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OPTION PRICING FORMULAS UNDER A CHANGE OF NUMÈRAIRE

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Abstract. We present some formulations of the Cox–Ross–Rubinstein and Black–Scholes formulas for European options obtained through a suitable change of measure, which corresponds to a change of numèraire for the underlying price process. Among other consequences, a closed formula for the price of an European call option at each node of the multi-period binomial tree is achieved, too. Some of the results contained herein, though comparable with analogous ones appearing elsewhere in the financial literature, provide however a supplementary widening and deepening in view of useful applications in the more challenging framework of incomplete markets. This last issue, having the present paper as a preparatory material, will be treated extensively in a forthcoming paper.

Keywords: Black–Scholes formula, binomial model, martingale measures, numèraire.

Mathematics Subject Classification: 91B25, 60G46.

1. INTRODUCTION

The present paper deals with the possibility of expressing well-known formulas in the theory of option pricing by using probability measures different from the risk-adjusted measure $\mathbb{Q}$, but strongly connected to $\mathbb{Q}$ through suitable Radon–Nikodym derivatives.

The procedure, which is basically based upon assuming the stock price itself as a numèraire for the underlying price process, far from being an artificial and fictitious mathematical issue, presents some advantages, turning out to be helpful to get a deep insight in the terms appearing in classical option pricing formulas as in Black–Scholes and Cox–Ross–Rubinstein's (see [2, 6] and [3]). On the other hand, in the case of the multi-period binomial model, it allows to write down, with some efforts, a closed formula (and, therefore, a computational algorithm) for the price of an European call option at each node of the binomial tree, not yet considered elsewhere as far as we know. Under the newly defined measures the (discounted) stock price becomes a submartingale: thus, with a change of measure, we mimic a world which lies between the $\mathbb{P}$-real world and the $\mathbb{Q}$-risk neutral world and which, however, resembles more the former than the latter.

At a first glance, the subject of the paper might seem too theoretical with no predictable concrete applications thereof. As a matter of fact, the new measures defined herein may be applied successfully in the framework of incomplete markets, providing contributions of some interest to the corresponding option pricing issue.

The above aspect, however, which has the submartingale property of the discounted stock price process as a key tool, leading further afield, will be addressed in a forthcoming paper.

We end this introduction with two notes: firstly, while proceeding in preparing the manuscript and checking carefully some references which should be consistent with our investigations, we casually came across the paper [8] by Lars Tyge Nielsen, which, as the reader may realize, carries out an analysis similar, in some way, to ours, but developed through different techniques. The paper is original and skillfully draws the reader's attention towards the deep meaning of option pricing formulas in the discrete and continuous case, preferring sometimes a more intuitive rather than a formally rigorous approach. We hope that the present work may contribute to awaken the fair interest toward the above cited paper, little-known, in our opinion, in the current literature.

Secondly, we point out that some results similar to ours may be found in [7]; however, in this regard, some important differences deserve to be remarked. Above all, our approach carries out a deeper theoretical study of the new measures and of the corresponding connections with the risk neutral measure $\mathbb{Q}$, both in the discrete and the continuous case: see, for instance, Propositions 2.1, 3.2, 3.4 and 3.5.

More than that, as far as the discrete case is concerned, we introduce the multi-step measure $\widehat{\mathbb{Q}}$ (see (3.22)) (which does not appear in [7]) and prove a detailed explicit formula pricing a derivative at each node of the binomial tree, and not solely at each time step of the tree, as performed in [7]. In other words, our analysis, though running in the wake of [7], as opposed to it, turns out to be somehow finer, fitting more accurately with the concrete applications following ahead.

The notation used through the paper is quite standard and do not need particular comments: following [10], while dealing with the multi-period binomial model with N steps, we denote by $\mathbb{E}_k[X]$ ($k \leq N$, X random variable) the conditional expectation of X based upon the information available at time k (and that we do not know before k). Such information are contained in the filtration $\mathcal{F}_k$. If Y is another random variable, $\mathrm{Var}_k(X)$, $cov_k(X, Y)$ and the conditional correlation coefficient $\rho_k(X, Y)$ are defined accordingly. Traditionally, if we are working under a probability measure $\mathbb{Q}$, the superscript "$\mathbb{Q}$" appears in all the values above and so we shall write $\mathbb{E}_k^{\mathbb{Q}}[X]$, $\mathrm{Var}_k^{\mathbb{Q}}(X)$ and so on. Each other notation not encompassed here will be specified at any occurrence.

Finally, let us recall the Black–Scholes formula for an European call option (see, e.g., [2, 7, 9] and [11]), given by

$$c(t, S_t) = S_t N(d_1(\tau, S_t)) - e^{-r\tau} X N(d_2(\tau, S_t)), \tag{1.1}$$

for any $t \in [0, T]$, where

— S_t is the current stock price,
— X is the strike price,
— T is the maturity and $\tau = T - t$ is the remaining expiration time,
— r is the risk-free interest rate,
— $N(\cdot)$ denotes the cumulative distribution function of a standard normal,
— $d_1(\tau, S_t) = \frac{1}{\sigma\sqrt{\tau}}\big(\ln\frac{S_t}{X} + \big(r + \frac{\sigma^2}{2}\big)\tau\big)$ and $d_2(\tau, S_t) = d_1(\tau, S_t) - \sigma\sqrt{\tau}$. In order to simplify the notation, we will shorten $d_1(\tau, S_t)$ and $d_2(\tau, S_t)$ by d_1 and d_2, respectively.

2. THE CONTINUOUS CASE

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and consider a dynamic market environment with a riskeless bond B_t with maturity T satisfying

$$dB_t = rB_t dt, \quad B_0 = 1,$$

and a risky asset whose price process $(S_t)_{0\leq t\leq T}$ follows a geometric Brownian motion, i.e., it is driven by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0; \tag{2.1}$$

here $\mu \geq 0$ and $\sigma > 0$ are respectively the drift and the volatility of the process and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion.

Assume that, at each time $t \in [0, T]$, we buy Δ_t shares of stock and invest the remaining part in the money market account at risk-free rate r. The evolution of this portfolio is described by the dynamics

$$dX_t = rX_t dt + \Delta_t(\mu - r)S_t dt + \Delta_t \sigma S_t dW_t.$$

We can replicate our portfolio by a short position in a derivative security, e.g., in a call option with strike X; the call pays $(S_T - X)^+ = \max(S_T - X, 0)$ at maturity T and its price c_t depends on t and S_t. The replication (hedging) happens if and only if

$$X_t = c_t, \tag{2.2}$$

for any t.

Now, after considering the discount factor $D_t = e^{-rt}$, from the Ito–Doeblin formula it follows that the discounted stock price satisfies

$$d(e^{-rt}S_t) = \sigma D_t S_t(\theta dt + dW_t),$$

where $\theta = \frac{\mu - r}{\sigma}$ is the market price of risk. Hence we can write

$$d(e^{-rt}S_t) = \sigma D_t S_t dW_t^{\mathbb{Q}}, \tag{2.3}$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big(-\int_0^T \theta\, dW_u - \frac{1}{2}\int_0^T \theta^2\, du\Big)$$

is the Radon–Nikodym derivative and, by virtue of Girsanov's Theorem (see, e.g., [11, Chapter V]), the process $W_t^{\mathbb{Q}} := W_t + \int_0^t \theta\, du$ $(0 \le t \le T)$ is a $\mathbb{Q}$-Brownian motion. The measure $\mathbb{Q}$ is said to be *risk-neutral* because it is equivalent to $\mathbb{P}$ and, in addition, turns the discounted price $e^{-rt}S_t$ into a martingale; indeed, according to (2.3), we have

$$e^{-rt}S_t = S_0 + \int_0^t \sigma e^{-ru} S_u d\,W_u^{\mathbb{Q}},$$

and the process $\left(\int_0^t \sigma e^{-ru} S_u d\,W_u^{\mathbb{Q}}\right)_{0\le t\le T}$ is an Ito integral and therefore a $\mathbb{Q}$-martingale (for more details see, e.g., [11, Chapter IV]). In particular the stock price process $(S_t)_{0\le t\le T}$ is described by the $\mathbb{Q}$-dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

and henceforth it will be supposed adapted to the filtration $(\mathcal{F}_t)_{0\le t\le T}$. Moreover, from the replication condition (2.2), we get $e^{-rt}X_t = e^{-rt}c_t$, where $e^{-rt}c_t = e^{-rT}\mathbb{E}^{\mathbb{Q}}[c_T|\mathcal{F}_t]$ due to the martingale property under $\mathbb{Q}$. Summing up, for any $t \in [0,T]$, the call price is given by

$$c_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - X)^+|\mathcal{F}_t], \tag{2.4}$$

or, equivalently, by

$$\begin{aligned} c_t &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - X)\cdot \mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] \\ &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\cdot \mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] - e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[X\cdot \mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] \\ &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\cdot \mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] - Xe^{-r\tau}\mathbb{Q}\{S_T > X|\mathcal{F}_t\}. \end{aligned}$$

In particular,

$$c_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T\cdot \mathbf{1}_{\{S_T>X\}}] - Xe^{-rT}\mathbb{Q}\{S_T > X\}.$$

Now let $\widetilde{\mathbb{Q}}$ be another probability measure related to $\mathbb{Q}$ by the following Radon–Nikodym derivative

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} := Z, \tag{2.5}$$

where Z is the random variable defined by

$$Z := e^{-rT}\frac{S_T}{S_0} = \exp\left(-\frac{\sigma^2}{2}T + \sigma W_T^{\mathbb{Q}}\right); \tag{2.6}$$

correspondingly, let us define the Radon–Nikodym derivative process $(Z_t)_{0\le t\le T}$ as

$$Z_t = \mathbb{E}^{\mathbb{Q}}[Z|\mathcal{F}_t] \tag{2.7}$$

with $Z_T = Z$. We note that

$$\mathbb{E}^{\mathbb{Q}}[Z] = e^{-rT}\frac{\mathbb{E}^{\mathbb{Q}}[S_T]}{S_0} = e^{-rT}e^{rT} = 1$$

and, in addition, since $\mathbb{Q}$ is risk-neutral,

$$Z_t = e^{-rt}\frac{S_t}{S_0} = \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t^{\mathbb{Q}}\right);$$

according to [11, Theorem 3.6.1], $(Z_t)_{0\le t\le T}$ is an exponential $\mathbb{Q}$-martingale process.

Under the change of measure performed in (2.5), we may prove that

$$e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] = S_t\,\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\}. \tag{2.8}$$

Indeed, by using [11, Lemma 5.2.2], since $\mathbf{1}_{\{S_T>X\}}$ is $\mathcal{F}_T$-measurable, we have

$$\begin{aligned} S_t\,\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\} &= S_t\,\mathbb{E}^{\widetilde{\mathbb{Q}}}[\mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] \\ &= S_t\frac{\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{S_T>X\}}Z_T|\mathcal{F}_t]}{Z_t} \\ &= S_t\frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{S_T>X\}}\,e^{-rT}\frac{S_T}{S_0}\middle|\mathcal{F}_t\right]}{e^{-rt}\frac{S_t}{S_0}} = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T\mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t]. \end{aligned}$$

Finally (see just below (2.4)), the Black–Scholes formula for the call option becomes

$$c_t = S_t\,\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\} - Xe^{-r\tau}\,\mathbb{Q}\{S_T>X|\mathcal{F}_t\}; \tag{2.9}$$

see also [7, Propostion 2.2.5].

It is clear that the above formula agrees with the classical formulation (1.1) with $\mathbb{Q}\{S_T>X|\mathcal{F}_t\} = N(d_2)$ and $\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\} = N(d_1)$, as better explained in the following proposition.

Proposition 2.1. *If $\widetilde{\mathbb{Q}}$ is the measure defined by* (2.5)*, the following properties hold true:*

(i) *the discounted process $(e^{-rt}S_t)_{0\le t\le T}$ is a $\widetilde{\mathbb{Q}}$-submartingale,*
(ii) *if $r\ge 0$*[1]*, the process $(S_t)_{0\le t\le T}$ is a $\widetilde{\mathbb{Q}}$-submartingale,*
(iii) *the Ito process*

$$W_t^{\widetilde{\mathbb{Q}}} := W_t^{\mathbb{Q}} - \int_0^t \sigma\,du \quad (0\le t\le T) \tag{2.10}$$

is a $\widetilde{\mathbb{Q}}$-Brownian motion. In particular the process $(S_t)_{0\le t\le T}$ obeys the following $\widetilde{\mathbb{Q}}$-dynamics

$$dS_t = (r+\sigma^2)S_t dt + \sigma S_t dW_t^{\widetilde{\mathbb{Q}}}, \tag{2.11}$$

[1] We note that in current financial markets there is the possibility of negative or near-zero interest rates (see, e.g., [1], [4]). However, if $r<0$, we can replace r by its absolute value $|r|$. The occurrence $r=0$ represents the ATM ("at the money") case.

(iv) *for any $t \in [0,T]$ the (unconditional) $\widetilde{\mathbb{Q}}$-expectation of S_t is given by*

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{\mathbb{E}^{\mathbb{Q}}[S_t]}, \tag{2.12}$$

i.e., by the ratio between the second and the first moment of S_t under $\mathbb{Q}$,

(v) *for every $t \in]0,T[$ one has $\widetilde{\mathbb{Q}}\{S_T > X|\mathcal{F}_t\} = N(d_1)$.*

Proof. (i) By virtue of [11, Lemma 5.2.2], for any s and t satisfying $0 \le s \le t \le T$ we compute

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{Q}}}[e^{-rt}S_t|\mathcal{F}_s] &= \frac{\mathbb{E}^{\mathbb{Q}}[e^{-rt}S_t\,Z_t|\mathcal{F}_s]}{Z_s} = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-2rt}\frac{S_t^2}{S_0}\middle|\mathcal{F}_s\right]}{e^{-rs}\frac{S_s}{S_0}} = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(2t-s)}\frac{S_t^2}{S_s}\middle|\mathcal{F}_s\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(2t-s)}S_s\exp\left(2\left(r-\frac{\sigma^2}{2}\right)(t-s)-2\sigma\sqrt{t-s}\,Y\right)\middle|\mathcal{F}_s\right],\end{aligned}$$

where

$$Y := -\frac{W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}}}{\sqrt{t-s}} \sim N(0,1). \tag{2.13}$$

Since S_s is $\mathcal{F}_s$-measurable and the random variable Y is $\mathcal{F}_s$-independent, we may avoid conditioning (see, for instance, [12, p. 88]) and therefore the above expectation $\mathbb{E}^{\widetilde{\mathbb{Q}}}[e^{-rt}S_t|\mathcal{F}_s]$ is equal to

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-r(2t-s)}S_s\exp\left(2\left(r-\frac{\sigma^2}{2}\right)(t-s)-2\sigma\sqrt{t-s}\,y\right)e^{-\frac{y^2}{2}}\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} S_s\exp(2r(t-s)-r(2t-s)-\sigma^2(t-s)-2\sigma\sqrt{t-s}\,y)e^{-\frac{y^2}{2}}\,dy \\ &= e^{-rs}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} S_s\exp(-\sigma^2(t-s)-2\sigma\sqrt{(t-s)}\,y-y^2/2)\,dy \\ &= e^{-rs}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} S_s e^{\frac{-(y+2\sigma\sqrt{t-s})^2+2\sigma^2(t-s)}{2}}\,dy \\ &= e^{\sigma^2(t-s)}e^{-rs}S_s\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\,dz = e^{\sigma^2(t-s)}e^{-rs}S_s,\end{aligned}$$

where we have set $z := y + 2\sigma\sqrt{t-s}$. Since $e^{\sigma^2(t-s)} \ge 1$, we finally get

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[e^{-rt}S_t|\mathcal{F}_s] \ge e^{-rs}S_s,$$

i.e., the discounted process $(e^{-rt}S_t)_{0\le t\le T}$ is a $\widetilde{\mathbb{Q}}$-submartingale, as announced.

(ii) Arguing as in (i), we may conclude that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t|\mathcal{F}_s] = e^{(t-s)(r+\sigma^2)}S_s,$$

where the term $e^{(t-s)(r+\sigma^2)}$ is surely greater than 1 if $r \geq 0$. However, the result is *a posteriori* quite obvious since, according to (2.11) (yet to be proved), the process $(S_t)_{0\leq t\leq T}$ is a $\widetilde{\mathbb{Q}}$-geometric Brownian motion.

(iii) The process Z_t defined in (2.7) can be written as

$$Z_t = \exp\left(-\int_0^t (-\sigma)\, dW_u^{\mathbb{Q}} - \frac{1}{2}\int_0^t \sigma^2\, du\right),$$

where $\mathbb{E}^{\mathbb{Q}}\big[e^{\frac{1}{2}\int_0^t \sigma^2\, du}\big] < +\infty$ for any $t \in [0,T]$. Therefore, applying Girsanov's Theorem (see, for instance, [11, Section 5.2]), yields that

$$W_t^{\widetilde{\mathbb{Q}}} := W_t^{\mathbb{Q}} - \int_0^t \sigma\, du \qquad (0 \leq t \leq T)$$

is a $\widetilde{\mathbb{Q}}$-Brownian motion.

In particular, we can derive the dynamics of the process S_t under $\widetilde{\mathbb{Q}}$ and obtain

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} = (r+\sigma^2)S_t dt + \sigma S_t dW_t^{\widetilde{\mathbb{Q}}}$$

whose solution is explicitly given by

$$S_t = S_0 \exp\left(\left(r + \frac{\sigma^2}{2}\right)t + \sigma W_t^{\widetilde{\mathbb{Q}}}\right) \qquad (0 \leq t \leq T).$$

(iv) [2] Let us fix t in $[0,T]$ and recall that the first moment of S_t under $\mathbb{Q}$ is given by

$$\mathbb{E}^{\mathbb{Q}}[S_t] = e^{rt}S_0;$$

therefore (see (2.6))

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t] = \mathbb{E}^{\mathbb{Q}}[S_t\, Z_T] = e^{-rT}\frac{\mathbb{E}^{\mathbb{Q}}[S_t\, S_T]}{S_0} = \frac{\mathbb{E}^{\mathbb{Q}}\left[S_t^2 \exp\left(-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}Y\right)\right]}{e^{rt}S_0}$$

where Y is defined by (2.13). From the $\mathbb{Q}$-independence of the random variables appearing in the $\mathbb{Q}$-expectation value it follows that

$$\mathbb{E}^{\mathbb{Q}}\left[S_t^2 \exp\left(-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}Y\right)\right] = \mathbb{E}^{\mathbb{Q}}[S_t^2]\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}Y\right)\right],$$

[2] As an alternative proof, by virtue of [11, Lemma 5.2.1], for any $0 \leq t \leq T$ we have soon

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t] = \mathbb{E}^{\mathbb{Q}}[S_t\, Z_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{e^{rt}S_0} = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{\mathbb{E}^{\mathbb{Q}}[S_t]},$$

as claimed.

where

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\frac{\sigma^2}{2}\tau-\sigma\sqrt{\tau}Y\right)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{\sigma^2}{2}\tau-\sigma\sqrt{\tau}y\right)e^{-\frac{y^2}{2}}\,dy$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{z^2}{2}}\,dz=1,$$

after setting $z:=(y+\sigma\sqrt{\tau})$. The expression (2.12) is now fully established.

(v) Conditioned on $\mathcal{F}_t$ (t fixed), on account of (2.11) S_T is $\widetilde{\mathbb{Q}}$-lognormally distributed with known parameters $m:=\ln S_t+\left(r+\frac{\sigma^2}{2}\right)\tau$ and $\sigma^2\tau$; consequentely, we can compute explicitly the probability $\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\}$ and obtain

$$\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\}=\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{X}^{+\infty}\frac{1}{y}\exp\left(-\frac{(\ln y-m)^2}{2\sigma^2\tau}\right)dy;$$

the change of variable

$$z:=-\frac{\ln y-m}{\sigma\sqrt{\tau}}$$

soon entails

$$\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1}e^{-\frac{z^2}{2}}\,dz=N(d_1),$$

which accomplishes the proof. □

Remark 2.2. We believe that the formulation (2.9) is rather significant because captures judiciously the difference between $N(d_1)$ and $N(d_2)$ appearing in the classic Black–Scholes formula (1.1). By virtue of (2.9), both $N(d_1)$ and $N(d_2)$ represent the (conditional) probability of the same event $\{S_T>X\}$ under the two different (but equivalent) measures $\widetilde{\mathbb{Q}}$ and $\mathbb{Q}$. In other words, it is just a change of measure that distinguishes $N(d_1)$ from $N(d_2)$. Incidentally observe that $N(d_1)>N(d_2)$ (because $d_1>d_2$) and, consequently, $\widetilde{\mathbb{Q}}\{S_T>X|\mathcal{F}_t\}>\mathbb{Q}\{S_T>X|\mathcal{F}_t\}$.

As clearly discussed in [8], $N(d_1)$ takes in charge not only the probability of exercise (or, equivalently, the probability of the event $\{S_T>X\}$ which is represented by $N(d_2)$), but also the fact that the exercise or, rather, the receipt of the stock on exercise, is strongly dependent on the conditional future values of the stock price at the maturity. Consequently, in evaluating $N(d_1)$, we must consider stock price higher than the exercise price as a given condition while computing the expected future value of the stock at maturity. Conversely, as already said, $N(d_2)$ is merely the risk-neutral probability that the option will be exercised at T and depends only on the event $\{S_T>X\}$.

We point out that in our analysis, however, unlike [8], the approach is quite different and relies upon a change of measure from the risk-neutral probability $\mathbb{Q}$ to $\widetilde{\mathbb{Q}}$, which equals to passing through a change of numèraire (for more details upon this issue, see, e.g., [11, Chapter IX], [7, Chapters I and II] and [5]). Specifically, in the case of the risk-neutral measure $\mathbb{Q}$ the numèraire is the money market account, i.e.,

$$M_t = e^{rt}, \tag{2.14}$$

for any $t \in [0, T]$, whereas, for the measure $\widetilde{\mathbb{Q}}$, the numèraire becomes

$$N_t = S_t, \tag{2.15}$$

i.e., the stock price itself. As a confirmation, one may easily verify that

1. the discounted process $e^{-rt}N_t$ is a $\mathbb{Q}$-martingale,
2. the Radon–Nikodym derivative that effects the change of measure from the numèraire-measure pair $(N, \widetilde{\mathbb{Q}})$ to the other pair $(M, \mathbb{Q})$ is given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{N_T}{N_0} \cdot \frac{M_0}{M_T},$$

   which agrees with (2.5) and (2.6),
3. the volatilities of M_t and N_t are (obviously) 0 and σ^2, respectively,
4. the discounted process $(e^{-rt}S_t)_{0 \le t \le T}$ is a submartingale under $\widetilde{\mathbb{Q}}$ as well as under the objective measure $\mathbb{P}$, which is quite realistic for (real) risky markets.

Remark 2.3. It is worthwhile noting that from (2.8) we easily get

$$\mathbb{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T > X\}} | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[S_T | \mathcal{F}_t] \cdot \mathbb{E}^{\widetilde{\mathbb{Q}}}[\mathbf{1}_{\{S_T > X\}} | \mathcal{F}_t]. \tag{2.16}$$

In other words, the random variables S_T and $\mathbf{1}_{\{S_T > X\}}$ are not $\mathbb{Q}$-independent (as one might expect, since $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{S_T > X\}} | \mathcal{F}_t] = N(d_2)$) and the $\mathbb{Q}$-expectation of their product may be expressed as the product of their expectations under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, respectively.

3. THE DISCRETE CASE

3.1. THE ONE-PERIOD BINOMIAL MODEL

We consider the one-period binomial model where $t = \{0, 1\}$. Let S_0 the initial stock price, known at time $t = 0$. The stock price at time $t = 1$ is a random variable defined as

$$S_1 = \begin{cases} S_1^u := uS_0, \\ S_1^d := dS_0, \end{cases}$$

where $d < 1 < u$ and S_1^u and S_1^d denote the stock price at time 1. Let p be the real probability that $S_1 = S_1^u$ and assume that the no-arbitrage condition holds true, i.e., $d < 1 + r < u$. Consider a derivative security, for instance a call option, with payoff $(S_1 - X)^+$, X being the strike price. Our aim is to compute the initial price of this derivative in order to replicate an investment portfolio made by risky assets. Suppose that at time $t = 0$ we start with wealth X_0, buy Δ_0 shares of stock S_0 and invest the remaining part at a risk-free rate r. The future value of our portfolio of stock and money market account is given by

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0) = (1 + r)X_0 + \Delta_0(S_1 - (1 + r)S_0).$$

Define $c_1^u := (S_1^u - X)^+$ as the value of the call if the stock price goes up, and $c_1^d := (S_1^d - X)^+$ as the value of the call if the stock price goes down. Observe that at time 0 the values c_1^u and c_1^d are known but we do not know which of them will really happen. The replication portfolio has to guarantee that

$$\begin{cases} \frac{1}{1+r}\, c_1^u = X_0 + \Delta_0\big(\frac{1}{1+r} S_1^u - S_0\big), \\ \frac{1}{1+r}\, c_1^d = X_0 + \Delta_0\big(\frac{1}{1+r} S_1^d - S_0\big). \end{cases} \tag{3.1}$$

By solving the above system in the unknowns X_0 and Δ_0, we can find the value of X_0, namely the initial price c_0 of the derivative security for the short position, as well as Δ_0; specifically one finds

$$X_0 = c_0 = \frac{1}{1 + r}[c_1^u p^{\mathbb{Q}} + c_1^d(1 - p^{\mathbb{Q}})], \tag{3.2}$$

and

$$\Delta_0 = \frac{c_1^u - c_1^d}{S_1^u - S_1^d}, \tag{3.3}$$

where $\mathbb{Q} := (p^{\mathbb{Q}}, 1 - p^{\mathbb{Q}})$ with

$$p^{\mathbb{Q}} := \frac{1 + r - d}{u - d} \tag{3.4}$$

is the risk-neutral probability. If we consider the interesting case[3] $dS_0 < X < uS_0$, in which the option will be exercised if the stock price goes up, and it will expire worthless if the stock price goes down, i.e., $c_1^u = uS_0 - X$ and $c_1^d = 0$, (3.2) becomes

$$c_0 = \frac{1}{1 + r}[S_0 p^{\mathbb{Q}} u - X p^{\mathbb{Q}}]. \tag{3.5}$$

Exactly as in Section 2 (see (2.5)–(2.7)), if we set

$$p^{\widetilde{\mathbb{Q}}} := \frac{1}{1 + r} \frac{S_1^u}{S_0} p^{\mathbb{Q}} = \frac{1}{1 + r} u p^{\mathbb{Q}} \tag{3.6}$$

[3] The other cases are trivial. Indeed, if $X \le dS_0$, we have $c_1^u = uS_0 - X$, $c_1^d = dS_0 - X$ and (3.2) gives

$$c_0 = S_0 - \frac{1}{1 + r} X,$$

i.e., $p^{\mathbb{Q}} = 1$ and $N(d_1) = N(d_2) = 1$. On the converse, if $uS_0 \le X$, we have $c_1^u = c_1^d = c_0 = 0$, i.e., $p^{\mathbb{Q}} = 0$ and $N(d_1) = N(d_2) = 0$. See, also, Remark 3.1.

and define correspondingly $\widetilde{\mathbb{Q}} = (p^{\widetilde{\mathbb{Q}}}, 1-p^{\widetilde{\mathbb{Q}}})$ as the following Radon–Nikodym derivative

$$\frac{\widetilde{\mathbb{Q}}}{\mathbb{Q}} := \frac{S_1}{(1+r)S_0}, \tag{3.7}$$

then (3.5) becomes, in turn,

$$c_0 = S_0 p^{\widetilde{\mathbb{Q}}} - \frac{1}{1+r} X p^{\mathbb{Q}}, \tag{3.8}$$

which resembles formula (2.9) for the continuous case.

Remark 3.1. By comparing (2.9) and (3.8) it is clear that $N(d_1)$ and $N(d_2)$ correspond to $p^{\widetilde{\mathbb{Q}}}$ and $p^{\mathbb{Q}}$, respectively. In particular, the no-arbitrage condition entails $\frac{u}{1+r} > 1$, which implies $p^{\widetilde{\mathbb{Q}}} > p^{\mathbb{Q}}$; this observation confirms once again that $N(d_1) > N(d_2)$.

Analogously to Proposition 2.1, we are now in a position to prove the next result.

Proposition 3.2. *The following properties hold true:*

(i) *the discounted stock price process* $\big(\frac{S_i}{(1+r)^i}\big)_{i=0,1}$ *is a* $\widetilde{\mathbb{Q}}$*-submartingale,*

(ii) *if* $r \geq 0$*, the stock price process* $(S_i)_{i=0,1}$ *is a* $\widetilde{\mathbb{Q}}$*-submartingale,*

(iii) *the* (*unconditional*) $\widetilde{\mathbb{Q}}$*-expectation of* S_1 *is given by*

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_1] = \frac{\mathbb{E}^{\mathbb{Q}}[S_1^2]}{\mathbb{E}^{\mathbb{Q}}[S_1]}, \tag{3.9}$$

i.e., by the ratio between the second and the first moment of S_1 *under* $\mathbb{Q}$*,*

(iv) *the probability* $p^{\widetilde{\mathbb{Q}}}$ *is different from the hedge-ratio* (*see* (3.3))

$$\Delta_0 = \frac{c_1^u - c_1^d}{S_1^u - S_1^d}.$$

Proof. As a first step, let us compute $\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_1]$. To this aim, observe that the explicit expression of $p^{\widetilde{\mathbb{Q}}}$ is given by

$$p^{\widetilde{\mathbb{Q}}} = \frac{u}{1+r} p^{\mathbb{Q}} = \frac{u}{1+r} \cdot \frac{1+r-d}{u-d}, \tag{3.10}$$

whence

$$1 - p^{\widetilde{\mathbb{Q}}} = \frac{(1+r)(u-d) - u(1+r-d)}{(1+r)(u-d)} = \frac{d}{1+r} \cdot \frac{u-1-r}{u-d} = \frac{d}{1+r}(1-p^{\mathbb{Q}}). \tag{3.11}$$

By definition of $\mathbb{E}^{\widetilde{\mathbb{Q}}}[\cdot]$, we easily compute[4)]

$$\begin{aligned} \mathbb{E}^{\widetilde{\mathbb{Q}}}[S_1] &= S_1^u p^{\widetilde{\mathbb{Q}}} + S_1^d (1-p^{\widetilde{\mathbb{Q}}}) = \frac{u S_1^u}{1+r} p^{\mathbb{Q}} + \frac{d S_1^d}{1+r}(1-p^{\mathbb{Q}}) \\ &= \frac{(S_1^u)^2}{(1+r)S_0} p^{\mathbb{Q}} + \frac{(S_1^d)^2}{(1+r)S_0}(1-p^{\mathbb{Q}}) = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_1^2}{(1+r)S_0}\right]. \end{aligned} \tag{3.12}$$

---

4) We can also derive $\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_1]$ noting that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_1] = \mathbb{E}^{\mathbb{Q}}\left[S_1 \frac{\widetilde{\mathbb{Q}}}{\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_1^2}{(1+r)S_0}\right].$$

(i) Since S_0 is known at time 0, by using (3.12) we plainly have

$$\mathbb{E}_0^{\widetilde{\mathbb{Q}}}\left[\frac{S_1}{1+r}\right] = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\frac{S_1}{1+r}\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_1}{1+r}\right)^2\right]}{S_0} \geq \frac{\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{S_1}{1+r}\right]\right)^2}{S_0}$$
$$= \frac{\left(\frac{S_0}{1+r}(up^{\mathbb{Q}} + d(1-p^{\mathbb{Q}}))\right)^2}{S_0} = \frac{S_0^2}{S_0} = S_0\left(= \frac{S_0}{(1+r)^0}\right),$$

where we have used the Jensen's inequality and the fact that the conditional expectation based on no information is simply the (unconditional) expectation, i.e., $\mathbb{E}_0[\cdot] = \mathbb{E}[\cdot]$. This proves that the discounted stock price process is a $\widetilde{\mathbb{Q}}$-submartingale.

(ii) By means of similar calculations as performed in *i*), it is easy to check that

$$\mathbb{E}_0^{\widetilde{\mathbb{Q}}}[S_1] \geq (1+r)S_0,$$

as well, where the term $(1+r)$ is surely greater than 1 if $r \geq 0$.

(iii) The equality (3.9) follows directly from (3.12), since the expression of the first moment of S_1 under $\mathbb{Q}$ is given by

$$\mathbb{E}^{\mathbb{Q}}[S_1] = S_1^u p^{\mathbb{Q}} + S_1^d(1-p^{\mathbb{Q}}) = (up^{\mathbb{Q}} + d(1-p^{\mathbb{Q}}))S_0 = (1+r)S_0.$$

(iv) Taking into account the last footnote 3 (page 460), the hedge-ratio in the interesting case $dS_0 < X < uS_0$ is given by (3.3), whereas the probability $p^{\widetilde{\mathbb{Q}}}$ is defined in (3.6). These two expressions are equal only when

$$X = \frac{S_0\, ud}{1+r},$$

otherwise they are different. The proof is now complete. □

3.2. THE MULTI-PERIOD BINOMIAL MODEL

In the multi-period binomial model we consider $N \geq 2$ periods and in each of them the stock price will either go up or down by the factors u and d, respectively. In particular, at time $t = n \leq N$ we have $n+1$ distinct configurations for S_n, which we write down as

$$S_n^{\rho^{(i,n-i)}(u,d)} := u^i d^{n-i} S_0$$

for any integer $i \in \{0, 1, \ldots, n\}$, where $\rho^{(i,n-i)}(u,d)$ denotes any permutation with repetition of the n-tuple

$$(\underbrace{u, \ldots, u}_{i}, \underbrace{d, \ldots, d}_{n-i}),$$

and corresponds to choosing i “up moves” (and $n-i$ “down moves”) for a total of $\binom{n}{i}$ different paths in the tree. More generally, for any $i \in \{0, \ldots, n\}$ and $k \leq n$, taking the step k as the starting point, we have

$$S_n^{\rho^{(i,n-i)}(u,d)} = S_k^{\rho^{(j,k-j)}(u,d)} u^{i-j} d^{(n-k)-(i-j)},$$

where $j \in \{(i-(n-k))^+, \ldots, \min(i,k)\}$. Correspondingly, for any k satisfying $0 \leq k \leq n \leq N$ the notation $c_k^{\rho^{(i,k-i)}(u,d)}$ denotes one of the possible $k+1$ values of the random variable c_k, i.e., one of the possible call prices for a fixed step k in the binomial tree. For a better understanding one can see Figure 1.

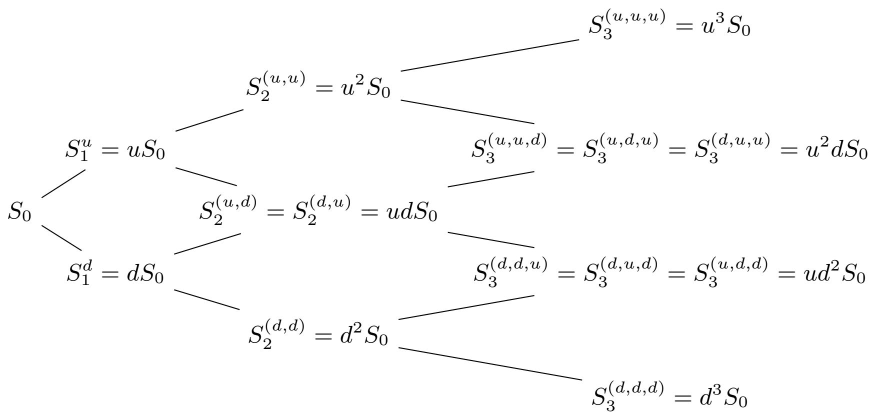

Fig. 1. Stock price tree for $N = 3$

As in the one period case, our aim is the replication of a risky portfolio by means of some derivative security, for instance a call option. For each period $n \leq N$ suppose we buy Δ_n shares of stock and invest the remaining part at a risk-free rate r (for more details, see, e.g., [10, Section 1.2]). Similarly to the one-period model explained in Subsection 3.1, we see that the price c_{n-1} for the short position is expressed recursively by

$$c_{n-1}^{\rho^{(i,(n-1)-i)}(u,d)} = \frac{1}{1+r}[c_n^{\rho^{(i+1,n-(i+1))}(u,d)} p^{\mathbb{Q}} + c_n^{\rho^{(i,n-i)}(u,d)}(1-p^{\mathbb{Q}})] \tag{3.13}$$

for any $i \in \{0, \ldots, n-1\}$, where the risk-neutral probability $\mathbb{Q} = (p^{\mathbb{Q}}, 1-p^{\mathbb{Q}})$ is still equal to (3.4). By induction, for any $n \in \{1, \ldots, N\}$ it is easy to see that

$$c_0 = \frac{1}{(1+r)^n}\left[\sum_{i=0}^{n}\binom{n}{i} c_n^{\rho^{(i,n-i)}(u,d)} (p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i}\right]. \tag{3.14}$$

On the other hand, if m_0 denotes the minimum number of upward moves necessary for the stock to end in the money at time n, (3.14) may be rephrased as

$$c_0 = \frac{1}{(1+r)^n}\left[\sum_{i=m_0}^{n}\binom{n}{i}(u^i d^{n-i} S_0 - X)(p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i}\right],$$

which, rearranging the terms, becomes also

$$c_0 = \frac{1}{(1+r)^n}\Bigg[\sum_{i=m_0}^{n}\binom{n}{i}u^i d^{n-i}(p^{\mathbb{Q}})^i(1-p^{\mathbb{Q}})^{n-i}S_0 - X\sum_{i=m_0}^{n}\binom{n}{i}(p^{\mathbb{Q}})^i(1-p^{\mathbb{Q}})^{n-i}\Bigg]. \tag{3.15}$$

Now let us introduce the complementary binomial distribution

$$\Phi(p^{\mathbb{Q}}, n, m_0) := \sum_{i=m_0}^{n}\binom{n}{i}(p^{\mathbb{Q}})^i(1-p^{\mathbb{Q}})^{n-i} \tag{3.16}$$

which represents the probability that the stock price will go up at least m_0 times so that the option will be exercised at time n, and corresponds to $N(d_2)$. Therefore, by using the measure $p^{\widetilde{\mathbb{Q}}}$ defined in (3.6), (3.15) becomes

$$c_0 = S_0\Phi(p^{\widetilde{\mathbb{Q}}}, n, m_0) - \frac{1}{(1+r)^n}X\Phi(p^{\mathbb{Q}}, n, m_0), \tag{3.17}$$

where again we have set

$$\begin{aligned}\Phi(p^{\widetilde{\mathbb{Q}}}, n, m_0) &:= \sum_{i=m_0}^{n}\binom{n}{i}(p^{\widetilde{\mathbb{Q}}})^i(1-p^{\widetilde{\mathbb{Q}}})^{n-i}\\ &= \frac{1}{(1+r)^n}\sum_{i=m_0}^{n}\binom{n}{i}u^i d^{n-i}(p^{\mathbb{Q}})^i(1-p^{\mathbb{Q}})^{n-i};\end{aligned}$$

this last term corresponds to $N(d_1)$ of the Black–Scholes formula.

Incidentally observe that, as a matter of fact, for a fixed n, $0 \le n \le N$,

$$c_0 = S_0\widetilde{\mathbb{Q}}\{S_n > X\} - \frac{X}{(1+r)^n}\mathbb{Q}\{S_n > X\},$$

and the event $\{S_n > X\}$ is evaluated under two different measures; thus the analogy with the continuous case is now quite evident.

More generally, for any $k \le n$, it may be proved that

$$c_k = S_k\widetilde{\mathbb{Q}}\{S_n > X|\mathcal{F}_k\} - \frac{X}{(1+r)^{n-k}}\mathbb{Q}\{S_n > X|\mathcal{F}_k\}.$$

Indeed, arguing as before concerning the equality (3.15) with slight modifications leads to

$$\begin{aligned}
c_k &= \frac{1}{(1+r)^{n-k}}\Bigg[\sum_{j=m_k}^{n-k}\binom{n-k}{j}\Big(S_n - X\Big)(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}\Bigg]\\
&= \frac{1}{(1+r)^{n-k}}\Bigg[\sum_{j=m_k}^{n-k}\binom{n-k}{j}\Big(S_k u^j d^{(n-k)-j} - X\Big)(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}\Bigg]\\
&= S_k\sum_{j=m_k}^{n-k}\binom{n-k}{j}(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j}\\
&\quad - \frac{X}{(1+r)^{n-k}}\sum_{j=m_k}^{n-k}\binom{n-k}{j}(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j},
\end{aligned}$$

where m_k is the minimum number of upward moves necessary for the option to end in the money at time n starting from the step k. Obviously $c_k = 0$ if $m_k > n-k$; see also [7, Proposition 2.1.2].

As an improvement, a closed formula[5)] for the value of the call option at each node of the binomial tree would be desiderable; indeed, for fixed n, k and i satisfying $0 \le i \le k \le n \le N$, a careful and non trivial inspection through the binomial tree shows that

$$\begin{aligned}
&c_k^{\rho^{(i,k-i)}(u,d)}\\
&= \frac{1}{(1+r)^{n-k}}\Bigg[\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}\Big(S_n^{\rho^{(i+j,n-(i+j))}(u,d)} - X\Big)(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}\Bigg]\\
&= \frac{1}{(1+r)^{n-k}}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}S_k^{\rho^{(i,k-i)}(u,d)}u^j d^{(n-k)-j}(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}\\
&\quad - \frac{X}{(1+r)^{n-k}}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}\\
&= S_k^{\rho^{(i,k-i)}(u,d)}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j}\\
&\quad - \frac{X}{(1+r)^{n-k}}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}(p^{\mathbb{Q}})^j(1-p^{\mathbb{Q}})^{(n-k)-j}.
\end{aligned}$$

5) Non iterative closed formula.

In the formula above m_k^i is a realization of the random variable m_k; more formally

$$m_k = \begin{cases} m_0, & \text{if}\, k = 0, \\ m_k^i \in \{0, \dots, (m_0 - i)^+\} \cup \{+\infty\}, & \text{if } k > 0,\ 0 \leq i \leq k, \end{cases}$$

where m_0 is the same (deterministic) positive integer appearing in (3.15). Obviously $c_k^{\rho^{(i,k-i)}(u,d)} = 0$ if $m_k^i = +\infty$.

Summing up, we can write

$$c_k = S_k \Phi(p^{\widetilde{\mathbb{Q}}}, n-k, m_k) - \frac{X}{(1+r)^{n-k}} \Phi(p^{\mathbb{Q}}, n-k, m_k) \tag{3.18}$$

and

$$c_k^{\rho^{(i,k-i)}(u,d)} = S_k^{\rho^{(i,k-i)}(u,d)} \Phi(p^{\widetilde{\mathbb{Q}}}, n-k, m_k^i) - \frac{X}{(1+r)^{n-k}} \Phi(p^{\mathbb{Q}}, n-k, m_k^i), \tag{3.19}$$

as generalizations of (3.17).

Remark 3.3. The importance of the last formula (3.19) lies on the fact that it allows to compute the call price $c_k^{\rho^{(i,k-i)}(u,d)}$ at any of the nodes i at time k only knowing the stock price $S_k^{\rho^{(i,k-i)}(u,d)}$ corresponding to the same node, together with the realization m_k^i. This procedure presents a remarkable advantage especially when dealing with binomial trees which price long expiring options and/or which use a huge number of time steps in order to get a nice accuracy. Specifically, from a computational point of view, while it is necessary to move in the tree "forward" until the end at time n for the stock price, it is pointless the full "backward" procedure from n up to time k, the unique information useful to compute m_k^i being the stock price (payoff) at time n.

The following proposition provides further information about the probability measure $\widetilde{\mathbb{Q}} = (p^{\widetilde{\mathbb{Q}}}, 1 - p^{\widetilde{\mathbb{Q}}})$.

Proposition 3.4. *For every $n \leq N$ the (unconditional) $\widetilde{\mathbb{Q}}$-expectation of S_n is given by*

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_n] = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{\mathbb{E}^{\mathbb{Q}}[S_n]}, \tag{3.20}$$

i.e., by the ratio between the second and the first moment of S_n under $\mathbb{Q}$.

Proof. Let us start by computing $\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_n]$. We know that $u^i d^{n-i} S_0 = S_n^{\rho^{(i,n-i)}(u,d)}$ and thus

$$\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_n] &= \sum_{i=0}^{n} \binom{n}{i} S_n^{\rho^{(i,n-i)}(u,d)} (p^{\widetilde{\mathbb{Q}}})^i (1-p^{\widetilde{\mathbb{Q}}})^{n-i} \\
&= S_0 \sum_{i=0}^{n} \binom{n}{i} u^i d^{n-i} (p^{\widetilde{\mathbb{Q}}})^i (1-p^{\widetilde{\mathbb{Q}}})^{n-i} \\
&= \frac{S_0}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} (u^i d^{n-i})^2 (p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i} \\
&= \frac{S_0}{(1+r)^n} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{S_n^{\rho^{(i,n-i)}(u,d)}}{S_0} \right)^2 (p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i} \\
&= \frac{1}{(1+r)^n S_0} \sum_{i=0}^{n} \binom{n}{i} (S_n^{\rho^{(i,n-i)}(u,d)})^2 (p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i}.
\end{aligned}$$

Consequently

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_n] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_n^2}{S_0(1+r)^n} \right], \tag{3.21}$$

whence the equality (3.20) directly follows, since the expression of the first moment of S_n under $\mathbb{Q}$ is given by

$$\mathbb{E}^{\mathbb{Q}}[S_n] = S_0 \sum_{i=0}^{n} \binom{n}{i} u^i d^{n-i} (p^{\mathbb{Q}})^i (1-p^{\mathbb{Q}})^{n-i} = S_0 (up^{\mathbb{Q}} + d(1-p^{\mathbb{Q}}))^n = S_0(1+r)^n. \quad \square$$

In analogy with the one-period binomial model, our next goal is to look for another equivalent measure under which the discounted process is a submartingale.

The underlying idea is to extend the definition of $\widetilde{\mathbb{Q}}$ in (3.7) to the multi-period binomial model with $N \geq 2$ steps. To this purpose, we are naturally led to consider the measure $\widehat{\mathbb{Q}}$ related to $\mathbb{Q}$ by the following Radon–Nikodym derivative

$$\frac{\widehat{\mathbb{Q}}}{\mathbb{Q}} := Z, \tag{3.22}$$

where Z is the random variable defined by

$$Z := \frac{S_N}{(1+r)^N S_0}. \tag{3.23}$$

For any $k \leq N$ define the Radon–Nikodym derivative process as

$$Z_k = \mathbb{E}_k^{\mathbb{Q}}[Z] \tag{3.24}$$

with $Z_N = Z$. One readily obtains

$$\mathbb{E}^{\mathbb{Q}}[Z] = \frac{\mathbb{E}^{\mathbb{Q}}[S_N]}{(1+r)^N S_0} = \frac{(1+r)^N S_0}{(1+r)^N S_0} = 1$$

and

$$Z_k = \frac{S_k}{(1+r)^k S_0},$$

since the discounted process is a $\mathbb{Q}$-martingale. Obviously $\widehat{\mathbb{Q}} \equiv \widetilde{\mathbb{Q}}$ if $N = 1$.

Now we may state the next proposition concerning the newly defined measure $\widehat{\mathbb{Q}}$.

Proposition 3.5. *The following properties hold true:*

(i) *the discounted stock price process* $\left(\frac{S_i}{(1+r)^i}\right)_{0\le i\le N}$ *is a* $\widehat{\mathbb{Q}}$*-submartingale,*

(ii) *if* $r \ge 0$*, the stock price process* $(S_i)_{0\le i\le N}$ *is a* $\widehat{\mathbb{Q}}$*-submartingale,*

(iii) *for any* $n \le N$ *the (unconditional)* $\widehat{\mathbb{Q}}$*-expectation of* S_n *is given by*

$$\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n] = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{\mathbb{E}^{\mathbb{Q}}[S_n]}, \tag{3.25}$$

i.e., by the ratio between the second and the first moment of S_n *under* $\mathbb{Q}$*. In particular, the expectation of* S_n *is the same under the probabilities* $\widetilde{\mathbb{Q}}$ *and* $\widehat{\mathbb{Q}}$ *(see Proposition 3.4).*

Proof. Fix n and k such that $0 \le k \le n \le N$.

(i) According to [10, Lemma 3.2.6] and with the help of Jensen's inequality, one has

$$\begin{aligned}\mathbb{E}_k^{\widehat{\mathbb{Q}}}\left[\frac{S_n}{(1+r)^n}\right] &= \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^n} Z_n\right]}{Z_k} = \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\left(\frac{S_n}{(1+r)^n}\right)^2\right]}{\frac{S_k}{(1+r)^k}}\\ &\ge \frac{\left(\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^n}\right]\right)^2}{\frac{S_k}{(1+r)^k}} = \frac{\left(\frac{S_k}{(1+r)^k}\right)^2}{\frac{S_k}{(1+r)^k}} = \frac{S_k}{(1+r)^k},\end{aligned}$$

since the discounted stock price process is a $\mathbb{Q}$-martingale.

(ii) Arguing as in (i), it is fairly easy to verify that

$$\mathbb{E}_k^{\widehat{\mathbb{Q}}}[S_n] \ge (1+r)^{n-k} S_k,$$

where surely $(1+r)^{n-k} \ge 1$ if $r \ge 0$.

(iii) By virtue of [10, Lemma 3.2.5], an easy calculation shows that

$$\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n] = \mathbb{E}^{\mathbb{Q}}[S_n\, Z_n] = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{(1+r)^n S_0} = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{\mathbb{E}^{\mathbb{Q}}[S_n]},$$

as announced. □

There are alternative ways to prove Proposition 3.5, based upon the general properties of the conditional expectation under a change of measure. We present these proofs in the Appendix A with the help of a preliminary lemma.

APPENDIX A. ALTERNATIVE PROOFS OF (i) AND (iii) IN PROPOSITION 3.5

Consider a discrete Bernoulli random variable U with parameter p and range $\{u, d\}$ and recall that

1. $\mathbb{E}[U] = up + d(1-p)$,
2. $\mathbb{E}[U^h] = u^h p + d^h(1-p) \quad (h \geq 2)$,
3. $\mathrm{Var}(U) = p(1-p)(u-d)^2$.

It is possible to express recursively the stock price process S_n in the N-times binomial model starting from S_0 by putting

$$S_n := S_0\, U_1 \cdot \ldots \cdot U_n \quad (n \geq 1), \tag{A.1}$$

where the U_i's are i.i.d. random variables with a Bernoulli distribution with parameter $p^{\mathbb{Q}}$ and range $\{u, d\}$; for simplicity we write $S_n = S_{n-1}U$ where U stands for U_n. For any $0 \leq k \leq n \leq N$, keeping in mind that now $\mathbb{E}^{\mathbb{Q}}[U] = up^{\mathbb{Q}} + d(1-p^{\mathbb{Q}}) = 1 + r$, it is easy to verify that

1. $\mathbb{E}_k^{\mathbb{Q}}[S_n] = S_k(1+r)^{n-k}$,
2. $\mathbb{E}_k^{\mathbb{Q}}[S_n^h] = S_k^h \left(\mathbb{E}^{\mathbb{Q}}[U^h]\right)^{n-k} \qquad (h \geq 2)$,
3. $\mathrm{Var}_k^{\mathbb{Q}}(S_n) = \mathbb{E}_k^{\mathbb{Q}}[S_n^2] - (\mathbb{E}_k^{\mathbb{Q}}[S_n])^2 = S_k^2 \left((u^2 p^{\mathbb{Q}} + d^2(1-p^{\mathbb{Q}}))^{n-k} - (1+r)^{2(n-k)}\right)$.

Before proceeding further, we need the following lemma which perhaps covers an interest on its own.

Lemma A1. *Let us consider the multi-period binomial model with $N \geq 2$ steps and denote by S_n the stock price at time $n \leq N$, defined recursively by* (A.1). *Then, for any $0 \leq k \leq n \leq N$, if $\mathbb{Q}$ is the risk-neutral measure and $r \geq 0$ is the risk-free interest rate, we have the following properties:*

(i) *the sequence $\left(\mathrm{Var}_k^{\mathbb{Q}}(S_n)\right)_{0 \leq n \leq N}$ is increasing;*

(ii) *for any $m \in \{0, \ldots, N\}$ with $n < m$ we have*

  (iia) $\mathbb{E}_k^{\mathbb{Q}}[S_n\, S_m] = (1+r)^{m-n} \cdot \mathbb{E}_k^{\mathbb{Q}}[S_n^2]$,

  (iib) $\mathrm{Var}_k^{\mathbb{Q}}(S_n\, S_m) = S_k^4 \big((\mathbb{E}^{\mathbb{Q}}[U^4])^{n-k} \cdot (\mathbb{E}^{\mathbb{Q}}[U^2])^{m-n}$
  $\qquad - (\mathbb{E}^{\mathbb{Q}}[U^2])^{2(n-k)} \cdot (1+r)^{2(m-n)}\big)$,

  (iic) $cov_k^{\mathbb{Q}}(S_n, S_m) = (1+r)^{m-n} \cdot \mathrm{Var}_k^{\mathbb{Q}}(S_n)$ *and therefore* $cov_k^{\mathbb{Q}}(S_n, S_m) \geq 0$,

  (iid) *if ρ denotes the* (*auto-*)*correlation coefficient, then*

$$\rho_k^{\mathbb{Q}}(S_n, S_m) = (1+r)^{m-n} \cdot \sqrt{\frac{\mathrm{Var}_k^{\mathbb{Q}}(S_n)}{\mathrm{Var}_k^{\mathbb{Q}}(S_m)}}.$$

Proof. (i) For any $n \geq 1$, due to the independence of the random variables S_n and U in $S_{n+1} = S_n U$, it follows that[6)]

$$\mathrm{Var}_k^{\mathbb{Q}}(S_{n+1}) = (\mathbb{E}_k^{\mathbb{Q}}[S_n])^2\, \mathrm{Var}^{\mathbb{Q}}(U) + \mathrm{Var}_k^{\mathbb{Q}}(S_n)\mathbb{E}^{\mathbb{Q}}[U^2] \geq \mathrm{Var}_k^{\mathbb{Q}}(S_n)$$

[6)] Each U_i is obviously independent of $\mathcal{F}_k$.

since

$$(\mathbb{E}_k^{\mathbb{Q}}[S_n])^2 \operatorname{Var}^{\mathbb{Q}}(U) \geq 0 \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}[U^2] \geq (\mathbb{E}^{\mathbb{Q}}[U])^2 = (1+r)^2 \geq 1.$$

(ii) For any $m \in \{0, \dots, N\}$, if $n < m$, a direct computation leads to:

$$
\begin{aligned}
&\text{(iia)}\ \ \mathbb{E}_k^{\mathbb{Q}}[S_n S_m] = \mathbb{E}_k^{\mathbb{Q}}[(S_0\, U_1 \cdot \ldots \cdot U_n)^2 \cdot U_{n+1} \cdot \ldots \cdot U_m] = (1+r)^{m-n} \cdot \mathbb{E}_k^{\mathbb{Q}}[S_n^2],\\
&\text{(iib)}\ \ \operatorname{Var}_k^{\mathbb{Q}}(S_n S_m) = \operatorname{Var}_k^{\mathbb{Q}}((S_0\, U_1 \cdot \ldots \cdot U_n)^2 \cdot U_{n+1} \cdot \ldots \cdot U_m)\\
&\qquad = S_k^4\, \mathbb{E}^{\mathbb{Q}}[(U_{k+1} \cdot \ldots \cdot U_n)^4]\, \mathbb{E}^{\mathbb{Q}}[(U_{n+1} \cdot \ldots \cdot U_m)^2]\\
&\qquad\quad - S_k^4\, (\mathbb{E}^{\mathbb{Q}}[(U_{k+1} \cdot \ldots \cdot U_n)^2 \cdot U_{n+1} \cdot \ldots \cdot U_m])^2\\
&\qquad = S_k^4 \Big((\mathbb{E}^{\mathbb{Q}}[U^4])^{n-k} \cdot (\mathbb{E}^{\mathbb{Q}}[U^2])^{m-n}\\
&\qquad\qquad - (\mathbb{E}^{\mathbb{Q}}[U^2])^{2(n-k)} \cdot (1+r)^{2(m-n)} \Big),\\
&\text{(iic)}\ \ cov_k^{\mathbb{Q}}(S_n, S_m) = \mathbb{E}_k^{\mathbb{Q}}[S_n^2 \cdot U_{n+1} \cdot \ldots \cdot U_m] - \mathbb{E}_k^{\mathbb{Q}}[S_n]\, \mathbb{E}_k^{\mathbb{Q}}[S_n \cdot U_{n+1} \cdot \ldots \cdot U_m]\\
&\qquad = \mathbb{E}_k^{\mathbb{Q}}[S_n^2]\, \mathbb{E}_k^{\mathbb{Q}}[U_{n+1} \cdot \ldots \cdot U_m] - (\mathbb{E}_k^{\mathbb{Q}}[S_n])^2\, \mathbb{E}_k^{\mathbb{Q}}[U_{n+1} \cdot \ldots \cdot U_m]\\
&\qquad = (1+r)^{m-n} \cdot (\mathbb{E}_k^{\mathbb{Q}}[S_n^2] - (\mathbb{E}_k^{\mathbb{Q}}[S_n])^2) = (1+r)^{m-n} \cdot \operatorname{Var}_k^{\mathbb{Q}}(S_n),\\
&\text{(iid)}\ \ \rho_k^{\mathbb{Q}}(S_n, S_m) = \frac{cov_k^{\mathbb{Q}}(S_n, S_m)}{\sqrt{\operatorname{Var}_k^{\mathbb{Q}}(S_n)\operatorname{Var}_k^{\mathbb{Q}}(S_m)}} = \frac{(1+r)^{m-n} \cdot \operatorname{Var}_k^{\mathbb{Q}}(S_n)}{\sqrt{\operatorname{Var}_k^{\mathbb{Q}}(S_n)\operatorname{Var}_k^{\mathbb{Q}}(S_m)}}\\
&\qquad = (1+r)^{m-n} \cdot \sqrt{\frac{\operatorname{Var}_k^{\mathbb{Q}}(S_n)}{\operatorname{Var}_k^{\mathbb{Q}}(S_m)}}.
\end{aligned}
$$

The proof is now complete. □

Remark A2. If $(W_t)_{t\geq 0}$ is a Brownian motion, then for any s and t satisfying $0 \leq s \leq t$ it is well-known that

(i) $\operatorname{Var}(W_t) \geq \operatorname{Var}(W_s)$,
(ii) $cov(W_t, W_s) = \operatorname{Var}(W_s)$,
(iii) $\rho(W_t, W_s) = \sqrt{\frac{\operatorname{Var}(W_s)}{\operatorname{Var}(W_t)}}$.

A quick look to the previous lemma highlights clearly the strong interplay between the discrete process $(S_n)_{0\leq n\leq N}$ and the continuous process $(W_t)_{t\geq 0}$.

Now we present the following alternative proofs of (i) and (iii) in Proposition 3.5, taking into account the definition (3.24) and the subsequent observation.

Proof. Fix n and k such that $0 \le k \le n \le N$.

(i) By virtue of Lemma A1-(iic) and the Bayes formula, one gets

$$\begin{aligned}
\mathbb{E}_k^{\widehat{\mathbb{Q}}}\left[\frac{S_n}{(1+r)^n}\right] &= \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^n} Z_N\right]}{Z_k} \\
&= \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^n}\right]\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_N}{(1+r)^N}\right] + cov_k^{\mathbb{Q}}\left(\frac{S_n S_N}{(1+r)^{n+N}}\right)}{\frac{S_k}{(1+r)^k}} \\
&= \frac{\left(\frac{S_k}{(1+r)^k}\right)^2 + \frac{(1+r)^{N-n}}{(1+r)^{n+N}} \cdot \mathrm{Var}_k^{\mathbb{Q}}(S_n)}{\frac{S_k}{(1+r)^k}} \ge \frac{\left(\frac{S_k}{(1+r)^k}\right)^2}{\frac{S_k}{(1+r)^k}} = \frac{S_k}{(1+r)^k},
\end{aligned}$$

since the discounted stock price is a $\mathbb{Q}$-martingale.

(i)' (alternative proof) Again the Bayes formula implies

$$\mathbb{E}_k^{\widehat{\mathbb{Q}}}\left[\frac{S_n}{(1+r)^n}\right] = \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n}{(1+r)^n} Z_N\right]}{Z_k} = \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n\, S_N}{(1+r)^{n+N}}\right]}{\frac{S_k}{(1+r)^k}},$$

where "taking out what is known" and Jensen's inequality, together with the independence of the U_i's, yield

$$\begin{aligned}
\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n S_N}{(1+r)^{n+N}}\right] &= \frac{S_k^2}{(1+r)^{n+N}}\,\mathbb{E}^{\mathbb{Q}}[U_{k+1}^2 \cdot \ldots \cdot U_n^2 \cdot U_{n+1} \cdot \ldots \cdot U_N] \\
&= \frac{S_k^2}{(1+r)^{n+N}}\,\mathbb{E}^{\mathbb{Q}}[U_{k+1}^2] \cdot \ldots \cdot \mathbb{E}^{\mathbb{Q}}[U_n^2] \cdot \mathbb{E}^{\mathbb{Q}}[U_{n+1}] \cdot \ldots \cdot \mathbb{E}^{\mathbb{Q}}[U_N] \\
&= \frac{S_k^2}{(1+r)^{n+N}}\,(\mathbb{E}^{\mathbb{Q}}[U^2])^{n-k}\,(1+r)^{N-n} \\
&\ge \frac{S_k^2}{(1+r)^{n+N}}\,(\mathbb{E}^{\mathbb{Q}}[U])^{2(n-k)}\,(1+r)^{N-n} \\
&= \frac{S_k^2}{(1+r)^{n+N}}\,(1+r)^{2(n-k)}\,(1+r)^{N-n} = \frac{S_k^2}{(1+r)^{2k}}.
\end{aligned}$$

Hence we can conclude that

$$\mathbb{E}_k^{\widehat{\mathbb{Q}}}\left[\frac{S_n}{(1+r)^n}\right] \ge \frac{\frac{S_k^2}{(1+r)^{2k}}}{\frac{S_k}{(1+r)^k}} = \frac{S_k}{(1+r)^k},$$

as desired.

(iii) By virtue of Lemma A1-(iia), one immediately gets

$$\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n] = \mathbb{E}^{\mathbb{Q}}[S_n\, Z_N] = \frac{\mathbb{E}^{\mathbb{Q}}[S_n S_N]}{(1+r)^N S_0} = \frac{(1+r)^{N-n}\cdot \mathbb{E}^{\mathbb{Q}}[S_n^2]}{(1+r)^N S_0} = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{(1+r)^n S_0} = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{\mathbb{E}^{\mathbb{Q}}[S_n]},$$

and the result follows. □

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