DOI 10.24425/aee.2021.137571

Invariance of reachability and observability for fractional positive linear electrical circuit with delays

TONG YUAN, HONGLI YANG

Shandong University of Science and Technology
China

e-mail: yhlmath@126.com

(Received: 27.07.2020, revised: 27.03.2021)

Abstract: This paper focuses on the invariance of the reachability and observability for fractional order positive linear electrical circuits with delays and their checking methods. By derivation and comparison, it shows that conditions and checking methods of reachability and observability for integer and fractional order positive linear electrical circuits with delays are invariant. An illustrative example is presented at the end of the paper.

Key words: invariance, observability, positive linear electrical circuits with delays, reachability

1. Introduction

Positive linear systems with delays refer to the systems in which a differential equation or a difference equation are positive and linear. In a variety of systems, such as communication, circuits [1, 2], power [3] and industrial engineering, state evolution depends not only on the current state, but also on the past state. This characteristic of the system is called time delay, which is usually brought by the measurement element or measurement process. Time delays can create many kinds of practical problems including the ones in positive electrical circuit systems, so it is necessary to study the properties of positive linear systems with delays. In the last decade, theoretical and applied researches of fractional calculus have achieved considerable advancements [4, 5], and the application of the fractional derivative in the circuit is evaluated [6–10], which makes the field of the fractional circuit better. What's more, many problems in large-scale systems with delays have been widely studied. References [11–13] studied the control and optimal allocation of power systems with delays.

The study of the linear systems and their properties has also achieved fruitful results. The stability of linear systems with delays is studied in [14–18]. The controllability of linear systems with delays and checking methods are proposed in [19–21]. Paper [22], mainly introduces the charge and discharge of capacitors. Paper [23], studies the reachability and controllability of fractional order positive discrete systems. In order to further learn more about fractional positive systems with time delays, it is necessary to compare observability and controllability of fractional positive systems with the one of integer positive systems [24]. By comparing paper [14] and paper [15], we find that conditions and the checking methods of stability for fractional positive linear electrical circuits with delays is invariant compared with integer systems. The reachability and observability of integer and fractional positive linear electrical circuits without delay is studied in [25] and the invariance properties between integer and fractional order positive systems are presented. Invariance properties are an interesting research topic, hence in this paper we study the invariance of reachability and observability for positive linear electrical circuits with delays.

The remainder of the paper is organized as follows. In Section 2, reachability conditions and checking methods for the integer and fractional positive linear electrical circuit with delays are investigated. Observability conditions and checking methods of the integer and fractional positive linear electrical circuit with delays are considered in Section 3. An example is presented in Section 4. Concluding remarks are given in Section 5.

The following notation will be used in this paper. R is the set of real numbers, $R^{n\times m}$ is the set of $n \times m$ real matrices, $R_+^{n\times m}$ is the set of $n \times m$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$, M_n are the sets of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

2. Invariance of reachability conditions and checking methods for positive linear electrical circuit with delays

Consider the integer linear electrical circuit systems with delays described by the following equation:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - w_k) + Bu(t), \tag{1}$$

where: $x(t) \in R^n$, $u(t) \in R^m$ are the state vector and input vector, respectively, and $A_0 \in R^{n\times n}$, $B \in R^{n\times m}$, $A_k \in R^{n\times n}$, $k = 0, 1, \ldots, q$, $w_k \geq 0$ $(k = 1, 2, \ldots, q)$ are the delays. The initial conditions for (1) has the form $x(t) = x_0(t)$, $t \in [-w, 0]$, $w = \max\{w_k\}$, where $x_0(t) \in R^n$ is a given initial state.

Definition 2.1. The linear electrical circuit with delays (1) is called (internally) positive if $x(t) \in R_+^n$ for any initial conditions $x_0(t) \in R_+^n$, $t \in [-w, 0]$ and all, $u(t) \in R_+^m$, $t \geq 0$.

Theorem 2.1. [14] The linear electrical circuit with delays (1) is (internally) positive if and only if

$$A_0 \in M_n, \quad A_k \in R_+^{n\times n}, \quad k = 1, 2, \ldots, q, \quad B \in R_+^{n\times m}.$$

Definition 2.2. The positive linear electrical circuit with delays (1) is called reachable in the time $[0, t_f]$, $t_f > 0$, if there exists an input $u(t) \in R_+^m$ for $t \in [0, t_f]$ which steers the state of the electrical circuit from $x(0) = 0$ to the given final state $x_f \in R_+^n$, i.e. $x(t_f) = x_f$.

Theorem 2.2. The positive linear electrical circuit with delays (1) is reachable in the time $[0, t_f]$ if and only if the reachability matrix

$$R(t_f) = \int_0^{t_f} e^{A_0 \tau} B B^T e^{A_0^T \tau} \, \mathrm{d}\tau \in R_+^{n \times n} \tag{2}$$

is a monomial matrix.

The input $u(t) \in R_+^m$, $t \in [0, t_f]$ which steers the state of the system from $x(0) = 0$ to the given final state $x_f \in R_+^n$, is given by

$$u(\tau) = B^T e^{A_0^T (t_f - \tau)} R^{-1}(t_f) \left(x_f - X \right) \in R_+^m, \quad \tau \in [0, t_f], \tag{3}$$

where

$$X = \int_0^{t_f} e^{A_0(t_f - \tau)} \sum_{k=1}^{q} A_k x_0(\tau - w_k) \, \mathrm{d}\tau.$$

Proof. The solution of (1) for $t \in [0, w]$ has the form

$$x(t) = e^{A_0 t} x(0) + \int_0^{t} e^{A_0(t - \tau)} \left[\sum_{k=1}^{q} A_k x_0(\tau - w_k) + Bu(t) \right] \mathrm{d}\tau. \tag{4}$$

Since $x(0) = 0$, thus we get

$$x(t_f) = \int_0^{t_f} e^{A_0(t_f - \tau)} \sum_{k=1}^{q} A_k x_0(\tau - w_k) \, \mathrm{d}\tau + \int_0^{t_f} e^{A_0(t_f - \tau)} B u(\tau) \, \mathrm{d}\tau. \tag{5}$$

It is known that $R^{-1}(t_f) \in R_+^{n \times n}$ if and only if the matrix (2) is monomial [2].
Substituting (3) into (5), we have

$$\begin{aligned} x(t_f) - X &= \int_0^{t_f} e^{A_0 (t_f - \tau)} B B^T e^{A_0^T (t_f - \tau)} R^{-1}(t_f) \left(x_f - X \right) \mathrm{d}\tau \\ &= \int_0^{t_f} e^{A_0 \tau} B B^T e^{A_0^T \tau} \, \mathrm{d}\tau R^{-1}(t_f)(x_f - X) = x_f - X. \end{aligned} \tag{6}$$

Therefore, the input (3) steers the state of the electrical circuit from $x(0) = 0$ to $x(t_f) = x_f$.

Theorem 2.3. The positive linear electrical circuit with delays (1) is reachable if and only if the following $n \times np$ size and $n \times nm$ size matrices are full rank.

$$\begin{aligned} &Q_{is} = A_0^i H \qquad (i = 0, 1, \ldots, n-1), \\ &Q_m = \left[B \;\; A_0 B \;\; A_0^2 B \;\; \cdots \;\; A_0^{n-1} B \right], \quad \operatorname{rank} Q_{is} = \operatorname{rank} Q_m = n, \end{aligned} \tag{7}$$

where $H = \left[A_1 \; A_2 \cdots A_q \right]$.

Proof. Expand the state transition matrix

$$e^{A\tau} = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^k}{k!} + \cdots,$$

where $A^0 = I$ is the identity matrix.

By the well-known Cayley–Hamilton theorem, it is possible to transform the transition matrix into the following form:

$$e^{A\tau} = a_0(\tau)I + a_1(\tau)A + \ldots + a_{n-1}(\tau)A^{n-1} = \sum_{i=0}^{n-1} a_i(\tau)A^i, \tag{8}$$

where $a_i(\tau)$ represents nonzero functions of τ, $i = 1, 2, \cdots n{-}1$.

Substitute the above Formula (8) into (5)

$$x(t_f) = \int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f - \tau) \sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau + \int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f - \tau) Bu(\tau)\,\mathrm{d}\tau.$$

From the above formula, $x(t_f)$ consists of two parts of integrals:

$$\int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f - \tau) \sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau \tag{9}$$

and

$$\int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f - \tau) Bu(\tau)\,\mathrm{d}\tau. \tag{10}$$

Formula (9) can be transformed into

$$\begin{aligned}
&\int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f - \tau) \sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau \\
&= \int_0^{t_f} \left(A_0^0 a_0(t_f - \tau) + A_0^1 a_1(t_f - \tau) + \cdots + A_0^{n-1} a_{n-1}(t_f - \tau)\right) \\
&\qquad \left(A_1 x_0\,(\tau - w_1) + a + A_2 x_0(\tau A_2 x_0(\tau - w_2) + \cdots + A_q x_0(\tau - w_q)\right)\mathrm{d}\tau \\
&= \int_0^{t_f} I a_0\left(t_f - \tau\right) A_1 x_0\,(\tau - w_1) + I a_0\left(t_f - \tau\right) A_2 x_0\,(\tau - w_2) + \cdots + \\
&\qquad A_0^{n-1} a_{n-1}\left(t_f - \tau\right) A_q x_0\left(\tau - w_q\right)\mathrm{d}\tau \\
&= I A_1 \int_0^{t_f} a_0(t_f - \tau) x_0(\tau - w_1)\,\mathrm{d}\tau + I A_2 \int_0^{t_f} a_0(t_f - \tau) x_0(\tau - w_2)\,\mathrm{d}\tau + \cdots + \\
&\qquad A_0^{n-1} A_q \int_0^{t_f} a_{n-1}\left(t_f - \tau\right) x_0\left(\tau - w_q\right)\mathrm{d}\tau
\end{aligned}$$

$$= \sum_{k=1}^{q} A_k \int_0^{t_f} a_0(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau + A_0^1 \sum_{k=1}^{q} A_k \int_0^{t_f} a_1(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau + \cdots +$$
$$A_0^{n-1} \sum_{k=1}^{q} A_k \int_0^{t_f} a_{n-1}\left(t_f-\tau\right) x_0\left(\tau-w_k\right)\mathrm{d}\tau$$
$$= \sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} a_i(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau.$$

Formula (10) can be transformed into

$$\int_0^{t_f} \sum_{i=0}^{n-1} A_0^i a_i(t_f-\tau)Bu(\tau)\,\mathrm{d}\tau = \sum_{i=0}^{n-1} A_0^i B \int_0^{t_f} a_i(t_f-\tau)u(\tau)\,\mathrm{d}\tau, \tag{11}$$

where

$$\int_0^{t_f} a_i(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau = \left[\beta_{ik}^{(1)}\beta_{ik}^{(2)}\cdots\beta_{ik}^{(n)}\right]^T = \beta_{ik} \qquad (i=0,1,\dots n{-}1,\quad k=1,2,\dots,q) \tag{12}$$

and

$$\int_0^{t_f} a_i(t_f-\tau)u(\tau)\,\mathrm{d}\tau = [\gamma_{i1}\gamma_{i1}\cdots\gamma_{im}]^T = \gamma_i \quad (i=0,1,\dots n{-}1). \tag{13}$$

Therefore,

$$\sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} a_i(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau = \sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k\beta_{ik}$$
$$= \left[I A_0 \cdots A_0^{n-1}\right] \begin{bmatrix} \sum_{k=1}^{q} A_k\beta_{0k} \\ \sum_{k=1}^{q} A_k\beta_{1k} \\ \vdots \\ \sum_{k=1}^{q} A_k\beta_{n-1k} \end{bmatrix} = \left[I\ A_0 \cdots A_0^{n-1}\right] \begin{bmatrix} \left[A_1\ A_2\cdots A_q\right]\eta_0 \\ \left[A_1\ A_2\cdots A_q\right]\eta_1 \\ \left[A_1\ A_2\cdots A_q\right]\eta_{n-1} \end{bmatrix}$$
$$= \left[A_1\ A_2\cdots A_q\right]\eta_0 + A_0\left[A_1\ A_2\cdots A_q\right]\eta_1 + \cdots + A_0^{n-1}\left[A_1\ A_2\cdots A_q\right]\eta_{n-1},$$

where

$$\eta_1 = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{iq} \end{bmatrix}, \quad i = 0, 1, \ldots, n{-}1.$$

Let $H = \begin{bmatrix} A_1 \, A_2 \cdots A_q \end{bmatrix}$, we get

$$\sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} a_i(t_f - \tau) x_0(\tau - w_k) \, \mathrm{d}\tau = H\eta_0 + A_0 H\eta_1 + \cdots + A_0^{n-1} H\eta_{n-1},$$

and replace (11) with (13)

$$\sum_{i=0}^{n-1} A_0^i B \int_0^{t_f} a_1 \left(t_f - \tau\right) u\,(\tau) \, \mathrm{d}\tau = \begin{bmatrix} B A_0 B \, A_0^2 B \cdots A_0^{n-1} B \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix}.$$

Then

$$x(t_f) = H\eta_0 + A_0 H\eta_1 + \cdots + A_0^{n-1} H\eta_{n-1} + \begin{bmatrix} B A_0 B \, A_0^2 B \cdots A_0^{n-1} B \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix}. \tag{14}$$

If the circuit is reachable, $\eta_0 \ldots \eta_{n-1}$ and $\gamma_0 \ldots \gamma_{n-1}$ can be obtained from Equation (14), when $\operatorname{rank} Q_{is} = \operatorname{rank} Q_m = n$.

Consider the fractional linear electrical circuit with delays described by the equation:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x\,(t - w_k) + Bu(t), \quad 0 < \alpha < 1, \tag{15}$$

where: $x(t) \in R^n$, $u(t) \in R^m$ are the state vector and input vector, respectively, $A_k \in R^{n\times n}$, $B \in R^{n\times m}$ and $w_k \geq 0\,(k = 1, 2, \ldots, q)$ are the delays, and

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} \, \mathrm{d}\tau, \qquad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t \tag{16}$$

is the Caputo derivation of the α order for $x(t)$. The initial conditions for (15) has the form $x(t) = x_0(t), t \in [-w, 0]$, $w = \max\{w_k\}$, where $x_0(t)$ is a given vector function.

For $\leq t \leq w$, the solution of the fractional linear electrical circuit with delays (15) has the form

$$x(t) = \Phi_0(t)x_0(0) \int_0^t \Phi(t-\tau) \left[\sum_{k=1}^{q} A_k x_0(\tau - w) + Bu(\tau) \right] \mathrm{d}\tau, \tag{17}$$

where

$$\Phi_0(t) = \sum_{j=0}^{\infty} \frac{A_0^j t^{j\alpha}}{\Gamma(j\alpha+1)}, \qquad \Phi(t) = \sum_{j=0}^{\infty} \frac{A_0^j t^{(j+1)\alpha-1}}{\Gamma[(j+1)\alpha]}.$$

Definition 2.3. The fractional linear electrical circuit with delays (15) is called positive if $x(t) \in R_+^n$ for any initial conditions $x_0(t) \in R_+^n$, $t \in [-w, 0]$ and all $u(t) \in R_+^m$, $t \geq 0$.

Theorem 2.4. [15] The fractional linear electrical circuit with delays (15) for $0 < \alpha < 1$ is positive if and only if

$$A_0 \in M_n, \quad A_k \in R_+^{n\times n}, \quad k = 1, 2, \ldots, q, \quad B \in R_+^{n\times m}.$$

Definition 2.4. The fractional positive electrical circuit with delays (15) is called reachable in the time $[0, t_f]$, $t_f > 0$, if there exists the input $u(t) \in R_+^m$ for $t \in [0, t_f]$ which steers the state of the electrical circuit from $x(0) = 0$ to the given final state $x_f \in R_+^n$, i.e. $x(t_f) = x_f$.

Theorem 2.5. The fractional positive linear electrical circuit with delays (15) is reachable in the time $[0, t_f]$ if and only if the reachability matrix

$$R_\alpha(t_f) = \int_0^{t_f} \Phi(\tau) B B^T \Phi^T(\tau)\, \mathrm{d}\tau \in R_+^{n\times n} \tag{18}$$

is a monomial matrix.

The input vector $u(t) \in R_+^m$, $t \in [0, t_f]$ which steers the state of the system from $x(0) = 0$ to the given final state $x_f \in R_+^n$ is given by

$$u(\tau) = B^T \Phi^T \left(t_f - \tau\right) R_\alpha^{-1}(t_f) \left(x_f - X_\alpha\right) \in R_+^m, \quad \tau \in [0, t_f], \tag{19}$$

where

$$X_\alpha = \int_0^{t_f} \Phi(t_f - \tau) \sum_{k=1}^{q} A_k x_0(\tau - w_k)\, \mathrm{d}\tau.$$

Proof. The solution of (15) for $t \in [0, w]$ has the form (17), since $x(0) = 0$, then we have

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) \sum_{k=1}^{q} A_k x_0(\tau - w_k)\, \mathrm{d}\tau + \int_0^{t_f} \Phi(t_f - \tau) Bu(\tau)\, \mathrm{d}\tau. \tag{20}$$

It is well known that $R_\alpha^{-1}(t_f) \in R_+^{n\times n}$ if and only if the matrix (18) is monomial [2].

Substituting (19) into (20) we obtain

$$x(t_f) - X_\alpha = \int_0^{t_f} \Phi\left(t_f - \tau\right) BB^T \Phi^T \left(t_f - \tau\right) R_\alpha^{-1}(t_f) \left(x_f - X_\alpha\right) \mathrm{d}\tau$$
$$= \int_0^{t_f} \Phi(\tau) BB^T \Phi^T(\tau) \,\mathrm{d}\tau R_\alpha^{-1}(t_f)(x_f - X_\alpha) = x_f - X_\alpha\,. \tag{21}$$

Hence, the input (19) steers the state of the electrical circuit from $x(0) = 0$ to $x(t_f) = x_f$.

Theorem 2.6. The fractional positive linear electrical circuit with delays (15) is reachable if and only if the following $n \times np$ size and $n \times nm$ size matrices are full rank.

$$\begin{aligned} Q_{is} &= A_0^i H \quad (i = 0, 1, \dots, n-1), \\ Q_m &= \left[B \;\; A_0 B \;\; A_0^2 B \;\; \cdots \;\; A_0^{n-1} B\right], \quad \operatorname{rank} Q_{is} = \operatorname{rank} Q_m = n, \end{aligned} \tag{22}$$

where

$$H = \left[A_1 \; A_2 \; \cdots \; A_q\right].$$

Proof. Using the well-known Cayley–Hamilton theorem it is possible to write the transition matrices into the follow form:

$$\Phi(\tau) = \sum_{i=0}^{n-1} \overline{a}_i(\tau) A^i, \tag{23}$$

where $\overline{a}_i(\tau)$ is a nonzero function of τ.

Substitute the (23) into (20), we can get

$$x(t_f) = \sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} \overline{a}_i(t_f - \tau) x_0(\tau - w_k)\,\mathrm{d}\tau + \sum_{i=0}^{n-1} A_0^i B \int_0^{t_f} \overline{a}_i(t_f - \tau) u(\tau)\,\mathrm{d}\tau. \tag{24}$$

From (24), it can be seen that $x(t_f)$ consists of two integrals:

$$\sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} \overline{a}_i(t_f - \tau) x_0(\tau - w_k)\,\mathrm{d}\tau \tag{25}$$

and

$$\sum_{i=0}^{n-1} A_0^i B \int_0^{t_f} \overline{a}_i(t_f - \tau) u(\tau)\,\mathrm{d}\tau, \tag{26}$$

where

$$\int_0^{t_f} \bar{a}_i \left(t_f - \tau\right) x_0 \left(\tau - w_k\right) \mathrm{d}\tau = \left[\bar{\beta}_{ik}^{(1)} \bar{\beta}_{ik}^{(2)} \cdots \bar{\beta}_{ik}^{(n)}\right]^T = \bar{\beta}_{ik} \tag{27}$$
$$(i = 0, 1, \dots, n-1, \quad k = 1, 2, \dots, q)$$

and

$$\int_0^{t_f} \bar{a}_i\left(t_f-\tau\right)u(\tau)\,\mathrm{d}\tau = [\bar{\gamma}_{i1}\bar{\gamma}_{i2}\cdots\bar{\gamma}_{im}]^T = \bar{\gamma}_i \quad (i=0,1,\ldots,n{-}1). \tag{28}$$

By replacing (25) with (27) we can get

$$\begin{aligned}
\sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} \bar{a}_i\left(t_f-\tau\right) x_0\left(\tau-w_k\right)\,\mathrm{d}\tau &= \sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \bar{\beta}_{ik} \\
&= \left[I\ A_0\cdots A_0^{n-1}\right]\begin{bmatrix} \sum_{k=1}^{q} A_k\bar{\beta}_{0k} \\ \sum_{k=1}^{q} A_k\bar{\beta}_{1k} \\ \vdots \\ \sum_{k=1}^{q} A_k\bar{\beta}_{n-1k}\end{bmatrix} = \\
&= \left[A_1\ A_2\cdots A_q\right]\bar{\eta}_0 + A_0\left[A_1\ A_2\cdots A_q\right]\bar{\eta}_1 + \cdots + A_0^{n-1}\left[A_1\ A_2\cdots A_q\right]\bar{\eta}_{n-1},
\end{aligned}$$

where

$$\bar{\eta}_1 = \begin{bmatrix} \bar{\beta}_{i1} \\ \bar{\beta}_{i2} \\ \vdots \\ \bar{\beta}_{iq}\end{bmatrix}, \quad i=0,1,\ldots,n{-}1.$$

By letting $H=\left[A_1\ A_2\ \cdots\ A_q\right]$, we can get

$$\sum_{i=0}^{n-1} A_0^i \sum_{k=1}^{q} A_k \int_0^{t_f} \overline{a}_i(t_f-\tau)x_0(\tau-w_k)\,\mathrm{d}\tau = H\overline{\eta}_0 + A_0H\overline{\eta}_1 + \cdots + A_0^{n-1}H\overline{\eta}_{n-1}\,.$$

Substitute the (28) into (26), thus

$$\sum_{i=0}^{n-1} A_0^i B \int_0^{t_f} \overline{a}_i(t_f-\tau)u(\tau)\,\mathrm{d}\tau = \left[B\ A_0B\ A_0^2B\cdots A_0^{n-1}B\right]\begin{bmatrix} \bar{\gamma}_0 \\ \bar{\gamma}_1 \\ \vdots \\ \bar{\gamma}_{n-1}\end{bmatrix}.$$

Then

$$x(t_f) = H\bar{\eta}_0 + A_0 H\bar{\eta}_1 + \cdots + A_0^{n-1} H\bar{\eta}_{n-1} + \begin{bmatrix} B & A_0 B & A_0^2 B \cdots A_0^{n-1} B \end{bmatrix} \begin{bmatrix} \bar{\gamma}_0 \\ \bar{\gamma}_1 \\ \vdots \\ \bar{\gamma}_{n-1} \end{bmatrix}. \tag{29}$$

If the circuit is reachable, $\overline{\eta}_0 \cdots \overline{\eta}_{n-1}$ and $\overline{\gamma}_0 \cdots \overline{\gamma}_{n-1}$ can be obtained from Equation (29), when $\operatorname{rank} Q_{is} = \operatorname{rank} Q_m = n$.

Theorem 2.7. The fractional positive linear electrical circuit with delays (15) is reachable in the time $[0, t_f]$ if and only if the positive linear electrical circuit with delays (1) is reachable in the same interval $[0, t_f]$.

Proof. From Theorems 2.3 and 2.6, we can find out that in the interval $[0, t_f]$, the fractional positive linear electrical circuit with delays (15) is reachable if and only if the positive linear electrical circuit with delays (1) is reachable.

Remark 2.1. From the above conclusions based on checking methods, it can be concluded that the reachability conditions of the positive linear electrical circuit with integer and fractional delays are invariant.

3. Invariance of observability conditions and checking methods for positive linear electrical circuit with delays

Consider the linear electrical circuit with delays described by the differential equation:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - w_k) + Bu(t), \tag{30a}$$

$$y(t) = Cx(t) + Du(t), \tag{30b}$$

where: $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ are the state, input and output vectors, respectively, $A_k \in R^{n \times n}$, $k = 0, 1, \ldots, q$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$ and $w_k \geq 0$ $(k = 1, 2, \ldots, q)$ are the delays. The initial conditions for (30) have the form $x(t) = x_0(t)$ for $t \in [-w, 0]$, $w = \max w_k$, where $x_0(t)$ is a given initial state.

Definition 3.1. The linear electrical circuit with delays (30) is called (internally) positive if $x(t) \in R_+^n$, $y(t) \in R_+^p$ for any initial conditions $x_0(t) \in R_+^n$, $t \in [-w, 0]$ and all $u(t) \in R_+^m$, $t \geq 0$.

Theorem 3.1. [2] The linear electrical circuit with delays (30) is (internally) positive if and only if

$$A_0 \in M_n, \quad A_k \in R_+^{n \times n}, \quad k = 1, 2, \ldots, q, \quad B \in R_+^{n \times m}, \quad C \in R_+^{p \times n}, \quad D \in R_+^{p \times m}.$$

Definition 3.2. The positive linear electrical circuit (30) with delays is called (strongly) observable in the interval of $[0, t_f]$ if by knowing the input $u(t)$ and output $y(t)$ for $[0, t_f]$, it is possible to find the unique $x(0) \in R_+^n$ of the electrical circuit.

Theorem 3.2. The positive linear electrical circuit (30) is observable in the interval $[0, t_f]$ if and only if the matrix

$$W_f = \int_0^{t_f} e^{A_0^T t} C^T C e^{A_0 t} \, \mathrm{d}t \in R_+^{n \times n} \tag{31}$$

is a monomial matrix.

Proof. Assuming $u(t) = 0$, we can get

$$y(t) = C e^{A_0 t} x(0) + C \int_0^t e^{A_0(t-\tau)} \sum_{k=1}^{q} A_k x_0(\tau - w_k) \, \mathrm{d}\tau \in R_+^p. \tag{32}$$

Let

$$Y(t) = y(t) - C \int_0^t e^{A_0(t-\tau)} \sum_{k=1}^{q} A_k x_0(\tau - w_k) \, \mathrm{d}\tau \in R_+^p. \tag{33}$$

Using the value of $y(t)$ in $[0, t_f]$, by weighting, i.e. multiply $e^{A_0^T t} C^T$ left on both sides of (33), then

$$e^{A_0^T t} C^T C e^{A_0 t} x(0) = e^{A_0^T t} C^T Y(t). \tag{34}$$

Integrating both sides of (34) in the interval $[0, t_f]$, we obtain

$$W_f x(0) = \int_0^{t_f} e^{A_0^T t} C^T Y(t) \, \mathrm{d}t \tag{35}$$

and

$$x(0) = W_f^{-1} \int_0^{t_f} e^{A_0^T t} C^T Y(t) \, \mathrm{d}t \in R_+^n \tag{36}$$

(36) holds if and only if the matrix (31) is monomial [2].

Theorem 3.3. The positive linear electrical circuit with delays (30) is observable if and only if the following $np \times n$ matrix is full rank.

$$Q_o = \begin{bmatrix} C \\ CA_0 \\ \vdots \\ CA_0^{n-1} \end{bmatrix} \quad \text{and} \quad \operatorname{rank} Q_o = n. \tag{37}$$

Proof. Let $u(t) = 0$, the solution of (30) is

$$x(t) = e^{A_0 t} x(0) + \int_0^t e^{A_0(t-\tau)} \sum_{k=1}^{q} A_k x_0(\tau - w_k) \, \mathrm{d}\tau, \tag{38a}$$

$$y(t) = C\left[e^{A_0 t}x(0) + \int\limits_0^t e^{A_0(t-\tau)}\sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau\right]. \tag{38b}$$

Using the well-known Cayley–Hamilton theorem, we have

$$e^{A\tau} = \sum_{i=0}^{n-1} a_i(\tau)A^i,$$

and by substituting it into (38b) we get

$$Y(t) = \sum_{i=0}^{n-1} a_i(t)CA_0^i x(0) = [a_0(t)\ a_1(t)\ \cdots\ a_{n-1}(t)]\begin{bmatrix} C \\ CA_0 \\ \vdots \\ CA_0^{n-1}\end{bmatrix}x(0). \tag{39}$$

Since $a_i(t)$ are known functions, the initial state $x(0)$ can be uniquely determined from $y(t)$ in the finite time $[0, t_f]$ if and only if the matrix Q_o is full rank.

Consider the fractional linear electrical circuit with delays described by the differential equation:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - w_k) + Bu(t), \tag{40a}$$

$$y(t) = Cx(t) + Du(t), \tag{40b}$$

where: $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ are the state, input and output vectors, respectively, $A_k \in R^{n\times n}$, $k = 0, 1, \ldots, q$ $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ and $w_k \geq 0$, $(k = 1, 2, \ldots, q)$ are the delays, $0 < \alpha < 1$. The initial conditions for (40) have the form $x(t) = x_0(t)$, $t \in [-w, 0]$, $w = \max\{w_k\}$, where $x_0(t)$ is a given vector function and the Caputo derivative of $x(t)$ is defined by (16).

Definition 3.3. The fractional linear electrical circuit with delays (40) is called (internally) positive if $x(t) \in R_+^n$, $y(t) \in R_+^p$ for any initial conditions $x_0(t) \in R_+^n$ for $t \in [-w, 0]$ and all $u(t) \in R_+^m$, $t \geq 0$.

Theorem 3.4. [2] The fractional linear electrical circuit with delays (40) is (internally) positive if and only if

$$A_0 \in M_n, \quad A_k \in R_+^{n\times n}, \quad k = 1, 2, \ldots, q, \quad B \in R_+^{n\times m}, \quad C \in R_+^{p\times n}, \quad D \in R_+^{p\times m}.$$

Definition 3.4. The fractional positive linear electrical circuit with delays (40) is called (strongly) observable in the interval $[0, t_f]$ if by knowing the input $u(t)$ and output $y(t)$ for $[0, t_f]$, it is possible to find the unique $x(0) \in R_+^n$ of the electrical circuit.

Theorem 3.5. The fractional positive linear electrical circuit with delays (40) is observable in the interval $[0, t_f]$ if and only if the matrix

$$W_\alpha = \int\limits_0^{t_f} \Phi_0^T(t)C^T C\Phi_0(t)\,\mathrm{d}t \in R_+^{n\times n} \tag{41}$$

is a monomial matrix.

Proof. Assuming $u(t) = 0$ then we can get

$$y(t) = C\Phi_0(t)x(0) + C\int_0^t \Phi(t-\tau)\sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau \in R_+^p. \tag{42}$$

Let

$$Y(t) = y(t) - C\int_0^t \Phi(t-\tau)\sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau \in R_+^p. \tag{43}$$

Using the value of $y(t)$ in $[0, t_f]$, by weighting, i.e. multiply $\Phi_0^T(t)C^T$ left on both sides of (42), then

$$\Phi_0^T(t)C^T C\Phi_0(t)x(0) = \Phi_0^T(t)C^T Y(t). \tag{44}$$

Integrating (44) on the interval $[0, t_f]$ we obtain

$$W_\alpha x(0) = \int_0^{t_f} \Phi_0^T(t)C^T Y(t)\,\mathrm{d}t \tag{45}$$

and

$$x(0) = W_\alpha^{-1}\int_0^{t_f} \Phi_0^T(t)C^T Y(t)\,\mathrm{d}t \in R_+^n \tag{46}$$

if and only if the matrix (41) is monomial [2].

Theorem 3.6. The fractional positive linear electrical circuit with delays (40) is observable if and only if the following $np \times n$ matrix is full rank.

$$Q_o = \begin{bmatrix} C \\ CA_0 \\ \vdots \\ CA_0^{n-1} \end{bmatrix} \quad \text{and} \quad \operatorname{rank} Q_o = n. \tag{47}$$

Proof. Let $u(t) = 0$, the solution of (40) is

$$x(t) = \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau)\sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau, \tag{48a}$$

$$y(t) = C\left[\Phi_0(t)x(0) + \int_0^t \Phi(t-\tau)\sum_{k=1}^{q} A_k x_0(\tau - w_k)\,\mathrm{d}\tau\right]. \tag{48b}$$

Using the well-known Cayley–Hamilton theorem we have

$$\Phi(\tau) = \sum_{i=0}^{n-1} \overline{a}_i(\tau)A^i$$

and by substituting it into (48b) we get

$$Y(t) = \sum_{i=0}^{n-1} \bar{a}_i(t) C A_0^i x(0) = [\bar{a}_0(t)\ \bar{a}_1(t) \cdots \bar{a}_{n-1}(t)] \begin{bmatrix} C \\ CA_0 \\ \vdots \\ CA_0^{n-1} \end{bmatrix} x(0). \tag{49}$$

Since $\overline{a}_i(t)$ is a known function, the initial state $x(0)$ can be uniquely determined according to $y(t)$ in the finite time $[0, t_f]$ if and only if the matrix Q_o is full rank.

Theorem 3.7. The fractional positive linear electrical circuit with delays (40) is observable in the interval $[0, t_f]$ if and only if the integer positive linear electrical circuit with delays (30) is observable in the same interval $[0, t_f]$.

Proof. From Theorems 3.3 and 3.6, we can find out that in the interval $[0, t_f]$ the fractional positive linear electrical circuit with delays (40) is observable if and only if the positive linear electrical circuit with delays (30) is observable.

Remark 3.1. The observability conditions based on checking methods shows that the positive linear electrical circuit with integer and fractional delays are invariant.

4. Example

We illustrate our conclusions by using the following example.

The fractional RLC circuit is shown in Fig. 1. It includes the fractional inductance L, capacitance C and C_1, which are the fractional α order, the resistances R_i, $i = 1, \ldots, 5$, source voltage $u(t) = E$, operational amplifier Y and delay element T. Denote $v(t)$ as voltage and $i(t)$ as current,

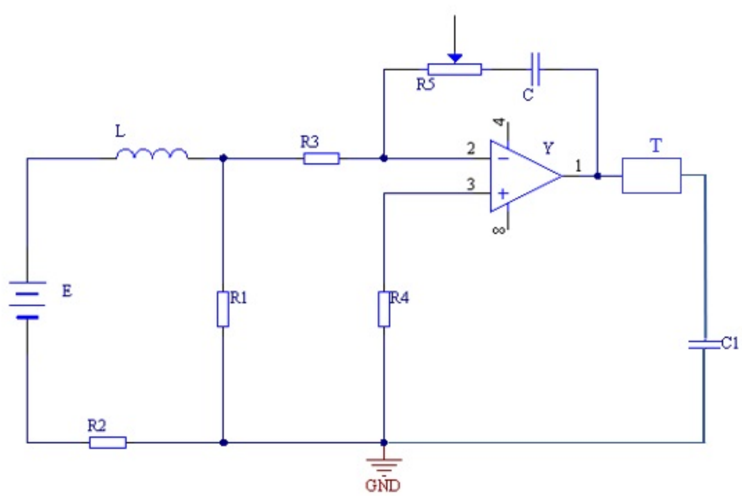

Fig. 1. The RLC circuit

where v_1 is the voltage of the resistance R_1. Let $x_1(t) = v_C(t)$, $x_2(t) = i_L(t)$ and $x_3(t) = v_{C_1}(t)$, $u(t)$ and $y(t)$ are input and output vectors, respectively. And σ is a parameter which is included to keep the equation dimensionally correct and $\sigma > 0$, see [7] for the details.

Now, $q = 1$, consider the circuit with a single delay shown in Fig. 1, and set its parameters as follows: $R_1 = 800\ \Omega$, $R_2 = 300\ \Omega$, $R_3 = 900\ \Omega$, $R_4 = 1\ \Omega$, $R_5 = 1000\ \Omega$, $L = 0.2$ H, $C = 3 \times 10^2$ µF, $C_1 = 6 \times 10^3$ µF, $w_k = 1.5$ s.

Using Kirchhoff's laws, we may write the equations:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x_1(t) = -\frac{\sigma^{1-\alpha}}{CR_5} x_1(t) + \frac{\sigma^{1-\alpha}\sqrt{2}}{2CR_5} x_1(t-1.5) + \frac{\sigma^{1-\alpha}}{CR_5} x_3(t), \tag{50a}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x_2(t) = -\frac{\sigma^{1-\alpha} R_1}{L} x_2(t) + \frac{\sigma^{1-\alpha} R_1}{L} x_2(t-1.5) + \frac{\sigma^{1-\alpha}}{L} u(t) + \frac{\sigma^{1-\alpha} R_1 v_1}{R_3 L}, \tag{50b}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x_3(t) = \frac{\sigma^{1-\alpha}}{C_1} x_2(t) + \frac{\sigma^{1-\alpha}}{C_1} x_2(t-1.5) - \frac{\sigma^{1-\alpha} v_1}{C_1 R_1} \tag{50c}$$

and choose

$$y(t) = x_1(t) + 0.5u(t). \tag{50d}$$

We assume, the circuit with a single delay has the following form:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} x(t) = A_{01} x(t) + A_{11} x(t-1.5) + Bu(t) + U, \tag{51a}$$

$$y(t) = Cx(t) + Du(t), \tag{51b}$$

where

$$A_{01} = \begin{bmatrix} -\dfrac{\sigma^1 - \alpha}{CR_5} & 0 & \dfrac{\sigma^1 - \alpha}{CR_5} \\ 0 & -\dfrac{\sigma^1 - \alpha R_1}{L} & 0 \\ 0 & \dfrac{\sigma^1 - \alpha}{C_1} & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} \dfrac{\sigma^1 - \alpha\sqrt{2}}{2CR_5} & 0 & 0 \\ 0 & \dfrac{\sigma^1 - \alpha R_1}{L} & 0 \\ 0 & \dfrac{\sigma^1 - \alpha}{C_1} & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ \dfrac{\sigma^1 - \alpha}{L} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, \quad D = [0.5], \quad U = \begin{bmatrix} 0 \\ \dfrac{\sigma^{1-\alpha} R_1 v_1}{R_3 L} \\ -\dfrac{\sigma^{1-\alpha} v_1}{C_1 R_1} \end{bmatrix}. \tag{52}$$

Let

$$\frac{\sigma^{1-\alpha} R_1 v_1}{R_3 L}, \quad -\frac{\sigma^{1-\alpha} v_1}{C_1 R_1}$$

be small disturbances, which are not considered here. The symbol T in (52) denotes transpose of the matrix.

From (52), we have

$$A_{01} = \begin{bmatrix} \frac{-3.33}{\sigma^{\alpha-1}} & 0 & \frac{3.33}{\sigma^{\alpha-1}} \\ 0 & \frac{-4}{\sigma^{\alpha-1}} & 0 \\ 0 & \frac{166.7}{\sigma^{\alpha-1}} & 0 \end{bmatrix}, \qquad A_{11} = \begin{bmatrix} \frac{2.36}{\sigma^{\alpha-1}} & 0 & 0 \\ 0 & \frac{4}{\sigma^{\alpha-1}} & 0 \\ 0 & \frac{166.7}{\sigma^{\alpha-1}} & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ \frac{5}{\sigma^{\alpha-1}} \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, \qquad D = [0.5]. \tag{53}$$

It can be seen from [15] that the positive fractional electrical circuit with delays (15) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in R_+^n$ satisfying the equality $A\lambda < 0$, where $A = \sum_{k=0}^{q} A_k$.

Since $A_{01} \in M_n$, $A_{11} \in R_+^{n\times n}$, $B \in R_+^{n\times m}$ and

$$A = A_{01} + A_{11} = \begin{bmatrix} \frac{-0.97}{\sigma^{\alpha-1}} & 0 & \frac{3.33}{\sigma^{\alpha-1}} \\ 0 & 0 & 0 \\ 0 & \frac{333.4}{\sigma^{\alpha-1}} & 0 \end{bmatrix},$$

there doesn't exist a strictly positive vector $\lambda \in R_+^n$ such that $A\lambda < 0$, then the fractional RLC circuit is not asymptotically stable.

According to (22),

$$Q_{0s} = A_{11} = \begin{bmatrix} \frac{2.36}{\sigma^{\alpha-1}} & 0 & 0 \\ 0 & \frac{4}{\sigma^{\alpha-1}} & 0 \\ 0 & \frac{166.7}{\sigma^{\alpha-1}} & 0 \end{bmatrix}, \qquad Q_{1s} = A_{01}A_{11} = \begin{bmatrix} \frac{-7.86}{\sigma^{2\alpha-2}} & \frac{555.1}{\sigma^{2\alpha-2}} & 0 \\ 0 & \frac{-16}{\sigma^{2\alpha-2}} & 0 \\ 0 & \frac{666.8}{\sigma^{2\alpha-2}} & 0 \end{bmatrix},$$

$$Q_m = \begin{bmatrix} B & A_{01}B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{5}{\sigma^{\alpha-1}} & \frac{-20}{\sigma^{2\alpha-2}} \\ 0 & \frac{833.5}{\sigma^{2\alpha-2}} \end{bmatrix}.$$

It shows that $\operatorname{rank} Q_{is} \neq \operatorname{rank} Q_m$ then from Theorem 2.6 it follows that the fractional RLC circuit is not reachable.

Also, from Theorem 3.6 it follows that

$$\operatorname{rank} Q_o = \operatorname{rank}\begin{bmatrix} C \\ CA_{01} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ \dfrac{-3.33}{\sigma^{\alpha-1}} & 0 & \dfrac{3.33}{\sigma^{\alpha-1}} \end{bmatrix} = 2. \tag{54}$$

The fractional RLC circuit is not observable.

5. Concluding remarks

The invariance of reachability and observability between the integer and fractional order positive linear electrical circuit with delays are investigated in this paper. The conclusion drawn from this paper shows that:

1. The checking methods have proven that the reachability of the positive linear electrical, circuit with integer and fractional delays is invariant.
2. The checking methods have proven that the observability of the positive linear electrical circuit with integer and fractional delays is invariant.
3. An example is analyzed at the end of this paper and it illustrates our conclusions.

Acknowledgement

This work was supported by the NSFC, grant 61503222.

References

[1] Dorf C.R., Svoboda A.J., *Introduction to electric circuits*, John Wiley & Sons (2010).

[2] Kaczorek T., Rogowski K., *Fractional linear systems and electrical circuit*, Springer (2015).

[3] Federico M., *Power system modelling and scripting*, Springer (2010).

[4] Kaczorek T., *Selected problems of fractional systems theory*, Springer (2011).

[5] Xin Z., Wenru L. *et al.*, *Application of fractional calculus in iterative sliding mode synchronization control*, Archives of Electrical Engineering, vol. 69, no. 3, pp. 499–519 (2020).

[6] Piotrowska E., *Analysis of linear continuous-time systems by the use of the conformable fractional calculus and Caputo*, Archives of Electrical Engineering, vol. 67, no. 3, pp. 629–639 (2018).

[7] Francisco G.A.J., Juan R.G., *Fractional RC and LC electrical circuits*, Ingeniera, Investigacin y Tecnologa, vol. 15, no. 2, pp. 311–319 (2014).

[8] Sikora R., *Fractional derivatives in electrical circuit theory-critical remarks*, Archives of Electrical Engineering, vol. 66, no. 1, pp. 155–163 (2017).

[9] Sikora R., Pawłowski S., *Problematic Applications of Fractional Derivatives in Electrotechnics and Electrodynamics*, Conference on Selected Issues of Electrical Engineering and Electronics, Szczecin, Poland, pp. 1–5 (2018).

[10] Sikora R., Pawłowski S., *Fractional derivatives and the laws of electrical engineering*, COMPEL-The international journal for computation and mathematics in electrical and electronic engineering, vol. 37, no. 4, pp. 1384–1391 (2018).

[11] Muthana T.A., Mohamed Z., *On the control of time delay power systems*, International Journal of Innovative Computing, Information and Control, vol. 9, no. 2, pp. 769–792 (2013).

[12] Zhaoyan L., Jun Q., *A simple method to compute delay margin of power system with single delay*, Automation of Electric Power System, vol. 32, no. 18, pp. 8–13 (2008).

[13] Jianjun Z., Yonggao Z., *Research on optimal configuration of fault current limiter based on reliability in large power network*, Archives of Electrical Engineering, vol. 69, no. 3, pp. 661–677 (2020).

[14] Kaczorek T., *Stability of positive continuous-time linear systems with delays*, Bulletin of The Polish Academy of Sciences-technical Sciences, vol. 57, no. 4, pp. 395–398 (2009).

[15] Kaczorek T., *Stability tests of positive fractional continuous-time linear systems with delays*, TransNav, the International Journal on Marine Navigation and Safety of Sea Transportation, vol. 7, no. 2, pp. 211–215 (2013).

[16] Xianming Z., Min W., *On delay-dependent stability for linear systems with delay*, Journal of Circuit and Systems, vol. 8, no. 3, pp. 118–120 (2003).

[17] Hai Z., Daiyong W., *Stability analysis for fractional-order linear singular delay differential systems*, Discrete Dynamics in Nature and Society, vol. 2014, no. 2014, pp. 1–8 (2014).

[18] Xianggeng Z., Yuxia L. *et al*., *Stability analysis of fractional-order Langford systems*, Journal of Shandong University of Science and Technology (Natural Science), vol. 38, no. 3, pp. 65–71(2019).

[19] Wei J., Zhicheng W., *Controllability of singular control systems with delay*, Journal of Hunan University, vol. 26, no. 4, pp. 6–9 (1999).

[20] Qiong W., Wei J., *The complete controllability, after all controllability, ultimate controllability and quasi controllability of delay control system*, College Mathematic, vol. 19, no. 3, pp. 63–66 (2003).

[21] Wei J., *The controllability of delay degenerate control systems with independent subsystems*, Applied Mathematics and Mechanics, vol. 24, no. 6, pp. 706–713 (2003).

[22] Peng L., Wenlong W. *et al*., *Alternate Charging and Discharging of Capacitor to Enhance the Electron Production of Bioelectrochemical Systems*, Environmental Science and Technology, vol. 45, no. 15, pp. 6647–6653 (2011).

[23] Kaczorek T., *Reachability and controllability to zero of positive fractional discrete-time systems*, European Control Conference, Kos, Greece, pp. 1708–1712 (2007).

[24] Xindong S., Hongli Y., *A new method for judgement computation of stability and stabilization of fractional order positive systems with constraints*, Journal of Shandong University of Science and Technology (Natural Science), vol. 40, no. 1, pp. 12–20 (2021).

[25] Kaczorek T., *Invariant properties of positive linear electrical circuit*, Archives of Electrical Engineering, vol. 68, no. 4, pp. 875–890 (2019).