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VECTOR PRODUCTS AS A TOOL FOR THE INVESTIGATION OF THE ISOMETRIC TRANSFORMATION OF OBJECTS

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Abstract

The identification of isometric displacements of studied objects with utilization of the vector product is the aim of the analysis conducted in this paper. Isometric transformations involve translation and rotation. The behaviour of distances between check points on the object in the first and second measurements is a necessary condition for the determination of such displacements. For every three check points about the measured coordinate, one can determine the vector orthogonal to the two neighbouring sides of the triangle that are treated as vectors, using the definition of the vector product in three-dimensional space. If vectors for these points in the first and second measurements are parallel to the studied object has not changed its position or experienced translation. If the termini of vectors formed from vector products treated as the vectors are orthogonal to certain axis, then the object has experienced rotation. The determination of planes symmetric to these vectors allows the axis of rotation of the object and the angle of rotation to be found. The changes of the value of the angle between the normal vectors obtained from the first and second measurements, by exclusion of the isometric transformation, are connected to the size of the changes of the coordinates of check points, that is, deformation of the object. This paper focuses mainly on the description of the procedure for determining the translation and rotation. The main attention was paid to the rotation, due to the new and unusual way in which it is determined. Mean errors of the determined parameters are often treated briefly, and this subject requires separate consideration.

Keywords. Isometric transformation, translation, rotation, axis and angle of rotation.

1. Introduction

Investigation of the deformations of technical objects is one of the main subjects of investigation in engineering geodesy. Many scientific works have been dedicated to this subject in the professional literature. As examples, one can cite the work of Nitschke and Knickmeyer (2000), in which the authors determined the parameters of the rotation of an object in three-dimensional space on the basis of general transformations of the space. Czaja (1990) described a test of the general formulation of the deformation of the object similarly. The main idea of the investigation of the deformation of objects consists in the establishment of check points on the studied object and determination of their coordinates in the first and second measurements. The changes of these coordinates allow the behaviour of the studied object in the interval of time between these measurements to be qualified. The method of investigation of the deformation of engineering objects with utilization of relationships between objects described in analytic geometry and utilization of the vector product was presented by Grabowski (2000, 2007). In some cases, these methods make it possible to broaden and enhance the interpretation of obtained results. Considerations are introduced in the present study for the further continuation of these works. They correct, to a considerable degree, specify, and broaden the understanding of this problem. Special attention will be paid to the identification of isometric transformations, with utilization of the vector product, including displacement and rotation.

2. Definition of the vector product

In mathematical analysis, the notion of a vector product $\vec{a}\times\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ is defined as the vector $\vec{c}$, whose length is $|\vec{a}\,||\vec{b}|\sin\varphi$, which equals the field of the parallelogram built on the vectors $\vec{a}$ and $\vec{b}$ that make up its sides. The vector $\vec{c}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$, where vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are right-hand oriented. If we express the vectors as rectangular coordinates $\vec{a}=[a_x,a_y,a_z]$, $\vec{b}=[b_x,b_y,b_z]$, $\vec{c}=[c_x,c_y,c_z]$, then the vector product of vectors $\vec{a}$ and $\vec{b}$ is found according to the formula

$$\vec{a}\times\vec{b}=\vec{c}=\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=\left(a_y b_z - a_z b_y\right)i+\left(a_z b_x - a_x b_z\right)j+\left(a_x b_y - a_y b_z\right)k \quad (1)$$

The vector product possesses the following properties:

- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$: when changing the order of the factors changes the sign of the vector product,
- $\alpha(\vec{a}\times\vec{b})=(\alpha\,\vec{a}\,)\times\vec{b}=\vec{a}\times(\alpha\,\vec{b})$: connectivity law is fulfilled by multiplying by a scalar vector product,
- $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$: the vector product does not conform to the associative law,
- $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$: the distributive law of multiplication is satisfied,
- $\vec{a}\times\vec{a}=\vec{0}$.

3. The idea of using the vector product as a method of investigating the object's deformation

Consider a solid body on which there are three check points whose coordinates are determined by geodetic methods. Let the coordinates of these points obtained in the first measuring epoch amount to $A^1=(x_A^1,y_A^1,z_A^1)$, $B^1=(x_B^1,y_B^1,z_B^1)$, and $C^1=(x_C^1,y_C^1,z_C^1)$. Creating vectors $\overrightarrow{(AB)}^1=[x_B^1-x_A^1,y_B^1-y_A^1,z_B^1-z_A^1]$ and $\overrightarrow{(AC)}^1=[x_C^1-x_A^1,y_C^1-y_A^1,z_C^1-z_A^1]$, we can calculate the vector product of these vectors according to Formula (1)

$$\overrightarrow{(AB)}^1\times\overrightarrow{(AC)}^1=\overrightarrow{N^1}=\begin{vmatrix} i & j & k \\ x_B^1-x_A^1 & y_B^1-y_A^1 & z_B^1-z_A^1 \\ x_C^1-x_A^1 & y_C^1-y_A^1 & z_C^1-z_A^1 \end{vmatrix} \tag{2}$$

which represents a normal vector at the point A^1, to the plane formed by points A^1,B^1, and C^1 in the first measuring epoch. If the coordinates are obtained in the second measuring epoch for the same points, $A=(x_A,y_A,z_A)$, $B=(x_B,y_B,z_B)$, and $C=(x_C,y_C,z_C)$, then by creating vectors $\overrightarrow{AB}=[x_B-x_A,y_B-y_A,z_B-z_A]$ and $\overrightarrow{AC}=[x_C-x_A,y_C-y_A,z_C-z_A]$, we can also calculate the vector product of these vectors $\overrightarrow{AB}\times\overrightarrow{AC}=\vec{N}$ using Formula (1).

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$$\overrightarrow{AB}\times\overrightarrow{AC}=\vec{N}=\begin{vmatrix} i & j & k \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix} \tag{3}$$

If the studied object experiences deformation between measuring epochs, then the vectors $\overrightarrow{N^1}$ and $\vec{N}$ will be created the angle between them, which is connected with the size of the deformation. We can found the size of this angle from the formula

$$\varphi=arc\ \cos\frac{\overrightarrow{N^1}\cdot\vec{N}}{\left|\overrightarrow{N^1}\right|\cdot\left|\vec{N}\right|} \tag{4}$$

If the angle φ is equal to zero, then points A^1,B^1, and C^1 do not change their positions or they all experience translation about the same vector (in particular, it is possible case that the points A^1,B^1,C^1,A,B,C lie on the same plane without translation). In the case of a lack of translation, the changes of coordinates of check points between the first and second measurements should not exceed their accuracy of determination. The basic condition that the studied object experiences only isometric transformations there is no changes in the distance between all check points in the first and second measurements, that is, $\overline{A_i^1A_j^1}=\overline{A_iA_j},\quad i\neq j,\quad i,j=1,2,...,n$.

3.1. Translation of the object

The object as a whole will experience only the translation, if there is an equal length of translation vectors for individual check points, $\sqrt{\left(x_i^1-x_i\right)^2+\left(y_i^1-y_i\right)^2+\left(z_i^1-z_i\right)^2}=\sqrt{\left(x_j^1-x_j\right)^2+\left(y_j^1-y_j\right)^2+\left(z_j^1-z_j\right)^2},\ i\neq j,\ i,j=1,2,...,n$,and have the same signs of the vector components. One can also determine of the translation using the vector product. It can be easily seen that if the object experiences only translation then the angle between vectors $\overrightarrow{N^1}$ and $\vec{N}$ will not change. It should be emphasized the possibility of becoming the unlikely event that no change in the angle between the vectors $\overrightarrow{N^1}$ and $\vec{N}$ if the points experienced rotation about an axis perpendicular to the plane of this triangle.

To determine this displacement, first of all it is necessary to found the differences of coordinate points A^1, B^1, C^1 and A, B, C. If the differences components have the same signs and not exceed the established accuracy of measurement, the coordinates of the translation vector to be determined as the arithmetic mean of the displacements of points, as follows

$$\vec{\Delta} = \left[\frac{x_A - x_A^1 + x_B - x_B^1 + x_C - x_C^1}{3},\ \frac{y_A - y_A^1 + y_B - y_B^1 + y_C - y_C^1}{3},\ \frac{z_A - z_A^1 + z_B - z_B^1 + z_C - z_C^1}{3} \right] \tag{5}$$

An additional test may be used to compare the lengths of the displacement vectors for the three studied points. Determination of the translation using the two methods will be presented for example 1. The mean error of the displacement vector coordinates can be determined using the known formulas for the root mean square error of the mean, for example, coordinate Δx, $m_{\Delta x} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$, where n is the number of studied points on the object, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Example 1. Consider three check points on the studied object for which the coordinates were determined in the first and second measuring epochs. The measured coordinates of these points are taken from Table 1.

Table 1. Coordinates of check points

Measurements							
First				Second			
points	x	y	z	points	x	y	z
A^1	1124,786	1456,397	1645,476	A	1124,885	1456,546	1645,675
B^1	1134,389	1471,645	1668,239	B	1134,488	1471,793	1668,435
C^1	1146,790	1445,765	1625,358	C	1146,892	1445,913	1625,555

The differences in coordinates between the first and second measuring epochs for the corresponding points are $\Delta x_A = 99mm$, $\Delta x_{\mathbf{B}} = 99mm$, $\Delta x_C = 102mm$, $\Delta y_A = 149mm$, $\Delta y_B = 148mm$ $\Delta y_C = 148mm$, $\Delta z_A = 199mm$, $\Delta z_B = 196mm$, and $\Delta z_C = 197mm$. If the accuracy of coordinates amount to $\pm 2mm$, then for the difference between coordinates it will be $\pm 2\sqrt{2} \approx \pm 3mm$. Because the calculated differences of coordinates do not exceed the abovementioned value, we can accept that the object experiences only the shift about the vector $\vec{\Delta} = [100{,}0;\ 148{,}3;\ 197{,}3]$ whose length is $\left|\vec{\Delta}\right| = 266{,}3mm$. Mean errors (a posteriori) of components of a translation vector amount to $m_{\Delta x} = \pm 1{,}71mm$, $m_{\Delta y} = \pm 0{,}58mm$, and $m_{\Delta z} = \pm 1{,}53mm$. The lengths of vector displacements for the three studied points are $d_A = 267{,}59\,mm$, $d_B = 264{,}80\,mm$, and $d_C = 266{,}68\,mm$. The length of the vector of the translation amounts to $\left|\vec{\Delta}\right| = \frac{d_A + d_B + d_C}{3} = 266{,}36mm$, and its mean error is $m_{\left|\vec{\Delta}\right|} = \pm 2{,}01mm$.

If we execute the calculations of vector products for the first and second measurements, then we will obtain

$$\overrightarrow{N^1} = \begin{vmatrix} i & j & k \\ 9{,}603 & 15{,}248 & 22{,}763 \\ 22{,}004 & -10{,}632 & -20{,}118 \end{vmatrix} = -64{,}743i + 694{,}070j - 437{,}616k$$

$$\vec{N} = \begin{vmatrix} i & j & k \\ 9{,}603 & 15{,}247 & 22{,}760 \\ 22{,}007 & -10{,}633 & -20{,}120 \end{vmatrix} = -64{,}763i + 694{,}092j - 437{,}649k$$

The angle between these vectors according to Formula (4) is $\varphi = \arccos\frac{677463{,}59}{823{,}0629 \cdot 823{,}1006} = \arccos 1{,}000000$. The obtained result $\varphi = 0^0$ confirms the parallelism of vectors $\overrightarrow{N^1}$ and $\vec{N}$, and also confirms the object's translation.

Deviation of the value of the angle φ from zero degrees can appear in practice as a result of errors of the coordinate points A, B, and C, even though the object has not experienced deformation. Here the question arises how to coordinate errors impact on value of the angle between the vectors. Given the complexity of the Formula (4), will be determined limits of these deviations. Taking into account the small value of the angle φ, we will accept, for the purpose of simplification, that the vectors $\vec{N} = a\,i + b\,j + c\,k$ and $\overrightarrow{N^1} = a^1 i + b^1 j + c^1 k$ possess the same coordinates, that is, $a = a^1,\ b = b^1, c = c^1$. The expression $\frac{\overrightarrow{N^1} \cdot \vec{N}}{\left|\overrightarrow{N^1}\right| \cdot \left|\vec{N}\right|}$ has a value close to one. The deviation from one will be largest if due to the adverse impact of the errors numerator will be as small as possible, and the denominator greatest. Let each of the vector coordinates $\vec{N} = a\,i + b\,j + c\,k$ be affected by the mistake δ. If all coordinates are positive, that is, $a > 0, b > 0, c > 0$, that's assuming unidirectional adverse impact of errors we get $\overrightarrow{N^1} \cdot \vec{N} = (a-\delta)^2 + (b-\delta)^2 + (c-\delta)^2 = a^2 + b^2 + c^2 - 2\,\delta\,(a+b+c) + 3\,\delta^2$, similarly the denominator will be the biggest, when

$\left|\overrightarrow{N^1}\right| \cdot \left|\vec{N}\right| = (a+\delta)^2 + (b+\delta)^2 + (c+\delta)^2 = a^2 + b^2 + c^2 + 2\,\delta\,(a+b+c) + 3\,\delta^2$.

Therefore the maximum deviation of the studied expression from one will happen when

$$\frac{\overrightarrow{N^1} \cdot \vec{N}}{\left|\overrightarrow{N^1}\right| \cdot \left|\vec{N}\right|} = \frac{a^2 + b^2 + c^2 - 2\,\delta\,(a+b+c) + 3\delta^2}{a^2 + b^2 + c^2 + 2\,\delta\,(a+b+c) + 3\delta^2} \approx \frac{a^2 + b^2 + c^2 - 2\,\delta\,(a+b+c)}{a^2 + b^2 + c^2 + 2\,\delta\,(a+b+c)} \tag{6}$$

The limit value of the deviation of an angle φ from zero is equal to

$$\Delta\varphi = \arccos\frac{a^2 + b^2 + c^2 - 2\,\delta\,(a+b+c)}{a^2 + b^2 + c^2 + 2\,\delta\,(a+b+c)} \tag{7}$$

If all vector coordinates are negative, that is, $a < 0, b < 0, c < 0$ then

$$\Delta\varphi = \arccos\frac{a^2 + b^2 + c^2 + 2\,\delta\,(a+b+c)}{a^2 + b^2 + c^2 - 2\,\delta\,(a+b+c)} \tag{8}$$

In other cases, different signs coordinates of the vector $\vec{N} = a\,i + b\,j + c\,k$ similar proceed. For example, if $a < 0, b > 0, c < 0$, then

$$\Delta\varphi = \arccos \frac{a^2 + b^2 + c^2 + 2\,\delta\,(a - b + c)}{a^2 + b^2 + c^2 - 2\,\delta\,(a - b + c)} \tag{9}$$

Example 2. For example 1, we will determine the value of the maximum deviation from zero of the angle φ between vectors. The vector $\vec{N} = -64{,}7i + 694{,}1j - 437{,}6k$. Let us accept that the coordinates of this vector are determined with the mistake $\delta = 3\,cm$. Using equation (9) we will obtain

$$\Delta\varphi = \arccos \frac{(-64{,}7)^2 + (694{,}1)^2 + (-437{,}6)^2 + 0{,}06\,(-64{,}7 - 694{,}1 - 437{,}6)}{(-64{,}7)^2 + (694{,}1)^2 + (-437{,}6)^2 - 0{,}06\,(-64{,}7 - 694{,}1 - 437{,}6)} = \arccos 0{,}999788$$

So $\Delta\varphi = 1{,}1795^0 = 1^0 10' 35''$. It should be underlined that this is the limit value whose probability of occurrence is very small. Crossing of this value unambiguously shows deformation of the studied object; it cannot be justified by adverse coordinate system errors.

If it is hard to estimate the average mistake of the coordinate vector $\vec{N} = a\,i + b\,j + c\,k$ then one can use the Gauss formula on the average error of a function for the realization of this task. Using the Gauss formula for the determination of the average mistake of coordinates described by Formula (3), we will obtain

$$m_a^2 = \left[3\,(y_A^2 + y_B^2 + y_C^2) - (y_A + y_B + y_C)^2 + 3\,(z_A^2 + z_B^2 + z_C^2) - (z_A + z_B + z_C)^2\right] m^2 \tag{10}$$

$$m_b^2 = \left[3\,(x_A^2 + x_B^2 + x_C^2) - (x_A + x_B + x_C)^2 + 3\,(z_A^2 + z_B^2 + z_C^2) - (z_A + z_B + z_C)^2\right] m^2 \tag{11}$$

$$m_c^2 = \left[3\,(x_A^2 + x_B^2 + x_C^2) - (x_A + x_B + x_C)^2 + 3\,(y_A^2 + y_B^2 + y_C^2) - (y_A + y_B + y_C)^2\right] m^2 \tag{12}$$

where m means the average mistake of the determination of coordinate check points.

3.2. The rotation of the object

If the studied object has only rotation relative to an axis, the rotation angles controlled points with respect to said axis between a first and second measurements are equal. In order to determine equation of the axis of rotation in a spatial coordinate system, consider the triangle A^1, B^1, C^1 in the first measurement that in the second measurement, after rotating through an angle of α, holds the position of A, B, C. Checkpoints have equal distance from the axis of rotation in the first and second measurement. As a result, rotation through angle α does not change the length of the vector formed by the vector product, that is, $\left|\vec{N^1}\right| = \left|\vec{N}\right|$. Therefore the terminus of these vectors will outline a circular arc about the central angle in the plane orthogonal to the axis of the rotation. The vector joining these points is perpendicular to the axis of the rotation. This allows the identification of the rotation of the object around the axis of rotation. Let us denote the vector calculated from Formula (2) as $\overrightarrow{N_A^1}$, while that calculated by Formula (3) is denoted as $\overrightarrow{N_A}$. The termini of these vectors have the coordinates

$$K_{A^1} = (x^K_{A^1}, y^K_{A^1}, z^K_{A^1}) = \Big(x^1_A + (y^1_B - y^1_A)\,(z^1_C - z^1_A) - (y^1_C - y^1_A)\,(z^1_B - z^1_A)\,,$$
$$, y^1_A + (x^1_C - x^1_A)\,(z^1_B - z^1_A) - (x^1_B - x^1_A)\,(z^1_C - z^1_A),\quad z^1_A + (x^1_B - x^1_A)\,(y^1_C - y^1_A) - (x^1_C - x^1_A)\,(y^1_B - y^1_A)\Big) =$$
$$= (x^1_A + a_{A^1},\ y^1_A + b_{A^1},\ z^1_A + c_{A^1}) \tag{14}$$

$$K_A = (x^K_A, y^K_A, z^K_A) = \Big(x_A + (y_B - y_A)(z_C - z_A) - (y_C - y_A)(z_B - z_A),$$
$$, y_A + (x_C - x_A)(z_B - z_A) - (x_B - x_A)(z_C - z_A)],\quad z_A + (x_B - x_A)(y_C - y_A) - (x_C - x_A)(y_B - y_A)\Big) =$$
$$= (x_A + a_A, y_A + b_A, z_A + c_A) \tag{15}$$

Let us consider the segment $\left|\overrightarrow{K_{A^1}K_A}\right|$. The centre of this segment possesses coordinates $S_A = (x^A_S, y^A_S, z^A_S) = (\frac{x^K_{A^1} + x^K_A}{2}, \frac{y^K_{A^1} + y^K_A}{2}, \frac{z^K_{A^1} + z^K_A}{2})$. The axis of rotation is in the plane normal to the section $\left|\overrightarrow{K_{A^1}K_A}\right|$ and go through the point S_A. Considering the orthogonality of the vector $\overrightarrow{K_{A^1}K_A} = [x^K_A - x^K_{A^1} \quad y^K_A - y^K_{A^1} \quad z^K_A - z^K_{A^1}]$ and any vector lying on this plane $\overrightarrow{PS_A} = [x - x^A_S, y - y^A_S, z - z^A_S]$, their scalar product is equal to zero. The equation of the plane $\pi_{K_{A^1}K_A}$ orthogonal to the section $\left|\overrightarrow{K_{A^1}K_A}\right|$ and passes through its center has the form $(\overrightarrow{PS_A} \cdot \overrightarrow{K_{A^1}K_A} = 0)$,

$$(x_A - x^1_A + a_A - a_{A^1})\,x + (y_A - y^1_A + b_A - b_{A^1})\,y + (z_A - z^1_A + c_A - c_{A^1})\,z + \frac{1}{2}[(x^1_A + a_{A^1})^2 + (y^1_A + b_{A^1})^2 + (z^1_A + c_{A^1})^2 - (x_A + a_A)^2 - (y_A + b_A)^2 - (z_A + c_A)^2] = 0 \tag{16}$$

By creating vector products for the considered triangle at the beginning in the points B^1 and B, we will find the equation of the plane $\pi_{K_{B^1}K_B}$ orthogonal to the section $\left|\overrightarrow{K_{B^1}K_B}\right|$ and crossed by its centre S_B.

$$(x_B - x^1_B + a_B - a_{B^1})\,x + (y_B - y^1_B + b_B - b_{B^1})\,y + (z_B - z^1_B + c_B - c_{B^1})\,z + \frac{1}{2}[(x^1_B + a_{B^1})^2 + (y^1_B + b_{B^1})^2 + (z^1_B + c_{B^1})^2 - (x_B + a_B)^2 - (y_B + b_B)^2 - (z_B + c_B)^2] = 0 \tag{17}$$

Equations A and B taken together determine the equation of the axis of rotation in the form of an edge (resulting from the intersection of the two planes), when it is certain that the rotation of the object on which lie the three control points was. The confirmation of this fact will be to check that this axis orthogonal to the plane belongs to the segment AB and passing through its center.

Equations (16) and (17), considered together, define the equation of the axis of rotation in the form of an edge (resulting from the intersection of the two planes), when it is certain that the rotation of the object on which lie the three control points was. The confirmation of this fact will be to check that this axis belongs to the plane orthogonal to the segment $\left|\overrightarrow{K_{C^1}K_C}\right|$ and crossed by its centre S_C, that is, orthogonal to the plane $\pi_{K_{C^1}K_C}$:

$$(x_C - x^1_C + a_C - a_{C^1})\,x + (y_C - y^1_C + b_C - b_{C^1})\,y + (z_C - z^1_C + c_C - c_{C^1})\,z + \frac{1}{2}[(x^1_C + a_{C^1})^2 + (y^1_C + b_{C^1})^2 + (z^1_C + c_{C^1})^2 - (x_C + a_C)^2 - (y_C + b_C)^2 - (z_C + c_C)^2] = 0 \tag{18}$$

By considering the axes of the rotation found from the pairs of equations {(16), (17)}, {(16), (18)}, and {(17), (18)}, we can infer about the behavior of the object. If the axes ideally covered or their position is similar (deviations caused by measurement errors are acceptable), we can accept that the rotation of the object happened. The large skew of these straight lines can also testify to the various changes of their positions.

One can use the fact that coefficients of the unknowns in the equation of the plane represent the coordinate vector orthogonal to the plane. Therefore by creating vector products for all the above mentioned three pairs of planes we will find three vectors. If they are collinear, this will be a confirmation of the rotation object whose axis will be parallel to these vectors.

Let us denote by $\vec{v} = [a,b,c]$ the vector of the axis of rotation. By treating the displacements of individual check points as vectors orthogonal to the axis of rotation, we can for the three test points calculate the coordinates of the vector $\vec{v} = [a,b,c]$ from the system of equations, as follows

$$\begin{aligned}(x_A - x_A^1)\ a + (y_A - y_A^1)\ b + (z_A - z_A^1)\ c = 0\\ (x_B - x_B^1)\ a + (y_B - y_B^1)\ b + (z_B - z_B^1)\ c = 0\\ (x_C - x_C^1)\ a + (y_C - y_C^1)\ b + (z_C - z_C^1)\ c = 0\end{aligned} \tag{19}$$

The abovementioned arrangement of homogeneous equations possesses non-zero solutions if

$$\begin{vmatrix}(x_A - x_A^1) & (y_A - y_A^1) & (z_A - z_A^1)\\ (x_B - x_B^1) & (y_B - y_B^1) & (z_B - z_B^1)\\ (x_C - x_C^1) & (y_C - y_C^1) & (z_C - z_C^1)\end{vmatrix} = 0 \tag{20}$$

is the basic condition of the rotation around three axis check points. Because the marked vector is free, this can be one of the coordinates chosen arbitrarily and two selected equations (the main determinant should be different from zero) determine the remaining two coordinates. For example, let $c = c_0$; then

$$a = -\frac{\begin{vmatrix}(z_A - z_{A^1})\,c_0 & y_A - y_{A^1}\\ (z_B - z_{B^1})\,c_0 & y_B - y_{B^1}\end{vmatrix}}{\begin{vmatrix}x_A - x_{A^1} & y_A - y_{A^1}\\ x_B - x_{B^1} & y_B - y_{B^1}\end{vmatrix}} \qquad b = -\frac{\begin{vmatrix}x_A - x_{A^1} & (z_A - z_{A^1})\,c_0\\ x_B - x_{B^1} & (z_B - z_{B^1})\,c_0\end{vmatrix}}{\begin{vmatrix}x_A - x_{A^1} & y_A - y_{A^1}\\ x_B - x_{B^1} & y_B - y_{B^1}\end{vmatrix}} \tag{21}$$

If the vectors $\vec{v}_i$ for the chosen triples of check points on the studied objects calculated by Formula (19) are parallel to one another this confirms the rotation of the object. The range of disturbances of parallelism vectors resulting from the inaccuracy of determination of coordinate check points requires separate consideration. To determine the equation of the axis of rotation should be considered pair of planes, for example, ($\pi_{K_{A^1}K_A}$, $\pi_{K_{B^1}K_B}$). They mark the edge equation of the rotation axis. To write a parametric equation of the axis of rotation, more convenient to use, should be determined the coordinates of any point $P_0 = (x_0, y_0, z_0)$ lying on this axis. For this purpose, one of the coordinate pick arbitrarily, we will find the remaining two coordinates from the solution of the edge equation. We will illustrate the abovementioned considerations using a numerical example.

To save the parametric, more convenient to use, should be determined the coordinates of any point.

Example 3. We will conduct an investigation of the rotation of a spatial object on which the coordinates of three check points A^1, B^1, and C^1 were measured in the first measurement and the coordinates of these same points A, B and C were measured in the second measurement. The measured coordinates of these points in the two measuring epochs are listed in Table 2.

Table 2. List of coordinates of check points

Measurements							
First				Second			
Points	x	y	z	Points	x	y	z
A^1	92,0000	102,0000	112,0000	A	90,0000	102,0000	110,0000
B^1	95,0378	108,9838	185,9784	B	100,9622	107,0162	183,2216
C^1	92,4964	60,7396	199,7640	C	95,5036	59,2604	197,8360

According to Formula (2), the vector normal to the plane of the triangle $A^1B^1C^1$ (the central point is A^1) has the form

$$\vec{N}_{B^1A^1C^1} = \begin{vmatrix} i & j & k \\ 3,0378 & 6,9838 & 73,9784 \\ 0,4964 & -41,2604 & 87,7640 \end{vmatrix} = 3665,3046i - 229,8866j - 128,8076k = a_{A^1}i + b_{A^1}j + c_{A^1}k$$

Similarly, a vector normal to the plane of the triangle ABC (the central point is A) is:

$$\vec{N}_{BAC} = \begin{vmatrix} i & j & k \\ 10,9622 & 7,0162 & 73,2216 \\ 5,5036 & -40,7396 & 87,8360 \end{vmatrix} = 3599,2936i - 559,8934j - 485,2100k = a_A i + b_A j + c_A k$$

According to Formulas (14) and (15), the coordinates of the termini of the abovementioned vectors will be $K_{A^1} = (3757,3046;\ -127,8866;\ -16,8076)$ and $K_A = (3689,2936;\ -459,8334;\ -375,2100)$; meanwhile the coordinates of the centre of the segment $\left|\overrightarrow{K_{A^1}K_A}\right|$ are $S_A = (x_S^A, y_S^A, z_S^A) = (3723,2991;\ -293,8600;\ -196,0088)$. The equation plane $\pi_{K_{A^1}K_A}$ orthogonal to the segment $\left|\overrightarrow{K_{A^1}K_A}\right|$ and passing through its center S_A in accordance with formula (16) has the form

$$-68,0110\,x - 332,0068\,y - 358,4024\,z + 85401,7923 = 0$$

Similarly we will obtain the equation of the plane $\pi_{K_{B^1}K_B}$ orthogonal to the segment $\left|\overrightarrow{K_{B^1}K_B}\right|$ and crossed by its centre S_B using the fact that the vector $\vec{N}_{A^1B^1C^1}$ is identical to the vector $\vec{N}_{B^1A^1C^1}$, and similarly $\vec{N}_{ABC} \equiv \vec{N}_{BAC}$. Using Formula (17) we will obtain

$$-60,0866\,x - 331,9744\,y - 359,1592\,z + 84936,6076 = 0$$

The equation of the plane $\pi_{K_{C^1}K_C}$ orthogonal to the segment $\left|\overrightarrow{K_{C^1}K_C}\right|$ and crossed by its centre S_C will be determined similarly using Formula (18), considering that $\vec{N}_{A^1B^1C^1} \equiv \vec{N}_{A^1C^1B^1}$ and $\vec{N}_{ABC} \equiv \vec{N}_{ACB}$:

$$-63,0038\,x - 331,4860\,y - 358,3304\,z + 84985,1541 = 0$$

To established the existence of the rotation, we utilize the fact that coefficients in the equation of the plane represent the coordinates of the vector orthogonal to the plane. Therefore by creating vector products for the three listed above pairs of planes ($\pi_{K_{A^1}K_A}, \pi_{K_{B^1}K_B}$), ($\pi_{K_{A^1}K_A}, \pi_{K_{C^1}K_C}$), and ($\pi_{K_{C^1}K_C}, \pi_{K_{B^1}K_B}$), we will obtain three vectors:

$$\vec{N}_{K_{A^1}K_A, K_{B^1}K_B} = \begin{vmatrix} i & j & k \\ -68{,}0110 & -332{,}0068 & -358{,}4024 \\ -60{,}0866 & -331{,}9744 & -359{,}1592 \end{vmatrix} = 262{,}8750\,i - 2891{,}5947\,j + 2628{,}7511\,k$$

$$\vec{N}_{K_{A^1}K_A, K_{C^1}K_C} = \begin{vmatrix} i & j & k \\ -68{,}0110 & -332{,}0068 & -358{,}4024 \\ -63{,}0038 & -331{,}4860 & -358{,}3304 \end{vmatrix} = 162{,}7515\,i - 1789.6957\,j + 1627{,}0043\,k$$

$$\vec{N}_{K_{C^1}K_C, K_{B^1}K_B} = \begin{vmatrix} i & j & k \\ -63{,}0038 & -331{,}4860 & -358{,}3304 \\ -60{,}0866 & -331{,}9744 & -359{,}1592 \end{vmatrix} = 99{,}7270\,i - 1097{,}5390\,j + 997{,}7820\,k$$

Because the abovementioned vectors $\vec{N}_{K_{A^1}K_A, K_{B^1}K_B} = 1{,}61549\,\vec{N}_{K_{A^1}K_A, K_{C^1}K_C}$ and $\vec{N}_{K_{A^1}K_A, K_{C^1}K_C} = 1{,}63108\,\vec{N}_{K_{C^1}K_C, K_{B^1}K_B}$ are collinear, the object has experienced the rotation. The abovementioned vectors are parallel to the axis of the rotation. Vector parallel to the axis of rotation is much easier to determine using the Formula (21), for example assuming $c = c_0 = 100$, we obtain $a = 10$ and $.b = -110$ Therefore the vector collinear to the axis of the rotation has the form $\vec{v} = 10\,i - 110\,j + 100\,k$. It is easy to check that it is parallel to the three vectors obtained from the vector products above. Because the point $P_0 = (x_0, y_0, z_0) = (70,\ 110,\ 123)$ lies on the axis of the rotation, the parametric equation of the axis of the rotation of the object has the form $x = 70 + 10t,\ y = 110 - 110t,\ z = 123 + 100t$. The distances of the test points to the rotation axis are $d_{A^1} = d_A = 25{,}8650m$, $d_{B^1} = d_B = 50{,}9529m$, $d_{C^1} = d_C = 28{,}8683m$. The angle of the rotation amounts to $\alpha = 7^0 40'46''$.

For n surveyed control points on the object, the value of the mean error of the angle of rotation can be calculated from the formula $m_\alpha = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\alpha_i - \overline{\alpha})^2}$, where $\overline{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\alpha_i$. To determine the accuracy define the axis of rotation (if necessary) requires separate consideration of the problem, that is, determination of the measure describing discrepancy the skew lines.

4. Investigation of the object with utilization of the vector product

We usually found a certain number n of check points on the studied object, depending on the shape and type of the object, the expected deformation, and so on. We can distinguish the following cases:

1. The object does not experience any displacement or deformation. The inadmissible changes of the coordinates of check points do not occur and therefore the distance between them is also equal to zero in two measuring epochs.
2. The object as a whole experiences only translation. The products vector created for each combination of three points from n points in the first and second

measurement give vectors parallel. In the analytic investigation, when only translation occurs, the lengths of the vectors of the translation for individual check points are equal.

$$\sqrt{\left(x_i^1 - x_i\right)^2 + \left(y_i^1 - y_i\right)^2 + \left(z_i^1 - z_i\right)^2} = \sqrt{\left(x_j^1 - x_j\right)^2 + \left(y_j^1 - y_j\right)^2 + \left(z_j^1 - z_j\right)^2},\ i \neq j,\ i, j = 1,2,\dots,n$$

and the components these vectors have the same signs. It should be underlined that all distances among points controlled in the first measurement will keep their lengths in the second measurement ($\overline{A_i^1 A_j^1} = \overline{A_i A_j},\quad i \neq j,\quad i, j = 1,2,\dots,n$).

3. The object as a whole experiences only rotation. Determine an occurrence rotation and the setting of the equation of rotation axes must be carried out to reasoning in Section 4. If we plot appointed vectors for every three check points, then we will obtain the vector field of parallel vectors.
4. The object experiences translation and rotation simultaneously. If the changes coordinates of controlled points not support a finding of translation or rotation separately, and all distances between check points in the first measurement retain their length in the second measurement ($\overline{A_i^1 A_j^1} = \overline{A_i A_j},\quad i \neq j,\quad i, j = 1,2,\dots,n$), then it can be concluded that translation and rotation occurred simultaneously. The singling out of the components vector of the translation requires separate considerations.
5. If the vector field possesses the various directions of vectors, by exclusion of isometric transformations, then various angles between these vectors formed for the chosen triples of check points can indicate the size of deformations. Moreover, various angles between vectors created from vector products calculated according to Formula (4) will also have various values. The angles are connected with the size of the changes of coordinates of check points, that is, deformation of the object.

5. Conclusion and final remarks

Vector products and the considerations made in the present paper can be helpful for preliminary and sometimes supplementary identification of displacements and deformations of an object. If, for all chosen triples of check points, normal vectors are parallel in the first and second measurements, it suggests that the studied points have not altered their positions or experienced translation. The analyses of coordinate check points performed here (e.g. the behavior of parallel directions of displacement vectors and equal their length) are helpful. If the termini of the normal vectors of the chosen triples of points orthogonal to a certain axis are treated as vectors, then the rotation of the object occurs in relation to an axis that is perpendicular to these directions. If the test does not confirm the presence of translation or rotation individually or in the aggregate, are different angles between vectors arising from the vector products of selected groups of three points for measuring first and second, will point to different deformations of the whole object or its fragments

Determining the vector products of the chosen triples of points and their comparison between the first and second measurements has an auxiliary investigative character and can sometimes deliver additional information about the behaviour of the object itself; for example, significant changes of the coordinates of the three points, which give no reason to determine of the translation, in the case of preserve parallelism vectors $\overline{N^1}$ and $\vec{N}$ confirm the change of position of points A, B, and C in the plane of triangle A^1, B^1, C^1 of the first measurement. It is obvious that for the comparisons of vector products for the chosen triples of points in the first and second measurements, we should choose the characteristic triples of points. This depends on the researcher's ingeniousness and experience. Identification of displacements and the deformations of engineering objects is not an easy task, but is continually current, although many works have been dedicated to this subject in the professional literature. Every new look at this problem delivers new, sometimes partial information useful in the identification of displacements and deformations of the tested objects. If in the given investigation it is possible to utilize various methods, then it is worth using them. The convergence of results makes it possible to be certain of the proper interpretation and gives a wider and deeper understanding of the solved problem.

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