1}(\delta[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)])$$

such that (2.12) holds. So, we can get that

$$\lim_{\delta\to 0^+} \lambda(\delta) = 0 \quad \text{and} \quad \lim_{\delta\to +\infty} \lambda(\delta) = +\infty. \tag{2.14}$$

By the combination of (2.14) and Lemma 2.2 (1), it follows that

$$\lim_{\delta\to 0^+} J(\lambda(\delta)u) = \lim_{\lambda\to 0^+} J(\lambda u) = 0, \tag{2.15}$$

and

$$\lim_{\delta\to +\infty} J(\lambda(\delta)u) = \lim_{\lambda\to +\infty} J(\lambda u) = -\infty. \tag{2.16}$$

From the definition of $d(\delta)$, we discover that

$$\lim_{\delta\to 0^+} d(\delta) = 0 \quad \text{and} \quad \lim_{\delta\to +\infty} d(\delta) = -\infty. \tag{2.17}$$

In view of (2.17) and part (1) of this lemma, we can easily obtain that there exists $\delta_0 \geq \frac{p+1}{2}$ satisfying $d(\delta_0) = 0$ and $d(\delta) > 0$ for $\delta \in (0, \delta_0)$.

(3) We will show that the inequality $d(\delta') < d(\delta'')$ is holding for $0 < \delta' < \delta'' \leq 1$ or $1 \leq \delta'' < \delta' < \delta_0$. Here, it suffices to prove that for any $u \in H_0^1(\Omega)$ with $I_{\delta''}(u) = 0$ ($\|u\|_{H^1} \neq 0$), there exists a $v \in H_0^1(\Omega)$ such that $I_{\delta'}(v) = 0$ ($\|v\|_{H^1} \neq 0$) and $J(u) - J(v) > 0$.

In fact, defining $\lambda(\delta)$ by (2.12), then we have $I_\delta(\lambda(\delta)u) = 0$. Considering $I_{\delta''}(u) = 0$, it follows that $\lambda(\delta'') = 1$. Furthermore, defining $q(\lambda) = J(\lambda u)$ and then a series of calculation gives that

$$\begin{aligned}\frac{dq(\lambda)}{d\lambda} &= \lambda l\|\nabla u\|_2^2 + \lambda(g \circ \nabla u)(t) - \int_\Omega f(\lambda u)u dx \\ &= (1-\delta)\lambda\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) + \frac{I_\delta(\lambda u)}{\lambda} \\ &= (1-\delta)\lambda\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big).\end{aligned} \tag{2.18}$$

We further take $v = \lambda(\delta')u$, then there appears relation $I_{\delta'}(v) = 0$ ($\|v\|_{H^1} \neq 0$). If $0 < \delta' < \delta'' \leq 1$, we can see that $0 < \lambda(\delta') < \lambda(\delta'') = 1$. And there exists $\delta^* \in [\delta', \delta'']$ such that

$$\begin{aligned}J(u) - J(v) &= q(1) - q(\lambda(\delta')) = [1 - \lambda(\delta')]q'(\lambda(\delta^*)) \\ &= [1 - \lambda(\delta')](1-\delta^*)\lambda(\delta^*)\big[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big] \\ &> \lambda(\delta')r(\delta'')[1 - \lambda(\delta')](1-\delta'') > 0,\end{aligned} \tag{2.19}$$

which together with definition of $d(\delta)$ yields that $d(\delta'') > d(\delta')$.

When $1 \leq \delta'' < \delta' < \delta_0$, we have that $\lambda(\delta') > \lambda(\delta'') = 1$. The analogous calculations as (2.19), we can deduce that

$$J(u) - J(v) > \lambda(\delta'')r(\delta'')[\lambda(\delta') - 1](\delta'' - 1) > 0 \quad \text{for} \quad \delta^\star \in [\delta'', \delta']. \tag{2.20}$$

Applying the definition of $d(\delta)$, we discover the relations $d(\delta'') > d(\delta')$. Thus, we end the proof of this lemma. □

Lemma 2.6. *Assume that the conditions* (A1), (A2) *hold and* $\delta \in (0, \frac{p+1}{2})$. *Furthermore,*

(1) *if*

$$I_\delta(u) > 0 \quad \textit{and} \quad J(u) \leq d(\delta),$$

then we have

$$0 < l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \frac{d(\delta)}{a(\delta)};$$

particularly, if

$$I(u) > 0 \quad \textit{and} \quad J(u) \leq d,$$

then we have

$$0 < l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \frac{2(p+1)}{p-1}d,$$

(2) *if*

$$J(u) \le d(\delta) \quad \textit{and} \quad l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) > \frac{d(\delta)}{a(\delta)},$$

then we have

$$I_\delta(u) < 0;$$

particularly, if

$$J(u) \le d \quad \textit{and} \quad l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) > \frac{2(p+1)}{p-1} d,$$

then we have

$$I(u) < 0,$$

(3) *if*

$$I_\delta(u) = 0\,(\|u\|_{H^1} \ne 0) \quad \textit{and} \quad J(u) \le d(\delta),$$

then we have

$$r(\delta) \le l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \le \frac{d(\delta)}{a(\delta)} \quad \textit{and} \quad J(u) = d(\delta);$$

particularly, if

$$I(u) = 0\,(\|u\|_{H^1} \ne 0) \quad \textit{and} \quad J(u) \le d,$$

then we have

$$r(\delta) \le l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \le \frac{2(p+1)}{p-1} d \quad \textit{and} \quad J(u) = d.$$

Proof. (1) For $\delta \in (0, \frac{p+1}{2})$, there appears the relation

$$\begin{aligned} &a(\delta)\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) + \frac{I_\delta(u)}{p+1} \\ &\le \frac{1}{2}\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) - \int\limits_\Omega F(u)dx \\ &= J(u) \le d(\delta), \end{aligned} \tag{2.21}$$

which together with $I_\delta(u) > 0$ gives that

$$0 < l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \frac{d(\delta)}{a(\delta)}.$$

When $I(u) > 0$ and $J(u) \le d$, we discover that

$$\begin{aligned} &\frac{p-1}{2(p+1)}\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) \\ &< \frac{p-1}{2(p+1)}\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) + \frac{I(u)}{p+1} \\ &\le J(u) \le d. \end{aligned} \tag{2.22}$$

Thus, we deduce that

$$0 < l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \frac{2(p+1)}{p-1} d.$$

(2) This conclusion can be immediately obtained from

$$d(\delta) \geq J(u) \geq a(\delta)[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] + \frac{I_\delta(u)}{p+1}. \tag{2.23}$$

So we have $I_\delta(u) < 0$. Using the similar arguments, we have $I(u) < 0$.

(3) By $I_\delta(u) = 0$ ($\|u\|_{H^1} \neq 0$) and the definition of $d(\delta)$, it follows that $J(u) \geq d(\delta)$. Hence, we deduce that $J(u) = d(\delta)$. The remainder of the proof can be got from Lemma 2.4 (2) and (2.23). This completes the proof of Lemma 2.6. □

Lemma 2.7. *If the hypotheses* (A1), (A2) *hold and* $0 < J(u) < d$ *for some* $u \in H_0^1(\Omega)$. *Further assume that* (δ_1, δ_2) *is the maximal interval such that* $d(\delta) > J(u)$ *for* $\delta \in (\delta_1, \delta_2)$. *Then, the sign of* $I_\delta(u)$ *is unchangeable for* $\delta \in (\delta_1, \delta_2)$.

Proof. By $J(u) > 0$, we can see that $\|u\|_{H^1} \neq 0$. If the sign of $I_\delta(u)$ is changeable, then it is found that there exists $\widetilde{\delta} \in (\delta_1, \delta_2)$ satisfying $I_{\widetilde{\delta}}(u) = 0$. Thus, applying the definition of $d(\delta)$, we can obtain that $J(u) \geq d(\widetilde{\delta})$ which contradicts $J(u) < d(\widetilde{\delta})$. This completes the proof of Lemma 2.7. □

Here, we shall give some invariant sets under the flow of (1.1). The next lemma is similar to the lemmas appeared in [7, 12] with slight modification.

Lemma 2.8. *If the hypotheses (A1)–(A3) hold and* $u_0(x), u_1(x) \in H_0^1(\Omega)$. *We also assume that* $0 < e < d$ *and* (δ_1, δ_2) *is the maximal interval such that* $d(\delta) > e$ *for* $\delta \in (\delta_1, \delta_2)$. *Then:*

(1) *all solutions of the problem* (1.1) *with* $E(0) = e$ *belong to* W_δ *for* $\delta \in (\delta_1, \delta_2)$ *and* $t \in [0, T)$, *provided* $I(u_0) > 0$ *or* $\|u_0\|_{H^1} = 0$,
(2) *all solutions of the problem* (1.1) *with* $E(0) = e$ *belong to* V_δ *for* $\delta \in (\delta_1, \delta_2)$ *and* $t \in [0, T)$, *provided* $I(u_0) < 0$.

Proof. (1) Let u be the solutions of the problem (1.1) with initial data $E(0) = e$ and $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$. When $\|u_0\|_{H^1} = 0$, it is easy to see that $u_0(x) \in W_\delta$ for $\delta \in (0, \delta_0)$. When $I(u_0) > 0$, by Lemma 2.7 and

$$\frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + J(u_0) \leq E(0) < d(\delta), \quad \delta \in (\delta_1, \delta_2), \tag{2.24}$$

we discover that $I_\delta(u_0) > 0$, i.e. $u_0(x) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$.

We will prove $u(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$. By contradiction, suppose that there exists a time $t_0 \in (0, T)$ satisfying $u(t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1, \delta_2)$, then we have $I_\delta(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$ or $E(t_0) = d(\delta)$. By (1.7) and (1.8), it follows that

$$\frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_t\|_2^2 + J(u) \leq E(t) \leq E(0) < d(\delta), \quad \delta \in (\delta_1, \delta_2), t \in (0, T). \tag{2.25}$$

So we see that $E(t_0) \neq d(\delta)$. When $I_\delta(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$, then we have form the definition of $d(\delta)$ that $J(u(t_0)) > d(\delta)$, which contradicts (2.25). This ends the proof of part (1).

(2) Let u be the solutions of the problem (1.1) with initial data $E(0) = e$ and $I(u_0) < 0$. By the combination of (2.24) and Lemma 2.7, it follows that $I_\delta(u_0) < 0$ and $E(0) < d(\delta)$, which implies that $u_0(x) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$. Furthermore, we will prove $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$. By contradiction and then suppose that there exists a time $t_0 \in (0, T)$ satisfying $u(t_0) \in \partial V_\delta$ for some $\delta \in (\delta_1, \delta_2)$, then we have $I_\delta(u(t_0)) = 0$ or $E(t_0) = d(\delta)$. Applying (2.25) again, we can see $E(t_0) \neq d(\delta)$. When $I_\delta(u(t_0)) = 0$, then we have from the definition of $d(\delta)$ that $J(u(t_0)) > d(\delta)$, which also contradicts (2.25). This completes the proof of Lemma 2.8. □

Remark 2.9. In view of Lemma 2.5 and Lemma 2.8, it is easy to see that if the hypothesis $E(0) = e$ in Lemma 2.8 is corrected to $0 < E(0) \leq e$, then all conclusions of Lemma 2.8 also hold.

Before giving next lemma, we introduce some notations as follows:

$$\lambda_1 = \left(\frac{l}{\alpha B_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}, \quad \tilde{d} = h(\lambda_1) = \frac{p-1}{2(p+1)} \alpha^{\frac{-2}{p-1}} \left(\frac{l}{B_{p+1}^2}\right)^{\frac{p+1}{p-1}}. \tag{2.26}$$

In fact, based on the definition of $J(u)$ and Sobolev inequality, we can discover that

$$\begin{aligned} J(u) &= \frac{1}{2} l \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega F(u)dx \\ &\geq \frac{1}{2} l \|\nabla u\|_2^2 - \frac{1}{p+1} \int_\Omega f(u)u dx \\ &\geq \frac{1}{2} l \|\nabla u\|_2^2 - \frac{\alpha}{p+1} \|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2} l \|\nabla u\|_2^2 - \frac{\alpha}{p+1} B_{p+1}^{p+1} \|\nabla u\|_2^{p+1}. \end{aligned} \tag{2.27}$$

Hence, let us define the function $h(\lambda) = \frac{1}{2} l \lambda^2 - \frac{\alpha}{p+1} B_{p+1}^{p+1} \lambda^{p+1}$, $\lambda > 0$. Through a series of direct computation, we can obtain that h is an increasing function for $0 < \lambda < \lambda_1$, decreasing function for $\lambda > \lambda_1$ and the constant

$$\lambda_1 = \left(\frac{l}{\alpha B_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}$$

is the maximum point of h satisfying

$$\tilde{d} = h(\lambda_1) = \frac{\alpha^{\frac{-2}{p-1}}(p-1)}{2(p+1)} \left(\frac{l}{B_{p+1}^2}\right)^{\frac{p+1}{p-1}}.$$

Lemma 2.10. *Assume that the conditions* (A1), (A2) *hold. For any fixed number* $\beta \in (0,1)$*, further assume* $I(u_0) < 0$ *and* $E(0) < \beta\tilde{d}$*, then we conclude that* $I(u) < 0$ *and*

$$\tilde{d} < \frac{p-1}{2(p+1)}[l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] < \frac{\alpha(p-1)}{2(p+1)}\|u\|_{p+1}^{p+1}, \tag{2.28}$$

for all $t \in [0,T)$.

Proof. Applying contradiction, we first obtain that $I(u) < 0$ for all $t \in [0,T)$. Indeed, if it is false, then there exist a time $t_0 > 0$ satisfying $I(u(t_0)) = 0$ and $I(u) < 0$ for $t \in [0,t_0)$. So we have

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \int_\Omega f(u)u dx \le \alpha\|u\|_{p+1}^{p+1}, \quad t \in [0,t_0). \tag{2.29}$$

Using (2.29) and the definition of $\tilde{d}$, there appears the relation

$$\begin{aligned}
\tilde{d} &= \frac{\alpha^{\frac{-2}{p-1}}(p-1)}{2(p+1)}\left(\frac{l}{B_{p+1}^2}\right)^{\frac{p+1}{p-1}} \\
&\le \frac{\alpha^{\frac{-2}{p-1}}(p-1)}{2(p+1)}\left\{\frac{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}{\|u\|_{p+1}^2}\right\}^{\frac{p+1}{p-1}} \\
&< \frac{p-1}{2(p+1)}\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big) \\
&< \frac{(p-1)}{2(p+1)}\int_\Omega f(u)u dx, \quad t \in [0,t_0).
\end{aligned} \tag{2.30}$$

Applying the time continuity of $\int_\Omega f(u)udx$, then the combination of (1.9) and (2.30) yields that

$$\begin{aligned}
\tilde{d} &\le \frac{(p-1)}{2(p+1)}\int_\Omega f(u(t_0))u(t_0)dx \\
&= \frac{1}{2}\int_\Omega f(u(t_0))u(t_0)dx - \frac{1}{p+1}\int_\Omega f(u(t_0))u(t_0)dx \\
&\le \frac{1}{2}\big(l\|\nabla u(t_0)\|_2^2 + (g \circ \nabla u)(t_0)\big) - \int_\Omega F(u(t_0))dx \\
&= J(u(t_0)).
\end{aligned} \tag{2.31}$$

Since $J(u(t_0)) \leq E(t_0) \leq E(0) < \tilde{d}$, this contradicts the above inequality. Thus, we deduce that $I(u(t)) < 0$ for $t \in [0,T)$. On the other hand, we have from (2.29) and (2.30) that

$$\begin{aligned}\tilde{d} &< \frac{p-1}{2(p+1)}\big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\big)\\ &< \frac{(p-1)}{2(p+1)}\int\limits_\Omega f(u)udx \leq \frac{(p-1)\alpha}{2(p+1)}\|u\|_{p+1}^{p+1},\end{aligned} \tag{2.32}$$

for all $t \in [0,T)$. We thereby end the proof of Lemma 2.10. □

3. GLOBAL EXISTENCE OF SOLUTIONS

The main goal in this section is devoted to discuss the existence of global weak solutions for the problem (1.1) at the low initial energy and critical initial energy by using the Galerkin approximation and improved potential well method involving with t.

3.1. GLOBAL EXISTENCE AT LOW INITIAL ENERGY LEVEL

In this subsection, when the initial data satisfies $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$, the existence of global weak solutions of the problem (1.1) is given in Theorem 1.3.

Proof of Theorem 1.3. Let $\{w_j(x)\}$ be a complete orthogonal basis in $H_0^1(\Omega)$. Then we construct the approximate solutions u_m for the problem (1.1) in the form

$$u_m(t) = \sum_{j=1}^m d_m^j(t)w_j(x), \quad m = 1,2,\dots, \tag{3.1}$$

satisfying

$$\begin{aligned}&(|u_m'|^{\rho-1}u_m'', w_j) + (\nabla u_m, \nabla w_j) - \left(\int\limits_0^t g(t-s)\nabla u_m(s)ds, \nabla w_j\right)\\ &+ (\nabla u'_m, \nabla w_j) + (\nabla u_m'', \nabla w_j) = (f(u_m), w_j),\end{aligned} \tag{3.2}$$

and

$$u_m(x,0) = \sum_{j=1}^m d_m^j(0)\omega_j(x) \to u_0(x) \ in \ H_0^1(\Omega), \quad m \to \infty, \tag{3.3}$$

$$u_m'(x,0) = \sum_{j=1}^m d_m'^j(0)\omega_j(x) \to u_1(x) \ in \ H_0^1(\Omega), \quad m \to \infty. \tag{3.4}$$

Now, multiplying (3.2) by $d_m'^j(t)$ and summing for $j = 1, \dots, m$, then we have

$$\begin{aligned}&(|u_m'|^{\rho-1}u_m'', u_m') + (\nabla u_m, \nabla u_m') - \left(\int_0^t g(t-s)\nabla u_m(s)ds, \nabla u_m'\right)\\ &+ (\nabla u'_m, \nabla u_m') + (\nabla u_m'', \nabla u_m') = (f(u_m), u_m').\end{aligned} \tag{3.5}$$

A series of calculation gives that

$$(|u_m'|^{\rho-1}u_m'', u_m') = \frac{1}{\rho+1}\frac{d}{dt}\|u_m'\|_{\rho+1}^{\rho+1}, \tag{3.6}$$

$$(\nabla u_m, \nabla u'_m) = \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|_2^2, \quad (\nabla u_m'', \nabla u_m') = \frac{1}{2}\frac{d}{dt}\|\nabla u_m'\|_2^2, \tag{3.7}$$

$$(f(u_m), u_m') = \frac{d}{dt}\int_\Omega \int_0^{u_m} f(s)dsdx = \frac{d}{dt}\int_\Omega F(u_m)dx, \tag{3.8}$$

and

$$\begin{aligned}&-\int_\Omega \int_0^t g(t-s)\nabla u_m(s)\nabla u'_m(t)dsdx\\ &= -\int_\Omega \int_0^t g(t-s)\big(\nabla u_m(s) - \nabla u_m(t)\big)\nabla u'_m(t)dsdx\\ &\quad -\int_\Omega \int_0^t g(t-s)\nabla u_m(t)\nabla u'_m(t)dsdx\\ &= \frac{1}{2}\int_\Omega \int_0^t g(t-s)\frac{d}{dt}\big(\nabla u_m(s) - \nabla u_m(t)\big)^2 dsdx\\ &\quad - \frac{1}{2}\int_\Omega \int_0^t g(t-s)\frac{d}{dt}\big(\nabla u_m(t)\big)^2 dsdx\\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^t g(t-s)\big(\nabla u_m(s) - \nabla u_m(t)\big)^2 dsdx\\ &\quad - \frac{1}{2}\int_\Omega \int_0^t g'(t-s)\big(\nabla u_m(s) - \nabla u_m(t)\big)^2 dsdx\\ &\quad - \frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u_m(t)\|_2^2 + \frac{1}{2}g(t)\|\nabla u_m\|_2^2.\end{aligned} \tag{3.9}$$

Inserting (3.6)–(3.9) into (3.5), there appears the relation

$$\begin{aligned}&\frac{1}{\rho+1}\frac{d}{dt}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}\frac{d}{dt}\left(l(t)\|\nabla u_m(t)\|_2^2\right)+\frac{1}{2}\frac{d}{dt}\|\nabla u_m'\|_2^2\\&+\frac{1}{2}\frac{d}{dt}(g\circ\nabla u_m)(t)-\frac{d}{dt}\int_\Omega F(u_m)dx\\&=-\|\nabla u_m'\|_2^2+\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\|\nabla u_m\|_2^2\le 0,\end{aligned}\tag{3.10}$$

which shows that

$$E_m'(t)=-\|\nabla u_m'\|_2^2+\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\|\nabla u_m(t)\|_2^2\le 0,\tag{3.11}$$

where

$$\begin{aligned}E_m(t)&=E(u_m,u_m')\\&=\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}\|\nabla u_m'\|_2^2+\frac{1}{2}l(t)\|\nabla u_m(t)\|_2^2\\&\quad+\frac{1}{2}(g\circ\nabla u_m)(t)-\int_\Omega F(u_m)dx.\end{aligned}$$

In view of (1.8), (1.9) and (3.11), we discover that

$$J(u_m)+\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}\|\nabla u_m'\|_2^2\le E_m(t)\le E_m(0),\quad 0\le t<\infty.\tag{3.12}$$

By the hypotheses $E(0)<d$ and $I(u_0)>0$ or $\|u_0\|_{H^1}=0$, we can see that $u_0(x)\in W$. So applying (3.3) and (3.4), we further get that $E_m(0)<d$, $I(u_m(0))>0$ and then $u_m(0)\in W$ for large enough m. Moreover, using (3.12) and same arguments as Lemma 2.8 (1), we obtain that $u_m(t)\in W$ and

$$\begin{aligned}E_m(t)\ge J(u_m)&=\frac{1}{2}l\|\nabla u_m\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)-\int_\Omega F(u_m)dx\\&\ge\frac{1}{2}l\|\nabla u_m\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)-\frac{1}{p+1}\int_\Omega f(u_m)u_m dx\\&=\frac{p-1}{2(p+1)}[l\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]+\frac{1}{p+1}I(u_m)\\&\ge\frac{p-1}{2(p+1)}[l\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]\ge 0,\end{aligned}\tag{3.13}$$

for m is large enough and $t\in[0,\infty)$. Hence, we have from (3.12) and (3.13) that

$$\begin{aligned}&\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}\|\nabla u_m'\|_2^2+\frac{p-1}{2(p+1)}[l\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]\\&\le E_m(t)<d,\end{aligned}\tag{3.14}$$

for m enough large and $t \in [0, \infty)$. By (3.13) and (3.14), we have that

$$\|\nabla u_m\|_2^2 < \frac{2(p+1)}{(p-1)l} d, \quad 0 \le t < \infty, \tag{3.15}$$

$$\|\nabla u'_m\|_2^2 < 2d, \quad 0 \le t < \infty, \tag{3.16}$$

$$\|u'_m\|_{\rho+1}^{\rho+1} \le (\rho+1)d, \quad 0 \le t < \infty, \tag{3.17}$$

$$|(f(u_m), u_m)| \le \alpha \|u_m\|_{p+1}^{p+1} \le \alpha B_{p+1}^{p+1} \left(\frac{2(p+1)}{(p-1)l} d \right)^{\frac{p+1}{2}}, \ 0 \le t < \infty. \tag{3.18}$$

Integrating (3.11) from 0 to t, there appears the relation

$$\frac{1}{2} \int_0^t g(t-s) \|\nabla u_m\|_2^2 ds = E_m(0) - E_m(t) - \int_0^t \|\nabla u'_m\|_2^2 ds + \frac{1}{2} \int_0^t (g' \circ \nabla u_m)(s) ds,$$

which together with (3.13), (3.14) and condition (A2) yields that

$$\int_0^t g(t-s) \|\nabla u_m\|_2^2 ds < 2d, \quad 0 \le t < \infty. \tag{3.19}$$

Furthermore, multiplying (3.2) by $d''^j_m(t)$ and summing for $j = 1, \dots, m$, we discover that

$$\begin{aligned} &(|u'_m|^{\rho-1} u''_m, u''_m) + (\nabla u_m, \nabla u''_m) - \left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla u''_m \right) \\ &+ (\nabla u'_m, \nabla u''_m) + (\nabla u''_m, \nabla u''_m) = (f(u_m), u''_m). \end{aligned} \tag{3.20}$$

Applying the Young, Hölder and Sobolev inequalities, we have form (3.20) and (A1) that

$$\begin{aligned} &\int_\Omega |u'_m|^{\rho-1} |u''_m|^2 dx + \|\nabla u''_m\|_2^2 \\ &= \left(\int_0^t g(t-s) \nabla u_m(s) ds, \nabla u''_m \right) \\ &\quad - (\nabla u_m, \nabla u''_m) + (\nabla u'_m, \nabla u''_m) + (f(u_m), u''_m) \\ &\le 3\epsilon \|\nabla u''_m\|_2^2 + \frac{B_{p+1}^2 \epsilon}{2} \|\nabla u''_m\|_2^2 + \frac{\alpha^2 B_{p+1}^{2p}}{8\epsilon} \|\nabla u_m\|_2^{2p} \\ &\quad + \frac{1}{4\epsilon} \|\nabla u_m\|_2^2 + \frac{1}{4\epsilon} \|\nabla u'_m\|_2^2 + \frac{1}{4\epsilon} \int_0^t g(s) ds \int_0^t g(t-s) \|\nabla u_m\|_2^2 ds, \end{aligned} \tag{3.21}$$

where B_{p+1} has been defined in Theorem 4.1. Here, let us take ϵ small enough such that $1-3\epsilon-\frac{B_{p+1}^2\epsilon}{2}>0$. So, by a simple calculation, (3.21) becomes

$$\begin{aligned}&\int_\Omega |u_m'|^{\rho-1}|u_m''|^2dx+\left(1-2\epsilon-\frac{B_{p+1}^2\epsilon}{2}\right)\|\nabla u_m''\|_2^2\\&\le\frac{\alpha^2B_{p+1}^{2p}}{8\epsilon}\|\nabla u_m\|_2^{2p}+\frac{1}{4\epsilon}\|\nabla u_m\|_2^2+\frac{1}{4\epsilon}\|\nabla u_m'\|_2^2\\&\quad+\frac{1}{4\epsilon}\int_0^t g(s)ds\int_0^t g(t-s)\|\nabla u_m\|_2^2ds.\end{aligned} \tag{3.22}$$

Inserting (3.15), (3.16) and (3.19) into (3.22), we can obtain the following inequality

$$\|\nabla u_m''\|_2^2<C,\quad 0\le t<\infty, \tag{3.23}$$

where C is a positive constant independent of time t. Applying Hölder inequality again, we have

$$(|u_m'|^{\rho-1}u_m'',u_m'')\le\|u_m'\|_{\rho+1}^{\rho-1}\|u_m''\|_{\rho+1}^2\le((\rho+1)d)^{\frac{\rho+1}{\rho-1}}\|u_m''\|_{\rho+1}^2,\quad 0\le t<\infty. \tag{3.24}$$

The estimates (3.15)–(3.18), (3.23) and (3.24) allow us to get a subsequence of $\{u_m\}$, which from now on will be denoted by $\{u_m\}$ and functions u, χ_1, χ_2, χ_3 such that

$$u_m\to u \text{ in } L^\infty(0,\infty;H_0^1(\Omega)) \text{ weakly star},\quad m\longrightarrow\infty, \tag{3.25}$$

$$u_m'\to u' \text{ in } L^\infty(0,\infty;H_0^1(\Omega)) \text{ weakly star},\quad m\longrightarrow\infty, \tag{3.26}$$

$$u_m'\to u' \text{ in } L^\infty(0,\infty;L^{\rho+1}(\Omega)) \text{ weakly star},\quad m\longrightarrow\infty, \tag{3.27}$$

$$u_m''\to u'' \text{ in } L^\infty(0,\infty;H_0^1(\Omega)) \text{ weakly},\quad m\longrightarrow\infty, \tag{3.28}$$

$$f(u_m)\to\chi_1 \text{ in } L^\infty(0,\infty;L^{\frac{p+1}{p}}(\Omega)) \text{ weakly star},\quad m\longrightarrow\infty, \tag{3.29}$$

$$|u_m'|^{\rho-1}u_m'\to\chi_2 \text{ in } L^\infty(0,\infty;L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly star},\quad m\longrightarrow\infty. \tag{3.30}$$

$$|u_m'|^{\rho-1}u_m''\to\chi_3 \text{ in } L^\infty(0,\infty;L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly},\quad m\longrightarrow\infty. \tag{3.31}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compactness, we have, thanks to the Aubin–Lions theorem, that

$$u_m \to u \text{ in } L^\infty(0,\infty; L^2(\Omega)) \text{ strongly}, \quad m \longrightarrow \infty, \tag{3.32}$$

$$u'_m \to u' \text{ in } L^\infty(0,\infty; L^2(\Omega)) \text{ strongly}, \quad m \longrightarrow \infty, \tag{3.33}$$

$$u''_m \to u'' \text{ in } L^\infty(0,\infty; L^2(\Omega)) \text{ strongly}, \quad m \longrightarrow \infty, \tag{3.34}$$

and further using Lemma 1.3 in [17], we can deduce

$$f(u_m) \to \chi_1 = f(u) \text{ in } L^\infty(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly star}, m \longrightarrow \infty, \tag{3.35}$$

$$|u'_m|^{\rho-1}u'_m \to \chi_2 = |u'|^{\rho-1}u' \text{ in } L^\infty(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly star}, m \longrightarrow \infty, \tag{3.36}$$

$$|u'_m|^{\rho-1}u''_m \to \chi_3 = |u'|^{\rho-1}u'' \text{ in } L^2(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly}, m \longrightarrow \infty. \tag{3.37}$$

Taking $m \to \infty$ in the (3.2) and then making use of (3.25)–(3.28) and (3.35)–(3.37), we discover that

$$\begin{aligned}(|u'|^{\rho-1}u'', w_j) + (\nabla u, \nabla w_j) - \left(\int_0^t g(t-s)\nabla u(s)ds, \nabla w_j\right) \\ + (\nabla u', \nabla w_j) + (\nabla u'', \nabla w_j) = (f(u), w_j).\end{aligned} \tag{3.38}$$

Considering that the basis $\{w_j(x)\}_{j=1}^\infty$ are dense in $H_0^1(\Omega)$, we choose a function $v \in H_0^1(\Omega)$ having the form $v = \sum_{j=1}^\infty d_j w_j(x)$, where $\{d_j\}_1^\infty$ are given functions. Multiplying (3.38) by d_j and then summing for $j = 1, \ldots$, it follows that

$$\begin{aligned}(|u'|^{\rho-1}u'', v) + (\nabla u, \nabla v) - \left(\int_0^t g(t-s)\nabla u(s)ds, \nabla v\right) \\ + (\nabla u', \nabla v) + (\nabla u'', \nabla v) = (f(u), v), \quad \forall\, v \in H_0^1(\Omega).\end{aligned}$$

Next, we shall prove that the total energy $E(t)$ satisfies (1.7). In terms of above discussion, for each fixed $t > 0$ we have

$$\begin{aligned}
&|(g \circ \nabla u)(t) - (g \circ \nabla u_m)(t)| \\
&= \Big| \int_0^t g(t-s) \int_\Omega |\nabla u(s) - \nabla u(t)|^2 dx ds \\
&\quad - \int_0^t g(t-s) \int_\Omega |\nabla u_m(s) - \nabla u_m(t)|^2 dx ds \Big| \\
&\le \int_0^t g(t-s) \|\nabla u(s) - \nabla u_m(s)\|_2 \|\nabla u(s) + \nabla u_m(s)\|_2 ds \\
&\quad + \int_0^t g(t-s) \|\nabla u(s) - \nabla u_m(s)\|_2 ds \|\nabla u(t) + \nabla u_m(t)\|_2 \\
&\quad + \int_0^t g(t-s) \|\nabla u(s) + \nabla u_m(s)\|_2 ds \|\nabla u(t) - \nabla u_m(t)\|_2 \\
&\quad + \int_0^t g(s) ds \|\nabla u(t) + \nabla u_m(t)\|_2 \|\nabla u(t) - \nabla u_m(t)\|_2 \\
&\le C \int_0^t g(t-s) \|\nabla u(s) - \nabla u_m(s)\|_2 ds \\
&\quad + C \int_0^t g(s) ds \|\nabla u(t) - \nabla u_m(t)\|_2 \to 0 \quad \text{as } m \to \infty.
\end{aligned} \tag{3.39}$$

By a simple calculation, we also discover that

$$\begin{aligned}
&\Big| \int_\Omega F(u_m) dx - \int_\Omega F(u) dx \Big| = \Big| \int_\Omega f(u + \theta_m u_m)(u_m - u) dx \Big| \\
&\le \|f(u + \theta_m u_m)\|_{\frac{p+1}{p}} \|u_m - u\|_{p+1} \le C \|u_m - u\|_{p+1} \to 0 \quad \text{as } m \to \infty,
\end{aligned} \tag{3.40}$$

where $0 < \theta_m < 1$. Thus, we have from (3.39) and (3.40) that

$$\lim_{m\to\infty} (g \circ \nabla u_m)(t) = (g \circ \nabla u)(t), \quad \lim_{m\to\infty} \int_\Omega F(u_m) dx = \int_\Omega F(u) dx. \tag{3.41}$$

On the other hand, (3.3) and (3.4) implies that $E_m(0) \to E(0)$, as $m \to \infty$. Thus, using Fatou's Lemma and (3.12), we can deduce that

$$\begin{aligned}
&\frac{1}{\rho+1}\|u'\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u'\|_2^2 + \frac{1}{2}l(t)\|\nabla u\|_2^2 \\
&\leq \liminf_{m\to\infty}\left(\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_m'\|_2^2 + \frac{1}{2}l(t)\|\nabla u_m\|_2^2\right) \\
&= \liminf_{m\to\infty}\left(E_m(t) + \int_\Omega F(u_m)dx - \frac{1}{2}(g\circ\nabla u_m)(t)\right) \\
&\leq \lim_{m\to\infty}\left(E_m(0) + \int_\Omega F(u_m)dx - \frac{1}{2}(g\circ\nabla u_m)(t)\right) \\
&= E(0) + \int_\Omega F(u)dx - \frac{1}{2}(g\circ\nabla u)(t),
\end{aligned} \tag{3.42}$$

which shows $E(t) \leq E(0)$. We thereby prove that u is a global weak solution of the problem (1.1) in Definition 1.1. Finally, applying Lemma 2.8 (1) again, we can easily conclude that $u(t) \in W$ for $0 \leq t < \infty$. This completes the proof of Theorem 1.3. □

3.2. GLOBAL EXISTENCE AT CRITICAL INITIAL ENERGY LEVEL

Our aim of this subsection is to prove the existence of global weak solutions for the problem (1.1) under the critical initial conditions $I(u_0) \geq 0$ and $E(0) = d$. The detailed content is shown in Theorem 1.4.

Proof of Theorem 1.4. We prove Theorem 1.4 by considering two cases

$$\|\nabla u_0\|_2^2 = 0 \quad \text{and} \quad \|\nabla u_0\|_2^2 \neq 0.$$

(1) $\|\nabla u_0\|_2^2 \neq 0$.
(i) If $I(u_0) > 0$, then a direct calculation gives that

$$\frac{d}{d\lambda}E(\lambda u_0, u_1)|_{\lambda=1} = \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + \|\nabla u_0\|_2^2 - \int_\Omega f(u_0)u_0dx,$$

which together with $l \leq l(t) \leq 1$ yields that

$$\frac{d}{d\lambda}E(\lambda u_0, u_1)|_{\lambda=1} \geq \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + I(u_0) > 0. \tag{3.43}$$

Hence, there exists an interval (λ_1, λ_2) such that $\frac{d}{d\lambda}E(\lambda u_0, u_1) > 0$ and $I(\lambda u_0) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$. We take a sequence $\{\lambda_m\}$ such that $\lambda_1 < \lambda_m < 1$, $m = 1, 2, \ldots$ and $\lambda_m \to 1$ as $m \to \infty$. Let us define $u_{0m}(x) = \lambda_m u_0(x)$, then we consider the problem (1.1) with the initial conditions

$$u(x,0) = u_{0m}(x), \quad u_t(x,0) = u_1(x), \tag{3.44}$$

where the initial data u_0, u_1 in (1.1) are replaced by $u_{0m}(x), u_1(x)$.

Firstly, by the combination of $I(u_0) > 0$ and Lemma 2.3 (3), we obtain that there exists a $\lambda^* = \lambda^*(0) > 1$ and $I(\lambda u_0) > 0$ for $0 \le \lambda < \lambda^*$. So we have that $I(u_{0m}) = I(\lambda_m u_0) > 0$ and

$$\begin{aligned} E_m(0) &= E(u_{0m}, u_1) \\ &= \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + \frac{1}{2}\|\nabla u_{0m}\|_2^2 - \int_\Omega F(u_{0m})dx \\ &\ge \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 + \frac{p-1}{2(p+1)}l\|\nabla u_{0m}\|_2^2 + \frac{1}{p+1}I(u_{0m}) > 0, \end{aligned} \tag{3.45}$$

for sufficiently large m. Furthermore, we have from (3.44) that

$$0 < E_m(0) = E(u_{0m}, u_1) = E(\lambda_m u_0, u_1) < E(u_0, u_1) = E(0) = d. \tag{3.46}$$

(ii) If $I(u_0) = 0$, by the definitions of d and $E(t)$, then we have

$$E(0) = E(u_0, u_1) \ge J(u_0) \ge d, \tag{3.47}$$

which along with $E(0) = d$ gives $J(u_0) = d$. Furthermore, applying Lemma 2.3 again, it follows that $\lambda^* = \lambda^*(0) = 1$, $J(\lambda u_0)$ is an increasing function on $\lambda \in [0,1]$, and $I(\lambda u_0) > 0$ for $\lambda \in (0,1)$. we choose a sequence $\{\lambda_m\}$ satisfying $\lambda_1 < \lambda_m < 1$, $m = 1, 2, \dots$ and $\lambda_m \to 1$ as $m \to \infty$. Let $u_{0m}(x) = \lambda_m u_0(x)$ and then investigate the problem (1.1) with initial conditions $u(x,0) = u_{0m}(x)$, $u_t(x,0) = u_1(x)$. It is easy to see that $I(u_{0m}) = I(\lambda_m u_0) > 0$. And we deduce that (3.46) also holds here.

Hence, applying Theorem 1.3, we obtain that the problem (1.1) (the initial conditions u_0, u_1 are corrected to $u_{0m}(x), u_1(x)$) admits a global weak solution $u_m \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_m' \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_m'' \in L^\infty(0,\infty; H_0^1(\Omega))$ and $u_m \in W$ satisfying

$$\begin{aligned} &(|u_m'|^{\rho-1}u_m'', v) + (\nabla u_m, \nabla v) - \left(\int_0^t g(t-s)\nabla u_m(s)ds, \nabla v\right) \\ &+ (\nabla u_m', \nabla v) + (\nabla u_m'', \nabla v) = (f(u_m), v), \end{aligned} \tag{3.48}$$

for all $v \in H_0^1(\Omega)$ and $t \in [0,\infty)$. By a series of calculation, we also have that

$$\begin{aligned} d > E_m(0) &\ge \frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_m'\|_2^2 + \frac{1}{2}(g \circ \nabla u_m)(t) \\ &\quad + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_m(t)\|_2^2 - \int_\Omega F(u_m)dx \\ &\ge \frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_m'\|_2^2 + J(u_m) \\ &\ge \frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_m'\|_2^2 \\ &\quad + \frac{p-1}{2(p+1)}\big(l\|\nabla u_m\|_2^2 + (g \circ \nabla u_m)(t)\big), \end{aligned} \tag{3.49}$$

and

$$\int_\Omega |u'_m|^{\rho-1}|u''_m|^2 dx + \left(1 - 2\epsilon - \frac{B_{p+1}^2\epsilon}{2}\right)\|\nabla u''_m\|_2^2$$
$$\leq \frac{\alpha^2 B_{p+1}^{2p}}{8\epsilon}\|\nabla u_m\|_2^{2p} + \frac{1}{4\epsilon}\|\nabla u_m\|_2^2 + \frac{1}{4\epsilon}\|\nabla u'_m\|_2^2 \tag{3.50}$$
$$+ \frac{1}{4\epsilon}\int_0^t g(s)ds \int_0^t g(t-s)\|\nabla u_m\|_2^2 ds,$$

for some positive constant ϵ such that $1 - 3\epsilon - \frac{B_{p+1}^2\epsilon}{2} > 0$. In view of (3.49) and (3.50), we deduce that

$$\|\nabla u_m\|_2^2 < \frac{2(p+1)}{(p-1)l}d, \quad 0 \leq t < \infty, \tag{3.51}$$

$$\|\nabla u'_m\|_2^2 < 2d, \quad 0 \leq t < \infty, \tag{3.52}$$

$$\|\nabla u''_m\|_2^2 < C, \quad 0 \leq t < \infty. \tag{3.53}$$

$$\|u'_m\|_{\rho+1}^{\rho+1} \leq (\rho+1)d, \quad 0 \leq t < \infty, \tag{3.54}$$

$$|(f(u_m), u_m)| \leq \alpha\|u_m\|_{p+1}^{p+1} \leq \alpha B_{p+1}^{p+1}\left(\frac{2(p+1)}{(p-1)l}d\right)^{\frac{p+1}{2}}, \; 0 \leq t < \infty, \tag{3.55}$$

$$(|u'_m|^{\rho-1}u''_m, u''_m) \leq ((\rho+1)d)^{\frac{\rho+1}{\rho-1}}\|u''_m\|_{\rho+1}^2, \quad 0 \leq t < \infty. \tag{3.56}$$

In view of estimates (3.51)–(3.56) and then using similar arguments as Theorem 1.3, we can see that there exists a subsequences of $\{u_m\}$ which will be denoted by the form $\{u_m\}$ and function $u \in \overline{W}$ satisfying

$$u_m \to u \text{ in } L^\infty(0,\infty; H_0^1(\Omega)) \text{ weakly star}, \; m \longrightarrow \infty, \tag{3.57}$$

$$u'_m \to u' \text{ in } L^\infty(0,\infty; H_0^1(\Omega)) \text{ weakly star}, \; m \longrightarrow \infty, \tag{3.58}$$

$$u'_m \to u' \text{ in } L^\infty(0,\infty; L^{\rho+1}(\Omega)) \text{ weakly star}, \; m \longrightarrow \infty, \tag{3.59}$$

$$u''_m \to u'' \text{ in } L^\infty(0,\infty; H_0^1(\Omega)) \text{ weakly}, \; m \longrightarrow \infty, \tag{3.60}$$

$$f(u_m) \to f(u) \text{ in } L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega) \text{ weakly star}, \; m \longrightarrow \infty, \tag{3.61}$$

$$|u'_m|^{\rho-1}u'_m \to |u'|^{\rho-1}u' \text{ in } L^\infty(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly star}, \; m \longrightarrow \infty. \tag{3.62}$$

$$|u'_m|^{\rho-1}u''_m \to |u'|^{\rho-1}u'' \text{ in } L^\infty(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \text{ weakly}, \; m \longrightarrow \infty. \tag{3.63}$$

Taking $m \to \infty$ in (3.48), it follows that (1.6) is holding. Finally, the analogous arguments as the proof of Theorem 1.3, we can conclude that $E(t) \leq E(0)$. Therefore, we prove that u is a global weak solution to the problem (1.1) in the sense of Definition 1.1.

(2) $\|\nabla u_0\|_2^2 = 0$.

In this case, it is easy to see that $J(u_0) = 0$ and

$$\frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 = E(0) = d.$$

We choose a sequence $\{\lambda_m\}$ again, which satisfy $0 < \lambda_m < 1$ and $\lambda_m \to 1$ as $m \to \infty$. Let $u_{1m}(x) = \lambda_m u_1(x)$. Then, we consider the problem (1.1) with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_{1m}(x). \tag{3.64}$$

We note that

$$\begin{aligned} E_m(0) = E(u_0, u_{1m}) &= \frac{1}{\rho+1}\|u_{1m}\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_{1m}\|_2^2 \\ &< \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_1\|_2^2 = E(0) = d. \end{aligned} \tag{3.65}$$

Hence, applying Theorem 1.3, we know that the problem (1.1) (the initial conditions u_0, u_1 are corrected to $u_0(x), u_{1m}(x)$) admits a global weak solution $u_m \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_m' \in L^\infty(0,\infty; H_0^1(\Omega))$, $u_m'' \in L^\infty(0,\infty; H_0^1(\Omega))$ with $u_m \in W$ for $0 \leq t < \infty$, which also satisfies (3.57)–(3.63). The remainder of the proof is the same as part (1) of this theorem. We thereby complete the proof of Theorem 1.4. □

4. FINITE TIME BLOW-UP

This section is focused on the finite time blow-up phenomenal of the solutions for the problem (1.1). Here, we not only derive the finite time blow-up result for certain solutions with initial data at non-positive energy and positive energy, but also establish the upper and lower bounds for the blow-up time T^*.

4.1. UPPER BOUND FOR BLOW-UP TIME

In this subsection, a blow-up criterion which ensures that the solutions u can not exist all time is derived, and an upper bound for blow-up time is given in Theorem 1.7.

To obtain the results of this section, we need to introduce the following lemma.

Lemma 4.1. *Let p satisfy the condition* (A1) *and $2 \leq s \leq p+1$. Then there exists a positive constant $C > 1$ depending on Ω only such that*

$$\|u\|_{p+1}^s \leq C(\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}) \tag{4.1}$$

for any $u \in H_0^1(\Omega)$.

Proof. If $\|u\|_{p+1} \leq 1$, then

$$\|u\|_{p+1}^{s} \leq \|u\|_{p+1}^{2} \leq B_{p+1}^{2} \|\nabla u\|_{2}^{2}$$

by the Sobolev embedding theorem. If $\|u\|_{p+1} > 1$, then

$$\|u\|_{p+1}^{s} \leq \|u\|_{p+1}^{p+1}.$$

Thus, the proof is completed. □

Now, we give the detailed proof of Theorem 1.7 as follows.

Proof of Theorem 1.7. Applying Lemma 2.10, we note that if $I(u_0) < 0$ and $E(0) < \beta\tilde{d}$, then $I(u) < 0$ for any $t \in [0, T_{\max})$. Arguing by contradiction, let us suppose that the solutions u of the problem (1.1) is global existence. Hence, for any $T > 0$, we may introduce the functional $\theta : [0, T] \to R^{+}$ in the form

$$\theta(t) = \|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u_t\|_{2}^{2} + \|u\|_{p+1}^{p+1}. \tag{4.2}$$

Since $\theta(t)$ is continuous on $[0, T]$, there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq \theta(t) \leq \delta_2$. Moreover, let us define

$$H(t) = \beta\tilde{d} - E(t), \tag{4.3}$$

for $t \in [0, \infty)$. Taking a derivative to (4.3), it follows that

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \|\nabla u_t\|_{2}^{2} + \frac{1}{2} g(t) \|\nabla u(t)\|_{2}^{2} \\ &\quad - \frac{1}{2} \int_{\Omega} \int_{0}^{t} g'(t-s) [\nabla u(s) - \nabla u(t)]^{2} ds dx \geq 0, \end{aligned} \tag{4.4}$$

and

$$H(t) \geq H(0) = \beta\tilde{d} - E(0) > 0. \tag{4.5}$$

In view of (1.8) and conditions (A1)(iii), we can obtain that

$$-E(t) \leq \int_{\Omega} F(u) dx \leq \frac{1}{p+1} \int_{\Omega} f(u) u dx \leq \frac{\alpha}{p+1} \|u\|_{p+1}^{p+1},$$

which along with (4.3) and Lemma 2.10 yields

$$H(t) \leq \beta\tilde{d} + \frac{\alpha}{p+1} \|u\|_{p+1}^{p+1} \leq \alpha \left(\frac{\beta(p-1)}{2(p+1)} + \frac{1}{p+1} \right) \|u\|_{p+1}^{p+1}, \tag{4.6}$$

for $t \in [0, \infty)$.

Next, we further define

$$\begin{aligned}\mathcal{G}(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho}\int\limits_{\Omega} |u_t|^{\rho-1}u_t u dx \\ + \varepsilon\int\limits_{\Omega} \nabla u_t \cdot \nabla u dx + \frac{\varepsilon}{2}\int\limits_{\Omega} |\nabla u|^2 dx, \quad t \geq 0,\end{aligned} \tag{4.7}$$

where

$$0 < \varepsilon \ll 1 \quad \text{and} \quad 0 < \sigma < \frac{1}{\rho+1} - \frac{1}{p+1}. \tag{4.8}$$

Taking derivative to (4.7), we have from (1.1) that

$$\begin{aligned}\mathcal{G}'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\|\nabla u_t\|_2^2 \\ &\quad + \varepsilon(|u_t|^{\rho-1}u_{tt}, u) + \varepsilon(\nabla u, \nabla u_{tt}) + \varepsilon(\nabla u, \nabla u_t) \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\|\nabla u_t\|_2^2 - \varepsilon\|\nabla u\|_2^2 \\ &\quad + \varepsilon\int\limits_{\Omega} f(u)u dx + \varepsilon\int\limits_{\Omega}\nabla u(t)\int\limits_0^t g(t-s)\nabla u(s) ds dx.\end{aligned} \tag{4.9}$$

Applying the following relation

$$\begin{aligned}(p+1-\vartheta)H(t) &= (p+1-\vartheta)\beta\tilde{d} \\ &\quad - \frac{p+1-\vartheta}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} - \frac{p+1-\vartheta}{2}\|\nabla u_t\|_2^2 \\ &\quad - \frac{p+1-\vartheta}{2}\left(l(t)\|\nabla u\|_2^2 + (g\circ\nabla u)(t) - 2\int\limits_{\Omega} F(u) dx\right),\end{aligned} \tag{4.10}$$

and further by the Young inequality

$$\begin{aligned}&\int\limits_{\Omega}\nabla u(t)\int\limits_0^t g(t-s)[\nabla u(s) - \nabla u(t)] ds dx \\ &\leq \frac{1}{4\gamma}\int\limits_0^t g(s) ds\|\nabla u\|_2^2 + \gamma(g\circ\nabla u)(t),\end{aligned} \tag{4.11}$$

then a series of calculation gives that

$$\begin{aligned}
\mathcal{G}'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon(p+1-\vartheta)\beta\tilde{d} + \varepsilon(p+1-\vartheta)H(t) \\
&\quad + \frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\|\nabla u_t\|_2^2 + \frac{\varepsilon(p+1-\vartheta)}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{\varepsilon(p+1-\vartheta)}{2}\|\nabla u_t\|_2^2 \\
&\quad + \frac{\varepsilon(p+1-\vartheta)}{2}l(t)\|\nabla u\|_2^2 + \frac{\varepsilon(p+1-\vartheta)}{2}(g\circ\nabla u)(t) \\
&\quad - \varepsilon(p+1-\vartheta)\int_\Omega F(u)dx - \varepsilon\|\nabla u\|_2^2 + \varepsilon\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx \\
&\quad + \varepsilon\int_\Omega f(u)udx \\
&\geq (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon(p+1-\vartheta)\beta\tilde{d} + \varepsilon(p+1-\vartheta)H(t) \\
&\quad + \varepsilon\left(\frac{1}{\rho} + \frac{p+1-\vartheta}{\rho+1}\right)\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\left(\frac{p+1-\vartheta}{2}l(t) - 1\right)\|\nabla u\|_2^2 \\
&\quad + \frac{\varepsilon(p+3-\vartheta)}{2}\|\nabla u_t\|_2^2 + \varepsilon\left(\int_\Omega f(u)udx - (p+1-\vartheta)\int_\Omega F(u)dx\right) \\
&\quad - \frac{\varepsilon}{4\gamma}\int_0^t g(s)ds\|\nabla u\|_2^2 - \varepsilon\left(\frac{p+1-\vartheta}{2} - \gamma\right)(g\circ\nabla u)(t) \\
&\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(p+1-\vartheta)H(t) - \varepsilon(p+1-\vartheta)\beta\tilde{d} \\
&\quad + \frac{\varepsilon(p+3-\vartheta)}{2}\|\nabla u_t\|_2^2 + \varepsilon\left(\frac{1}{\rho} + \frac{p+1-\vartheta}{\rho+1}\right)\|u_t\|_{\rho+1}^{\rho+1} \\
&\quad + \varepsilon\left(\left(\frac{p+1-\vartheta}{2} - 1\right) - \left(\frac{p+1-\vartheta}{2} + \frac{1}{4\gamma}\right)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\
&\quad + \varepsilon\left(\frac{p+1-\vartheta}{2} - \gamma\right)(g\circ\nabla u)(t) + \frac{\varepsilon\vartheta\alpha}{p+1}\|u\|_{p+1}^{p+1},
\end{aligned} \tag{4.12}$$

where the constants $\vartheta, \gamma > 0$ will be determined later, $\alpha > 0$ is shown in conditions (A1). Applying Lemma 2.10 again, there appears the relation

$$\begin{aligned}
-\varepsilon(p+1-\vartheta)\beta\tilde{d} &\geq \frac{-\varepsilon\beta(p-1)}{2}\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right) \\
&\geq \frac{-\varepsilon\beta(p-1)}{2}\left(l(t)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)\right),
\end{aligned} \tag{4.13}$$

where we use the fact that $l(t) \geq l > 0$. Inserting (4.13) into (4.12), it follows that

$$\begin{aligned}\mathcal{G}'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(p+1-\vartheta)H(t) + \varepsilon\left(\frac{1}{\rho} + \frac{p+1-\vartheta}{\rho+1}\right)\|u_t\|_{\rho+1}^{\rho+1} \\ & + \frac{\varepsilon(p+3-\vartheta)}{2}\|\nabla u_t\|_2^2 + \frac{\varepsilon\vartheta\alpha}{p+1}\|u\|_{p+1}^{p+1} \\ & + \varepsilon\left(\frac{p+1-\vartheta}{2} - \frac{\beta(p-1)}{2} - \gamma\right)(g\circ\nabla u)(t) \\ & + \varepsilon\left(\frac{p-1-\vartheta}{2} - \frac{\beta(p-1)}{2} - \left(\frac{p+1-\vartheta}{2} - \frac{\beta(p-1)}{2} + \frac{1}{4\gamma}\right)\int_0^t g(s)ds\right)\|\nabla u\|_2^2.\end{aligned} \tag{4.14}$$

Choosing

$$0 < \gamma \leq \frac{(1-\beta)(p-1)}{2} + \frac{2-\vartheta}{2} \quad \text{and} \quad 0 < \vartheta < (p-1)(1-\beta),$$

we discover from (1.20) that

$$\begin{aligned}\mathcal{G}'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(p+1-\vartheta)H(t) \\ & + \varepsilon\left(\frac{1}{\rho} + \frac{p+1-\vartheta}{\rho+1}\right)\|u_t\|_{\rho+1}^{\rho+1} + \frac{\varepsilon(p+3-\vartheta)}{2}\|\nabla u_t\|_2^2 \\ & + \varepsilon K_1\|\nabla u\|_2^2 + \varepsilon K_2(g\circ\nabla u)(t) + \frac{\varepsilon\vartheta\alpha}{p+1}\|u\|_{p+1}^{p+1},\end{aligned} \tag{4.15}$$

where

$$K_1 = \frac{(p-1)(1-\beta)-\vartheta}{2} - \left(\frac{(1-\beta)(p-1)}{2} + \frac{2-\vartheta}{2} + \frac{1}{4\gamma}\right)\int_0^t g(s)ds > 0, \tag{4.16}$$

$$K_2 = \frac{(1-\beta)(p-1)}{2} + \frac{2-\vartheta}{2} - \gamma \geq 0. \tag{4.17}$$

So we obtain that

$$\mathcal{G}'(t) \geq \varepsilon\zeta\left(H(t) + \|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \|u\|_{p+1}^{p+1}\right) \geq 0, \tag{4.18}$$

for some small constant $\zeta > 0$. Here, choosing ε small enough, we have from (4.5) that

$$\mathcal{G}(0) = H^{1-\sigma}(0) + \frac{\varepsilon}{\rho}\int_\Omega |u_1|^{\rho-1}u_1u_0dx + \varepsilon\int_\Omega \nabla u_1\cdot\nabla u_0dx + \frac{\varepsilon}{2}\int_\Omega |\nabla u_0|^2dx > 0, \tag{4.19}$$

which along with (4.18) yields

$$\mathcal{G}(t) \geq \mathcal{G}(0) > 0, \quad t \geq 0. \tag{4.20}$$

Now, using the Hölder inequality and Sobolev inequality, one has

$$\begin{aligned}\left|\int_\Omega |u_t|^{\rho-1}u_t u\right|^{\frac{1}{1-\sigma}} &\le \|u_t\|_{\rho+1}^{\frac{\rho}{1-\sigma}}\|u\|_{\rho+1}^{\frac{1}{1-\sigma}}\\ &\le C\|u_t\|_{\rho+1}^{\frac{\rho}{1-\sigma}}\|u\|_{p+1}^{\frac{1}{1-\sigma}} \le C(\|u_t\|_{\rho+1}^{\frac{\rho q}{1-\sigma}}+\|u\|_{p+1}^{\frac{q'}{1-\sigma}}),\end{aligned}\tag{4.21}$$

where $\rho < p$ and $\frac{1}{q}+\frac{1}{q'}=1$. Taking $q=\frac{(\rho+1)(1-\sigma)}{\rho}>1$, then we have

$$\frac{q'}{1-\sigma}=\frac{\rho+1}{(\rho+1)(1-\sigma)-\rho}<p+1.\tag{4.22}$$

So using Lemma 4.1 and (4.21), there appears the relation

$$\left|\int_\Omega |u_t|^{\rho-1}u_t u\right|^{\frac{1}{1-\sigma}} \le C[\|u_t\|_{\rho+1}^{\rho+1}+\|u\|_{p+1}^{p+1}+\|\nabla u\|_2^2].\tag{4.23}$$

Applying the Young inequality and Hölder inequality again, we obtain that

$$\left|\int_\Omega \nabla u_t\cdot\nabla u dx\right|^{\frac{1}{1-\sigma}} \le \|\nabla u\|_2^{\frac{1}{1-\sigma}}\|\nabla u_t\|_2^{\frac{1}{1-\sigma}} \le C(\|\nabla u\|_2^{\frac{2}{1-2\sigma}}+\|\nabla u_t\|_2^2).\tag{4.24}$$

Hence, by the combination of (4.7), (4.23) and (4.24), we get that

$$\begin{aligned}\mathcal{G}^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t)+\frac{\varepsilon}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u dx+\varepsilon\int_\Omega \nabla u_t\cdot\nabla u dx+\frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{1-\sigma}}\\ &\le C\big(H(t)+\|u_t\|_{\rho+1}^{\rho+1}+\|\nabla u_t\|_2^2+\|\nabla u\|_2^{\frac{2}{1-2\sigma}}+\|\nabla u\|_2^{\frac{2}{1-\sigma}}+\|u\|_{p+1}^{p+1}\big).\end{aligned}\tag{4.25}$$

By (4.5) and $\delta_1 \le \theta(t) \le \delta_2$, it follows that

$$\|\nabla u\|_2^{\frac{2}{1-2\sigma}}+\|\nabla u\|_2^{\frac{2}{1-\sigma}} \le {\delta_2}^{\frac{1}{1-2\sigma}}+\delta_2^{\frac{1}{1-\sigma}} \le \frac{{\delta_2}^{\frac{1}{1-2\sigma}}+\delta_2^{\frac{1}{1-\sigma}}}{H(0)}H(t).\tag{4.26}$$

Inserting (4.6), (4.26) into (4.25), we discover that

$$\mathcal{G}^{\frac{1}{1-\sigma}}(t) \le C\big(\|u_t\|_{\rho+1}^{\rho+1}+\|\nabla u_t\|_2^2+\|u\|_{p+1}^{p+1}\big), \quad \forall t \ge 0,\tag{4.27}$$

In view of (4.18) and (4.27), there appears the relation

$$\mathcal{G}'(t) \ge Q\mathcal{G}^{\frac{1}{1-\sigma}}(t), \quad \forall t \ge 0,\tag{4.28}$$

where the positive constant Q is depending only on C, ε and ζ. Integrating (4.28) form 0 to t, we get that

$$\mathcal{G}^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\mathcal{G}^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma Q t}{1-\sigma}}, \quad \forall t \geq 0, \tag{4.29}$$

which implies that $\mathcal{G}(t)$ blows up in finite time

$$T^* \leq \frac{1-\sigma}{\sigma Q \mathcal{G}^{\frac{\sigma}{1-\sigma}}(0)}. \tag{4.30}$$

By (4.27), we further deduce that

$$\lim_{t \to T^{*-}} \left(\|u_t\|_{\rho+1}^{\rho+1} \|\nabla u_t\|_2^2 + \|u\|_{p+1}^{p+1} \right) = +\infty,$$

which contradicts $T_{\max} = \infty$. Therefore, the solutions u can not exists all time, i.e. blow up in finite time. We thereby complete the proof of Theorem 1.7. □

4.2. LOWER BOUND FOR BLOW-UP TIME

With the blow-up criterion of Theorem 1.7 in hand, attention in this section is turned to the investigation on the lower bound for blow-up time of the problem (1.1), which is described by Theorem 1.8.

Proof. Let us introduce the auxiliary function

$$\begin{aligned} F(t) &= E(t) + \int_\Omega F(u)dx \\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}l(t)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t). \end{aligned} \tag{4.31}$$

By differentiating $F(t)$ with respect to t and using (3.11), we have

$$F'(t) = E'(t) + \int_\Omega f(u)u_t dx \leq \int_\Omega f(u)u_t dx. \tag{4.32}$$

Applying the conditions (A1), Young inequality and Hölder inequality, there appears the relation

$$F'(t) \leq \int_\Omega |f(u)u_t|dx \leq \alpha \int_\Omega |u|^p|u_t|dx \leq \alpha\|u\|_{p+1}^p\|u_t\|_{p+1} \leq \frac{\alpha}{2}\|u\|_{p+1}^{2p} + \frac{\alpha}{2}\|u_t\|_{p+1}^2. \tag{4.33}$$

Furthermore, by Sobolev inequality we get that

$$F'(t) \leq \frac{\alpha B_{p+1}^{2p}}{2}\|\nabla u\|_2^{2p} + \frac{\alpha B_{p+1}^2}{2}\|\nabla u_t\|_2^2, \tag{4.34}$$

which together with (4.31) yields that

$$F'(t) \leq \alpha 2^{p-1} B_{p+1}^{2p} l^{-p} F(t)^p + \alpha B_{p+1}^2 F(t), \tag{4.35}$$

here we used the fact that the function $l(t) \geq l$. Then, integrating above inequality from 0 to t, we have

$$\int\limits_{F(0)}^{F(t)} \frac{d\eta}{\alpha 2^{p-1} B_{p+1}^{2p} l^{-p} \eta^p + \alpha B_{p+1}^2 \eta}, \tag{4.36}$$

where

$$F(0) = \frac{1}{\rho+1} \|u_1\|_{\rho+1}^{\rho+1} + \frac{1}{2} \|\nabla u_1\|_2^2 + \frac{1}{2} \|\nabla u_0\|_2^2.$$

From the results of Theorem 1.7, we know that there exists a finite time T^* such that the solutions u blow up with $\lim_{t \to T^{*-}} F(t) = +\infty$. Therefore, we obtain a lower bound for T^* given by

$$T^* \geq \int\limits_{F(0)}^{\infty} \frac{d\eta}{\alpha 2^{p-1} B_{p+1}^{2p} l^{-p} \eta^p + \alpha B_{p+1}^2 \eta}.$$

Clearly, the integral is bound since exponents $p > 1$. This completes the proof of Theorem 1.8 □

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REFERENCES

[1] A.B. Al'shin, M.O. Korpusov, A.G. Siveshnikov, *Blow up in nonlinear Sobolev type equations*, De Gruyter Series in Nonlinear Aanlysis and Applications, vol. 15, Berlin, 2011.

[2] K. Anaya, S.A. Messaoudi, *General decay rate of a weakly dissipative viscoelastic equation with a general damping*, Opuscula Math. **40** (2020), no. 6, 647–666.

[3] J. Ball, *Remarks on blow up and nonexistence theorems for nonlinear evolutions equations*, Quart. J. Math. **28** (1977), 473–486.

[4] S. Berrimi, S.A. Messaoudi, *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonlinear Anal. TMA **64** (2006), 2314–2331.

[5] M.M. Cavalcanti, V.N.D. Cavalcanti, J. Ferreira, *Existence and uniform decay for nonlinear viscoelastic equation with strong damping*, Math. Methods Appl. Sci. **24** (2001), 1043–1053.

[6] Y.X. Chen, R.Z. Xu, *Global well-posedness of solutions for fourth order dispersive wave equation with nonlinear weak damping, linear strong damping and logarithmic nonlinearity*, Nonlinear Anal. TMA **192** (2020), 1–39.

[7] H.F. Di, Y.D. Shang, *Global well-posedness for a nonlocal semilinear pseudo-parabolic equation with conical degeneration*, J. Differential Integral Equations **269** (2020), no. 5, 4566–4597.

[8] H.F. Di, Y.D. Shang, X.M. Peng, *Global existence and nonexistence of solutions for a viscoelastic wave equation with nonlinear boundary source term*, Math. Nachr. **289** (2016), no. 3, 1408–1432.

[9] H.F. Di, Y.D. Shang, Z.F. Song, *Initial boundary value problem for a class of strongly damped semilinear wave equations with logarithmic nonlinearity*, Nonlinear Anal. RWA **51** (2020), 1–22.

[10] H.F. Di, Y.D. Shang, X.X. Zheng, *Global well-posedness for a fourth order pseudo--parabolic equation with memory and source terms*, Discrete Contin. Dyn. Syst. Ser. B **21** (2016), no. 3, 781–801.

[11] F. Gazzola, T. Weth, *Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level*, Differ. Integral Equ. **18** (2005), no. 9, 961–990.

[12] R. Ikehata, T. Suzuki, *Stable and unstable sets for evolution equations of parabolic and hyperbolic type*, Hiroshima Math. J. **26** (1996), no. 3, 475–491.

[13] V.K. Kalantarov, O.A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type*, J. Soviet Math. **10** (1978), 53–70.

[14] A.Y. Kolesov, E.F. Mishchenko, N.K. Rozov, *Asymptotic methods of investigation of periodic solutions of nonlinear hyperbolic equations*, Trudy Mat. Inst. Steklova **222** (1998), 3–191.

[15] W. Lian, M.S. Ahmed, R.Z. Xu, *Global existence and blow up of solution for semi-linear hyperbolic equation with the product of logarithmic and power-type nonlinearity*, Opuscula Math. **40** (2020), no. 1, 111–130.

[16] W. Lian, R.Z. Xu, *Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term*, Adv. Nonlinear Anal. **9** (2019), 613–632.

[17] J.L. Lions, *Quelques méthodes de résolutions des probléms aux limites non linéaires*, Dunod, Paris, 1969.

[18] W.J. Liu, *General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source*, Nonlinear Anal. TMA **73** (2010), 1890–1904.

[19] L.Q. Lu, S.J. Li, S.G. Chai, *On a viscoelastic equation with nonlinear boundary damping and source terms: Global existence and decay of the solution*, Nonlinear Anal. RWA **12** (2012), 295–302.

[20] S.A. Messaoudi, N.E. Tatar, *Global existence and uniform stability of solutions for a quasilinear viscoelastic problem*, Math. Methods Appl. Sci. **30** (2007), 665–680.

[21] P. Rosenau, *Evolution and breaking of the ion-acoustic waves*, Phys. Fluids **31** (1988), no. 6, 1317–1319.

[22] D.H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Ration. Mech. Anal. **30** (1968), no. 2, 148–172.

[23] B.K. Shivamoggi, *A symmetric regularized long wave equation for shallow water waves*, Phys. Fluids **29** (1986), no. 3, 890–891.

[24] H.T. Song, C.K. Zhong, *Blow-up of solutions of a nonlinear viscoelastic wave equation*, Nonlinear Anal. RWA **11** (2010), 3877–3883.

[25] H.T. Song, D.S. Xue, *Blow up in a nonlinear viscoelastic wave equation with strong damping*, Nonlinear Anal. TMA **109** (2014), 245–251.

[26] Y.J. Wang, *A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy*, Appl. Math. Lett. **22** (2009), 1394–1400.

[27] F.L. Wang, D. Hu, M.Q. Xiang, *Combined effects of Choquard and singular nonlinearities in fractional Kirchhoff problems*, Adv. Nonlinear Anal. **10** (2020), 636–658.

[28] X.C. Wang, R.Z. Ru, *Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation*, Adv. Nonlinear Anal. **10** (2021), 261–288.

[29] Y.J. Wang, Y.F. Wang, *Exponential energy decay of solutions of viscoelastic wave equations*, J. Math. Anal. Appl. **347** (2008), 18–25.

[30] R.Z. Xu, Y.B. Yang, Y.C. Liu, *Global well-posedness for strongly damped viscoelastic wave equation*, Appl. Anal. **92** (2013), no. 1, 138–157.

[31] Y.B. Yang, M.S. Ahmed, L.L. Qin, R.Z. Xu, *Global well-posedness of a class of fourth-order strongly damped nonlinear wave equations*, Opuscula Math. **39** (2019), no. 2, 297–313.

[32] J. Zhou, *Global existence and blow-up of solutions for a Kirchhoff type plate equation with damping*, Appl. Math. Comput. **265** (2015), 807–818.

Huafei Di (corresponding author)
dihuafei@yeah.net
https://orcid.org/0000-0002-8206-3800

School of Mathematics and Information Science
Guangzhou University
Guangzhou 510006, P.R. China

Zefang Song
song_zefang@163.com
https://orcid.org/0000-0001-7059-069X

School of Economics and Statistics
Guangzhou University
Guangzhou 510006, P.R. China

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