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ANALYTICAL AND NUMERICAL SOLUTIONS OF NOT EXACT DIFFERENTIAL EQUATIONS WITH INTERPRETATION IN *MATHEMATICA* PROGRAM

Abstract

Introduction and aim: This paper shows the analytical and numerical solutions of some not exact differential equations. Some short description of a search procedure for integral factor in all three cases has been shown in the considerations. The main aim of this paper is to use *Mathematica* program to solve the not exact differential equations.
Material and methods: In the paper have been analyzed exact differential equation and four not exact differential equations. In order to solve not exact differential equations and create some graphs of obtained solutions has been applied *Mathematica* program. Analytical and numerical methods have been used in the paper.
Results: In the case of integrating factor which dependent on two variables has been shown the way of its searching by using some *expectation method*. In particular case, when integrating factor has form $\mu(x,y)=x^a y^b$ the quantities a and b we can find by solving a system of two linear equations with unknown values a and b.
Conclusion: Program *Mathematica* allows us to look, for more difficult cases, some integrating factor dependent on two variables x and y by using a expectation method.

Keywords: Not exact differential equation, integrating factor, general and particular integral, analytical and numerical method, graphical interpretation of solution, *Mathematica.*
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ANALITYCZNO-NUMERYCZNE ROZWIĄZANIE RÓWNAŃ RÓŻNICZKOWYCH NIEZUPEŁNYCH Z INTERPRETACJĄ W PROGRAMIE MATHEMATICA

Streszczenie

Wstęp i cele: *W pracy pokazano rozwiązania analityczne dla równań różniczkowych niezupełnych. Przestawiono krótki opis procedury szukania czynnika całkującego we wszystkich trzech przypadkach. Głównych celem pracy jest zastosowanie programu Mathematica do rozwiązywania równań różniczkowych niezupełnych.*
Materiał i metody: *Zanalizowano równanie różniczkowe zupełne oraz cztery równania różniczkowe niezupełne. W celu wykonania wykresów otrzymanych rozwiązań szczególnych zastosowano program numeryczny Mathematica. W pracy zastosowano zarówno metodę analityczną jak i numeryczną.*
Wyniki: *W przypadku czynnika całkującego zależnego od dwóch zmiennych pokazano sposób jego wyznaczania stosując metodę przewidywań. W szczególności gdy czynnik całkujący ma postać* $\mu(x,y)=x^a y^b$ *wykładniki a oraz b znajduje się rozwiązując układ dwóch równań linowych o zmiennych a i b.*
Wnioski: Program *Mathematica* pozwala na analizę, dla bardziej trudniejszych przypadków, czynnika całkującego zależnego od dwóch zmiennych x oraz y stosując metodę przewidywań.

Słowa kluczowe: *Równanie różniczkowe niezupełne, czynnik całkujący, całka ogólna i szczególna, metoda analityczna i numeryczna, interpretacja graficzna rozwiązania, Mathematica.*
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1. Exact differential equation

A differential equation of the form [4]-[8], [10], [11]:

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

is said to be *a exact differential equation* if its left-hand side is exact differential of some function u = u(x, y), that is

$$M(x,y)dx + N(x,y)dy \equiv du \equiv \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{2}$$

Theorem 1.

For equation (1) to be exact differential equation it is necessary and sufficient that the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{3}$$

should be hold in some range D of the variables x and y.

The general integral of equation (1) is of the form [4]-[8], [10], [11]:

$$u(x,y) = C \tag{4}$$

or

$$\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x,y)dy = C. \tag{5}$$

Example 1.

Let us consider the following exact differential equation [1], [3], [4], [10]:

$$(2x\cos^2 y)\,dx + (8\sqrt[3]{y} - x^2 \sin 2y)\,dy = 0, \tag{6}$$

where

$$M_0(x,y) = 2x\cos^2 y, \tag{7}$$

$$N_0(x,y) = 8\sqrt[3]{y} - x^2 \sin 2y. \tag{8}$$

• Analytical solution

Let us verify if this differential equation is exact equation:

$$\frac{\partial M_0}{\partial y} = \frac{\partial}{\partial y}(2x\cos^2 y) = -2x \cdot 2\cos y \cdot \sin y = -2x\sin 2y, \tag{9}$$

$$\frac{\partial N_0}{\partial x} = \frac{\partial}{\partial x}(8\sqrt[3]{y} - x^2 \sin 2y) = -2x\sin 2y. \tag{10}$$

So that

$$\frac{\partial M_0}{\partial y} = \frac{\partial N_0}{\partial x}, \tag{11}$$

i.e. condition (3) is fulfilled. Thus the given equation (6) is exact differential equation. So exists a function u(x, y) that

$$\frac{\partial u}{\partial x} = 2x\cos^2 y, \qquad (12)$$

$$\frac{\partial u}{\partial y} = 8\sqrt[3]{y} - x^2 \sin 2y. \qquad (13)$$

Form the equations (12)-(13) we find the function u(x, y).

The first, let us integrate the equation (12) with respect to x, where y is constant:

$$u(x,y) = \int (2x\cos^2 y)dx = 2\cos^2 y \int (x)dx = x^2\cos^2 y + \varphi(y) \qquad (14)$$

where φ(y) is some differentiable function.

Now let us differentiate the equation (14) with respect to y, where x is constant:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2\cos^2 y + \varphi(y)\right] = -x^2 \sin 2y + \frac{d\varphi}{dy}. \qquad (15)$$

Next let us compare the equations (13) and (15), thus we obtain:

$$8\sqrt[3]{y} - x^2\sin 2y = -x^2 \sin 2y + \frac{d\varphi}{dy}, \qquad (16)$$

$$\frac{d\varphi}{dy} = 8\sqrt[3]{y}. \qquad (17)$$

So that

$$\varphi(y) = 6y\sqrt[3]{y} + C, \quad C \in \Re. \qquad (18)$$

Thus the general solution of exact differential equation (6) has the following form:

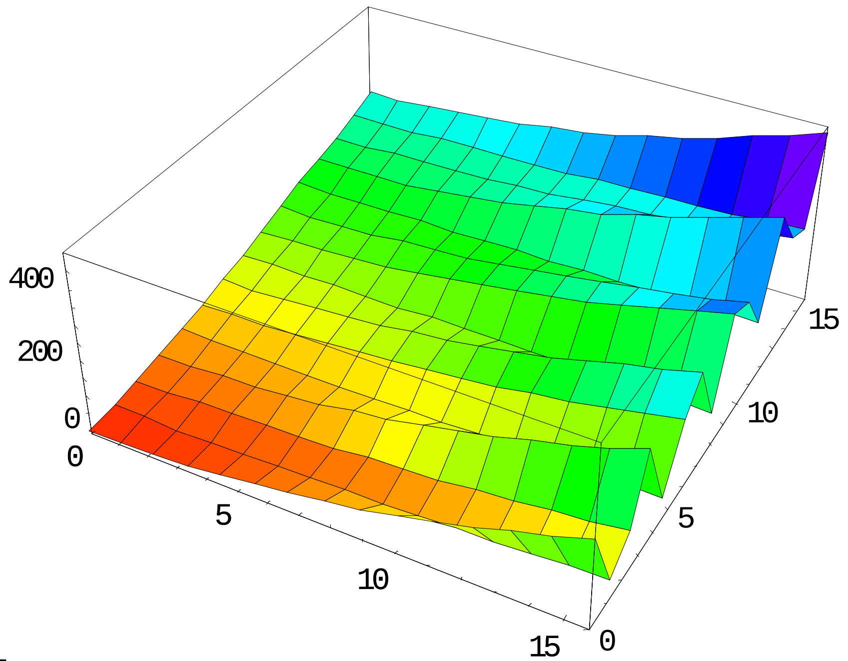
$$u(x,y) = x^2\cos^2 y + 6y\sqrt[3]{y} + C, \quad C \in \Re. \qquad (19)$$

• **Numerical solution**

Let us make a procedure 1 for the equation (6) and graph of function (19) for C=0 [12].

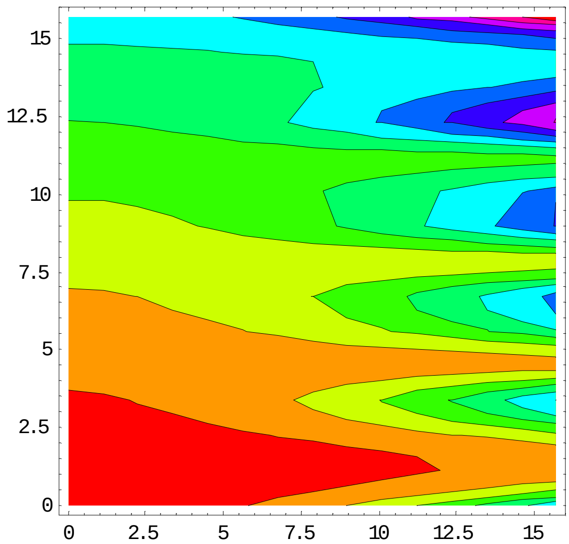
Program 1 *(Mathematica 4.1)*	Commentary
In[1]:= `M0[x_,y_]=2x*(Cos[y])^2`	*Function $M_0(x,y)$*
`N0[x_,y_]=8*y^(1/3)-(x^2)*Sin[2*y]`	*Function $N_0(x,y)$*
`D[M0[x,y],y]=D[N0[x,y],x]//Simplify`	*Checking derivatives*
`step1=Integrate[M0[x,y],x]`	*Integrating of function M_0*
`step2=D[step1+ϕ[y],y]`	*Differentiation*
`step3=Solve[step2==N0[x,y]+ϕ'[y]]//Simplify`	*Determining ϕ'[y] function*
`step4=Integrate[ϕ'[y]/.step3[[1]],y]`	*Integrating of function ϕ'[y]*
`solution=step1+step4+C`	*Determining the solution*
Out[1]= $2x\text{Cos}[y]^2$	*Function $M_0(x,y)$*
Out[2]= $8y^{1/3} - x^2\text{Sin}[2y]$	*Function $N_0(x,y)$*
Out[3]= True	*Checking if* $M0'_y = N0'_x$
Out[4]= $x^2\text{Cos}[y]^2$	*Result of integration M_0*
Out[5]= $-2x^2\text{Cos}[y]\text{Sin}[y] + \phi'[y]$	*Result of differentiation*
Out[6]= $\{\{\phi'[y] \to 8y^{1/3}\}\}$	*Obtained function ϕ'[y]*

`Out[7]=` $6y^{4/3}$	*Obtained function* $\phi[y]$
`Out[8]=` $C+6y^{4/3}+x^2 Cos[y]^2$	*General solution of equation (6)*
`Plot3D[solution/.C->0,{x,0,5Pi},{y,0,5Pi}, ColorFunction → Hue]`	*Creating* `Plot3D`
`ContourPlot[solution/.C->0,{x,0,5Pi}, {y,0,5Pi},ColorFunction → Hue]`	*Creating* `ContourPlot`
`DensityPlot[solution/.C->0,{x,0,5Pi}, {y,0,5Pi},ColorFunction → Hue]`	*Creating* `DensityPlot`



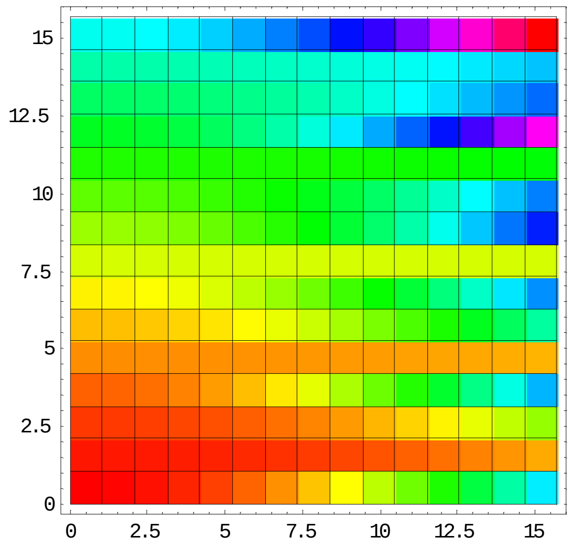
`Out[9]=`

Fig. 1a. `Plot3D` of solution (19) for C=0 *Mathematica (program 1)*
Source: Elaboration by the Authors



`Out[10]=`

Fig. 1b. `ContourPlot` of solution (19) for C=0 *Mathematica (program 1)*
Source: Elaboration by the Authors



`Out[11]=`

Fig. 1c. `DesnityPlot` of solution (19) for C=0 *Mathematica (program 1)*
Source: Elaboration by the Authors

2. Not exact differential equation and integrating factor

In some cases, when equation (1) is not exact differential equation, it may be possible to select a function $\mu = \mu(x, y)$ after multiplying by which the left-hand side of (1) turns into a total differential:

$$du = \mu M\,dx + \mu N\,dy. \tag{20}$$

This function $\mu = \mu(x, y)$ is called *an integrating factor*. From the definition of integrating factor we have [4]-[8], [10], [11]:

$$\frac{\partial \mu}{\partial y}(\mu M) = \frac{\partial \mu}{\partial x}(\mu N) \tag{21}$$

or

$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu, \tag{22}$$

whence

$$N\frac{\partial \ln \mu}{\partial x} - M\frac{\partial \ln \mu}{\partial y} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}. \tag{23}$$

We have obtained the partial differential equation to find an integrating factor. We discuss some particular cases where it is relatively easy to find a solution of equation (23), i.e. to find an integrating factor.

Case 1° If $\mu = \mu(x)$, then $\frac{\partial \mu}{\partial y} = 0$ and equation (23) will take the form [4]-[8], [10], [11]:

$$A(x) \equiv \frac{\partial \ln \mu}{\partial x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x, y)}. \tag{24}$$

The integration factor, depending on x alone, we can find from the formula:

$$\mu(x) = \exp[\int A(x)dx]. \tag{25}$$

Case 2° If $\mu = \mu(y)$, then $\frac{\partial \mu}{\partial x} = 0$ and equation (23) will take the form [4]-[8], [10], [11]:

$$B(y) \equiv \frac{\partial \ln \mu}{\partial y} = -\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}. \tag{26}$$

The integration factor, depending on y alone, we can find from the formula:

$$\mu(y) = \exp[\int B(y)dy]. \tag{27}$$

Case 3°. If $\mu = \mu(x, y)$, then [4]-[8], [10], [11]:

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = Nf(x) - Mg(y), \tag{28}$$

$$\varphi(x) = \exp[\int f(x)dx], \tag{29}$$

$$\psi(y) = \exp[\int g(y)dy]. \tag{30}$$

In that case the integrating factor depending on both x and y can be fund from the formula:

$$\mu(x, y) = \varphi(x) \cdot \psi(y). \tag{31}$$

3. Analytical and numerical analysis of not exact differential equations and integrating factors

Example 2.

Let us analyse the following not exact differential equation [1], [3], [4], [10]:

$$(2x^2 y + 2y + 5)dx + (2x^3 + 2x)\,dy = 0, \tag{32}$$

where

$$M_1(x, y) = 2x^2 y + 2y + 5, \tag{33}$$

$$N_1(x, y) = 2x^3 + 2x. \tag{34}$$

- **Analytical solution**

Let us verify if this equation is not exact differential equation:

$$\frac{\partial M_1}{\partial y} = 2x^2 + 2, \tag{35}$$

$$\frac{\partial N_1}{\partial x} = 6x^2 + 2. \tag{36}$$

So that

$$\frac{\partial M_1}{\partial y} \neq \frac{\partial N_1}{\partial x} \tag{37}$$

i.e. condition (3) is not fulfilled. Thus the given equation (32) is not exact differential equation.

Let us define the formula (24)

$$A(x) = \frac{(2x^2 + 2) - (6x^2 + 2)}{2x^3 + 2x} = \frac{-4x^2}{2x^3 + 2x} = -\frac{2x}{x^2 + 1}. \tag{38}$$

The function A(x) depends on x. The integrating factor $\mu(x)$ we find from the formula (25):

$$\mu(x) = \exp\left(-\int \frac{2x}{x^2 + 1} dx\right) = \exp[-\ln(x^2 + 1)] = \exp\left[\ln\left(\frac{1}{x^2 + 1}\right)\right] = \frac{1}{x^2 + 1}. \tag{39}$$

Now we multiply both sides of the equation (32) by $\frac{1}{x^2 + 1}$ and after simplification we have:

$$\left(2y + \frac{5}{x^2 + 1}\right)dx + 2x\,dy = 0, \tag{40}$$

where

$$M_2(x, y) = 2y + \frac{5}{x^2 + 1}, \tag{41}$$

$$N_2(x, y) = 2x. \tag{42}$$

Let us verify if the equation (40) is exact differential equation:

$$\frac{\partial M_2}{\partial y} = 2, \tag{43}$$

$$\frac{\partial N_2}{\partial x} = 2, \tag{44}$$

what means that

$$\frac{\partial M_2}{\partial y}=\frac{\partial N_2}{\partial x}. \tag{45}$$

Thus the obtained equation (40) is exact differential equation.

So exists a function u(x, y) that

$$\frac{\partial u}{\partial x}=2y+\frac{5}{x^2+1}, \tag{46}$$

$$\frac{\partial u}{\partial y}=2x. \tag{47}$$

The first, let us integrate the equation (46) with respect to x, where y is constant:

$$u(x,y)=\int\left(2y+\frac{5}{x^2+1}\right)dx+\varphi(y), \tag{48}$$

Taking into account the known formulae

$$\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx, \tag{49}$$

$$\int\frac{dx}{x^2+1}=\mathrm{arctg}(x)+C \tag{50}$$

and integrating the equation (48) with respect to x we have:

$$u(x,y)=2xy+5\mathrm{arctg}(x)+\varphi(y) \tag{51}$$

where $\varphi(y)$ is some differentiable function.

Now let us differentiate the equation (51) with respect to y, where x is constant:

$$\frac{\partial u}{\partial y}=2x+\frac{d\varphi}{dy}. \tag{52}$$

Next let us compare the equations (47) and (52), thus we obtain:

$$2x=2x+\frac{d\varphi}{dy}, \tag{53}$$

$$\frac{d\varphi}{dy}=0. \tag{54}$$

After integration (54) with respect to y, we have:

$$\varphi(y)=C,\quad C\in\Re. \tag{55}$$

Thus the general solution of the differential equation (32) has the following form:

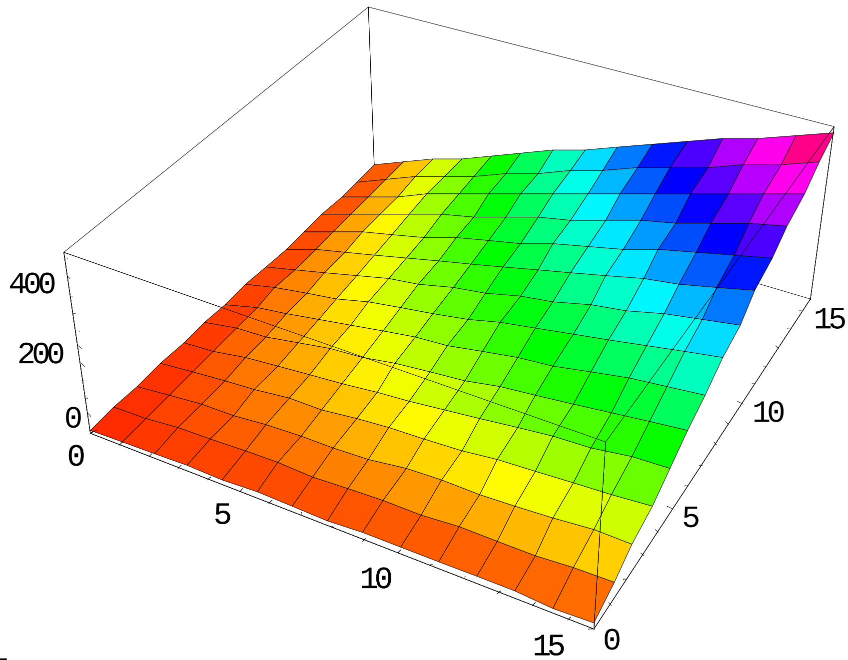
$$u(x,y)=2xy+5\mathrm{arctg}(x)+C,\quad C\in\Re. \tag{56}$$

• Numerical interpretation of solution

Let us make a procedure 2 of solution for equation (32) and graphical illustrate of the solution (56) for constant C=0 [2], [9], [12].

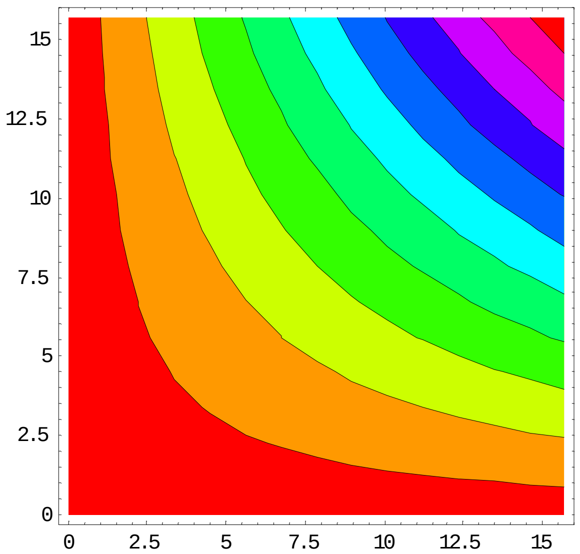
Program 2 *(Mathematica 4.1)*	Commentary
`In[1]:= M1[x_,y_]=2x^2*y+2y+5`	*Function $M_1(x,y)$ of Eq.(32)*
`N1[x_,y_]=2*x^3+2x`	*Function $N_1(x,y)$ of Eq.(32)*
`D[M1[x,y],y]=D[N1[x,y],x]//Simplify`	*Checking derivatives: False*
`A[x_]=(D[M1[x,y],y]-D[N1[x,y],x)/ N1[x,y]//Simplify`	*Definition of a function A(x)*
`step0=Exp[Integrate[A[x],x]]`	*Definition of a function $\mu(x)$*
`M2[x_,y_]=step0*M1[x,y]`	*Function $M_2(x,y)$ of Eq.(40)*
`N2[x_,y_]=step0*N1[x,y]`	*Function $N_2(x,y)$ of Eq.(40)*
`D[M2[x,y],y]=D[N2[x,y],x]//Simplify`	*Checking derivatives: True*
`step1=Integrate[M2[x,y],x]`	*Integration of function M_2*
`step2=D[step1+ϕ[y],y]`	*Differentiation*
`step3=Solve[step2==N2[x,y]+ϕ′[y]]//Simplify`	*Determining $\phi'[y]$ factor*
`step4=Integrate[ϕ′[y]/.step3[[1]],y]`	*Integration of factor $\phi'[y]$*
`solution=step1+step4+C`	*Determining the solution*
Out[1]= $5+2y+2x^2y$	*Function $M_1(x,y)$ of Eq.(32)*
Out[2]= $2x+2x^3$	*Function $N_1(x,y)$ of Eq.(32)*
Out[3]= $2+2x^2 == 2+6x^2$	*Checking derivatives: False*
Out[4]= $-\frac{2x}{1+x^2}$	*Function A(x)*
Out[5]= $\frac{5+2(1+x^2)y}{1+x^2}$	*Integrating factor $\mu(x)$*
Out[6]= $\frac{1}{1+x^2}$	*Function $M_2(x,y)$ of Eq.(40)*
Out[7]= $2x$	*Function $N_2(x,y)$ of Eq.(40)*
Out[8]= True	*Checking if $M2'_y = N2'_x$*
Out[9]= $2xy+5\text{ArcTan}[x]$	*Result for integration $M_2(x,y)$*
Out[10]= $2x+\phi'[y]$	*Result for differentiation*
Out[11]= $\{\{\phi'[y]\to 0\}\}$	*Obtained function $\phi'[y]$*
Out[12]= 0	*Obtained function $\phi[y]$*
Out[13]= $C+2xy+5\text{ArcTan}[x]$	*General solution of equation (32)*

`Plot3D[solution/.C->0,{x,0,2Pi},{y,0,2Pi}, ColorFunction → Hue]`	*Creating* `Plot3D`
`ContourPlot[solution/.C->0,{x,0,2Pi}, {y,0,2Pi},ColorFunction → Hue]`	*Creating* `ContourPlot`
`DensityPlot[solution/.C->0,{x,0,2Pi}, {y,0,2Pi},ColorFunction → Hue]`	*Creating* `DensityPlot`



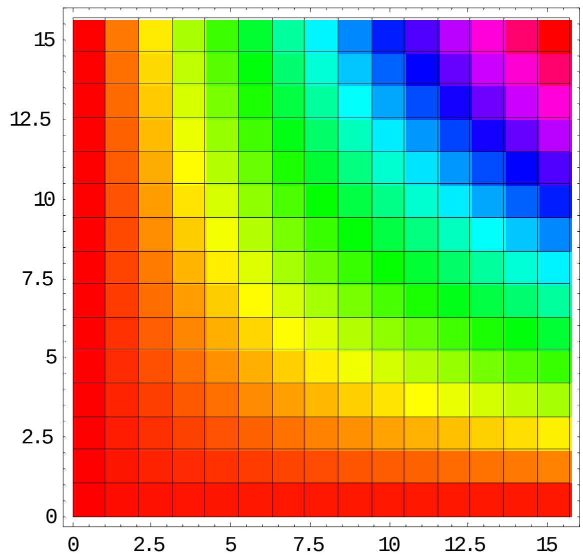
Out[14]=

Fig. 2a. `Plot3D` of solution (56) for C=0, *Mathematica (program 2)*
Source: Elaboration by the Authors



Out[15]=

Fig. 2b. `ContourPlot` of solution (56) for C=0 *Mathematica (program 2)*
Source: Elaboration by the Authors



Out[16]=

Fig. 2c. `DensityPlot` of solution (56) for C=0 *Mathematica (program 2)*
Source: Elaboration by the Authors

Example 3.

Let us analyse the following not exact differential equation [1], [3], [4], [10]:

$$(2xy^2 - 3y^3)dx + (7 - 3xy^2)dy = 0, \tag{57}$$

where

$$M_3(x, y) = 2xy^2 - 3y^3, \tag{58}$$

$$N_3(x, y) = 7 - 3xy^2. \tag{59}$$

For differential equation it is taken into account the initial condition:

$$u(x = 1, y = 1) = 1. \tag{60}$$

• **Analytical solution**

Let us verify if this equation is not exact differential equation:

$$\frac{\partial M_3}{\partial y} = 4xy - 9y^2, \tag{61}$$

$$\frac{\partial N_3}{\partial x} = -3y^2. \tag{62}$$

So that

$$\frac{\partial M_3}{\partial y} \neq \frac{\partial N_3}{\partial x} \tag{63}$$

i.e. condition (3) is not fulfilled. Thus the given equation (57) is not exact differential equation.

Let us define the formula (24)

$$A(x) = \frac{(4xy - 9y^2) - (-3y^2)}{7 - 3xy^2} = \frac{4xy - 6y^2}{7 - 3xy^2} = \frac{2(2x - 3y)y}{7 - 3xy^2}. \tag{64}$$

The function A(x) depends on x and y.

Let us define the formula (26)

$$B(y) = \frac{(-3y^2) - (4xy - 9y^2)}{2xy^2 - 3y^3} = \frac{-4xy + 6y^2}{2xy^2 - 3y^3} = \frac{-2y(2x - 3y)}{y^2(2x - 3y)} = -\frac{2}{y}. \tag{65}$$

The function B(y) depends only on variable y. The integrating factor $\mu(y)$ we can find from the formula (27):

$$\mu(y) = \exp\left(-2\int\frac{dy}{y}\right) = \exp(-2\ln y) = \exp\left(\ln\frac{1}{y^2}\right) = \frac{1}{y^2}. \tag{66}$$

Now we multiply both sides of equation (57) by expression $1/y^2$:

$$(2x - 3y)dx + \left(\frac{7}{y^2} - 3x\right)dy = 0, \tag{67}$$

where

$$M_4(x, y) = 2x - 3y, \tag{68}$$

$$N_4(x, y) = \frac{7}{y^2} - 3x. \tag{69}$$

Let us verify if the equation (67) is exact differential equation:

$$\frac{\partial M_4}{\partial y} = -3, \tag{70}$$

$$\frac{\partial N_4}{\partial x} = -3. \tag{71}$$

what means that

$$\frac{\partial M_4}{\partial y} = \frac{\partial N_4}{\partial x}. \tag{72}$$

Thus the equation (67) is exact total differential equation. So exists a function u(x, y) that

$$\frac{\partial u}{\partial x} = 2x - 3y, \tag{73}$$

$$\frac{\partial u}{\partial y} = \frac{7}{y^2} - 3x. \tag{74}$$

The first, let us integrate the equation (73) with respect to x, where y is constant:

$$u(x, y) = \int (2x - 3y) dx + \varphi(y), \tag{75}$$

$$u(x, y) = x^2 - 3xy + \varphi(y), \tag{76}$$

where $\varphi(y)$ is some differentiable function. Now let us differentiate the equation (76) with respect to y, where x is constant:

$$\frac{\partial u}{\partial y} = -3x + \frac{d\varphi}{dy}. \tag{77}$$

Next let us compare the equations (74) and (77), thus we obtain:

$$\frac{7}{y^2} - 3x = -3x + \frac{d\varphi}{dy}, \tag{78}$$

$$\frac{d\varphi}{dy} = \frac{7}{y^2}. \tag{79}$$

After integration (79) with respect to y, we have:

$$\varphi(y) = -\frac{7}{y} + C, \quad C \in \Re. \tag{80}$$

Thus the general solution of exact differential equation (67) has the following form:

$$u(x, y) = x^2 - 3xy - \frac{7}{y} + C, \quad C \in \Re. \tag{81}$$

Let us take the initial condition (60) in general solution (81), then we obtain the following particular solution for the differential equation (57):

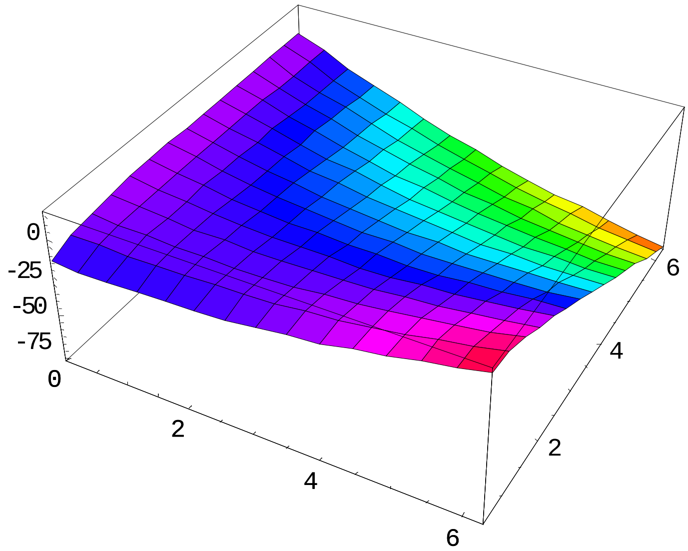
$$u(x, y) = x^2 - 3xy - \frac{7}{y} + 10. \tag{82}$$

• **Numerical interpretation of solution**

Let us make a procedure 3 for equation (57) and graphical illustrate of the solution (82) for constant C=0 [2], [9], [12].

Program 3 *(Mathematica 4.1)*	Commentary
In[1]:= M3[x_,y_]=2*x*y^2-3*y^3	*Function $M_3(x,y)$ of Eq.(57)*
M3[x_,y_]=7-3*x*y^2	*Function $N_3(x,y)$ of Eq.(57)*
u0:=1 x0:=1 y0:=1	*Initial conditions for u(x,y)*
D[M3[x,y],y]=D[N3[x,y],x]//Simplify	*Checking derivatives*
A[x_]=(D[M3[x,y],y]-D[N3[x,y],x])/ N3[x,y]//Simplify	*Definition of a function A(x)*
B[y_]=(D[N3[x,y],x]-D[M3[x,y],y])/ M3[x,y]//Simplify	*Definition of a function B(y)*
step0=Exp[Integrate[B[y],y]]	*Defining function $\mu(y)$*
M4[x_,y_]=step0*M3[x,y]//Simplify	*Function $M_4(x,y)$ of Eq.(67)*
N4[x_,y_]=step0*N3[x,y]//Simplify	*Function $N_4(x,y)$ of Eq.(67)*
D[M4[x,y],y]=D[N4[x,y],x]//Simplify	*Checking derivatives*
step1=Integrate[P[x,y],x	*Integrating of function P*
step2=D[step1+ϕ[y],y]	*Differentiating of step1*
step3=Solve[step2==N4[x,y]+ϕ′[y]]//Simplify	*Determining $\phi'[y]$ factor*
step4=Integrate[ϕ′[y]/.step3[[1]],y]	*Integrating of factor $\phi'[y]$*
Solution1=step1+step4+C	*General solution of Eq.(57)*
solution2=solution1==u0/.{x→x0,y→y0}	*Using initial condition (60)*
solution3=Solve[solution2,C]	*Determining constant C*
solution4=step1+step4+C/.solution3	*Particular solution of Eq.(57)*
Out[1]= $2xy^2+3y^3$	*Function $M_3(x,y)$ of Eq.(57)*
Out[2]= $7-3xy^2$	*Function N_3 (x,y) of Eq.(57)*
Out[6]= $4xy-9y^2==-3y^2$	*Checking derivatives: False*
Out[7]= $\frac{2(2x-3y)y}{7-3xy^2}$	*Function A(x,y)*
Out[8]= $-\frac{2}{y}$	*Function B(y)*
Out[9]= $\frac{1}{y^2}$	*Integrating factor $\mu(y)$*
Out[10]= $2x-3y$	*Function $M_4(x,y)$ of Eq.(67)*
Out[11]= $-3x+\frac{7}{y^2}$	*Function $N_4(x,y)$ of Eq.(67)*
Out[12]= True	*Checking if* $M4'_y=N4'_x$
Out[13]= x^2-3xy	*Result of integration $M_4(x,y)$*
Out[14]= $-3x+\phi'[y]$	*Result of differentiation*
Out[15]= $\left\{\left\{\phi'[y]\to\frac{7}{y^2}\right\}\right\}$	*Obtained function $\phi'[y]$*
Out[16]= $-\frac{7}{y}$	*Obtained function $\phi[y]$*

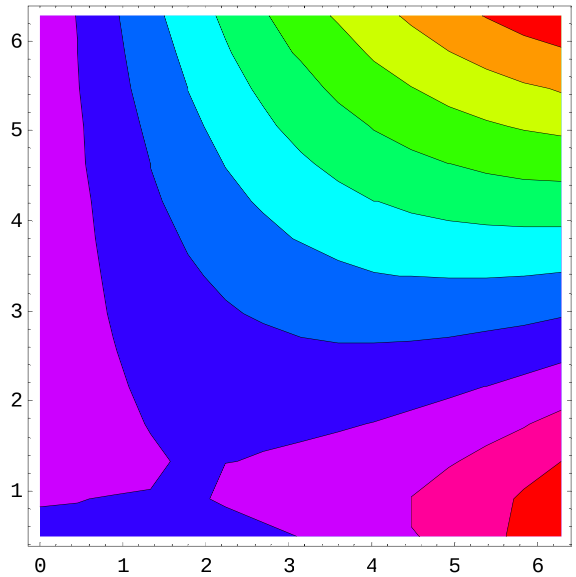
Out[17]= $C+x^2-\frac{7}{y}-3xy$	*General solution of Eq. (57)*
Out[18]= $-9+C==1$	*Equation with constant C*
Out[19]= $\{\{C\to 10\}\}$	*Determining the constant C*
Out[20]= $\{10+x^2-\frac{7}{y}-3xy\}$	*Particular solution of Eq. (57)*
`Plot3D[solution4,{x,0,2Pi},{y,0.05,2Pi}, ColorFunction → Hue]`	*Creating* `Plot3D`
`ContourPlot[solution4,{x,0,2Pi}, {y,0.05,2Pi},ColorFunction → Hue]`	*Creating* `ContourPlot`
`DensityPlot[solution4,{x,0,2Pi}, {y,0.05,2Pi},ColorFunction → Hue]`	*Creating* `DensityPlot`



Out[21]=

Fig. 3a. `Plot3D` of solution (82) for C=10, *Mathematica (program 3)*

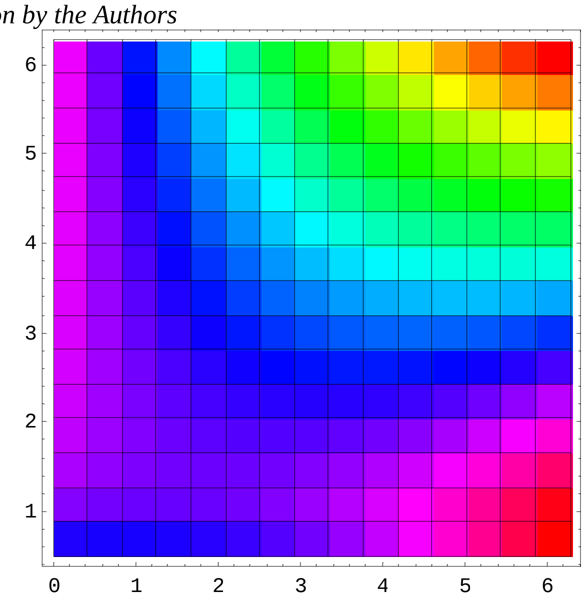
Source: Elaboration by the Authors



Out[22]=

Fig. 3b. `ContourPlot` of solution (82) for C=10 *Mathematica (program 3)*

Source: Elaboration by the Authors



Out[23]=

Fig. 3c. `DensityPlot` of solution (82) for C=10 *Mathematica (program 3)*

Source: Elaboration by the Authors

Example 4

Let us analyse the following not exact differential equation [1], [3], [4], [10]:

$$\left(\frac{y}{x}+3x^2\right)dx+\left(1+\frac{x^3}{y}\right)dy=0 \tag{83}$$

where

$$M_5(x,y)=\frac{y}{x}+3x^2, \tag{84}$$

$$N_5(x,y)=1+\frac{x^3}{y}. \tag{85}$$

• **Analytical solution**

Let us verify if the equation (83) is not exact differential equation:

$$\frac{\partial M_5}{\partial y}=\frac{1}{x}, \tag{86}$$

$$\frac{\partial N_5}{\partial x}=\frac{3x^2}{y}. \tag{87}$$

So that

$$\frac{\partial M_5}{\partial y}\neq\frac{\partial N_5}{\partial x} \tag{88}$$

i.e. condition (3) is not fulfilled. Thus the equation (83) is not exact differential equation.

Let us define the formula (24)

$$A(x)=\frac{\frac{1}{x}-\frac{3x^2}{y}}{1+\frac{x^3}{y}}=\frac{y-3x^3}{x(y+x^3)}. \tag{89}$$

The function A(x) depends on x and y. Let us define the formula (26)

$$B(y)=\frac{\frac{3x^2}{y}-\frac{1}{x}}{\frac{y}{x}+3x^2}=\frac{3x^3-y}{y(3x^3+y)}. \tag{90}$$

The function B(y) also depends on x and y.
Let us see to the third case. Thus in accordance with the formula (28) we have:

$$\frac{1}{x}-3\frac{x^2}{y}\equiv\left(1+\frac{x^3}{y}\right)\cdot f_1(x)-\left(\frac{y}{x}+3x^2\right)\cdot g_1(y). \tag{91}$$

Using *expectation method* for the functions $f_1(x)$ and $g_1(y)$ we propose that:

$$f_1(x)\equiv\frac{A_1}{x}, \tag{92}$$

$$g_1(y) \equiv \frac{B_1}{y}. \tag{93}$$

Inserting functions (92) and (93) into (91) we obtain:

$$\frac{1}{x} - 3\frac{x^2}{y} \equiv \left(1 + \frac{x^3}{y}\right) \cdot \frac{A_1}{x} - \left(\frac{y}{x} + 3x^2\right) \cdot \frac{B_1}{y}. \tag{94}$$

After transformation in (94) we have:

$$\frac{1}{x} - 3\frac{x^2}{y} \equiv (A_1 - B_1) \cdot \frac{1}{x} + (A_1 - 3B_1)\frac{x^2}{y}. \tag{95}$$

If we compare the right functions in above equation (95) we have the following system of unknown coefficients:

$$\begin{cases} A_1 - B_1 = 1 \\ A_1 - 3B_1 = -3 \end{cases} \Leftrightarrow \begin{cases} A_1 = 3 \\ B_1 = 2. \end{cases} \tag{96}$$

The functions $f_1(x)$ and $g_1(y)$ have the following forms:

$$f_1(x) \equiv \frac{3}{x}, \tag{97}$$

$$g_1(y) \equiv \frac{2}{y}. \tag{98}$$

So the functions $\varphi(x)$ and $\psi(y)$, in accordance with (29)-(30), are defined by the formulae:

$$\varphi(x) = \exp\left(3\int \frac{dx}{x}\right) = \exp(3\ln x) = \exp(\ln x^3) = x^3, \tag{99}$$

$$\psi(y) = \exp\left(2\int \frac{dy}{y}\right) = \exp(2\ln y) = \exp(\ln y^2) = y^2. \tag{100}$$

The integrating factor $\mu(x, y) = \varphi(x)\cdot\psi(y)$ has the following form:

$$\mu(x, y) = \varphi(x) \cdot \psi(y) = x^3 y^2. \tag{101}$$

Multiplying both side of equation (83) by the expression x^3y^2 we obtain the following differential equation:

$$(x^2y^3 + 3x^5y^2)dx + (x^3y^2 + x^6y)\,dy = 0. \tag{102}$$

where

$$M_6(x, y) = x^2y^3 + 3x^5y^2, \tag{103}$$

$$N_6(x, y) = x^3y^2 + x^6y. \tag{104}$$

Let us see if the equation (102) is exact differential equation:

$$\frac{\partial M_6}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + 3x^5y^2) = 3x^2y^2 + 6x^5y, \tag{105}$$

$$\frac{\partial N_6}{\partial x} = \frac{\partial}{\partial x}(x^3y^2 + x^6y) = 3x^2y^2 + 6x^5y. \tag{106}$$

what means

$$\frac{\partial M_6}{\partial y} = \frac{\partial N_6}{\partial x}. \tag{107}$$

Thus the given equation (102) is exact differential equation. So exists a function u(x,y) that

$$\frac{\partial u}{\partial x} = M_6(x,y) \quad \Leftrightarrow \quad \frac{\partial u}{\partial x} = x^2y^3 + 3x^5y^2, \tag{108}$$

$$\frac{\partial u}{\partial y} = N_6(x,y) \quad \Leftrightarrow \quad \frac{\partial u}{\partial y} = x^3y^2 + x^6y. \tag{109}$$

- **The first way to obtain the general solution of differential equation (83)**

Let us integrate the equation (108) with respect to x where y is constant and add φ(y):

$$u(x,y) = \int (x^2y^3 + 3x^5y^2)\,dx + \varphi(y), \tag{110}$$

$$u(x,y) = \frac{1}{3}x^3y^3 + \frac{1}{2}x^6y^2 + \varphi(y), \tag{111}$$

where φ(y) is some differentiable function.

Now let us differentiate the equation (111) with respect to y, where x is constant:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{3}x^3y^3 + \frac{1}{2}x^6y^2 + \varphi(y)\right] = x^3y^2 + x^6y + \frac{d\varphi}{dy}. \tag{112}$$

Next let us compare the equations (109) and (112), thus we obtain:

$$x^3y^2 + x^6y = x^3y^2 + x^6y + \frac{d\varphi}{dy}, \tag{113}$$

$$\frac{d\varphi}{dy} = 0. \tag{114}$$

After integration (114) with respect to y, we have:

$$\varphi(y) = C, \quad C \in \Re. \tag{115}$$

The obtained function φ(y) we put into the formula (111). Thus the general solution of the exact differential equation (102) and not exact equation (83) has the following form:

$$u(x,y) = \frac{1}{3}x^3y^3 + \frac{1}{2}x^6y^2 + C, \quad C \in \Re. \tag{116}$$

- **The second way to obtain the general solution of differential equation (83)**

The first, let us integrate the equation (109) with respect to y, where x is constant:

$$u(x,y) = \int (x^3y^2 + x^6y)dy + \zeta(x), \tag{117}$$

$$u(x,y) = \frac{1}{3}x^3y^3 + \frac{1}{2}x^6y^2 + \zeta(x) \tag{118}$$

where ζ(x) is some differentiable function.

Now let us differentiate the equation (118) with respect to x, where y is constant:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[\frac{1}{3}x^3y^3+\frac{1}{2}x^6y^2+\zeta(x)\right]=x^2y^3+3x^5y^2+\frac{d\zeta}{dx}. \tag{119}$$

Next let us compare the equations (108) and (119), thus we obtain:

$$x^2y^3+3x^5y^2=x^2y^3+3x^5y^2+\frac{d\zeta}{dx}, \tag{120}$$

$$\frac{d\zeta}{dx}=0. \tag{121}$$

After integration (121) with respect to x, we have:

$$\zeta(x)=C,\quad C\in\Re. \tag{122}$$

The obtained function ζ(x) we put into the formula (118). Thus the general solution of the exact differential equation (102) and also not exact equation (83) has the following form:

$$u(x,y)=\frac{1}{3}x^3y^3+\frac{1}{2}x^6y^2+C,\quad C\in\Re. \tag{116′}$$

As we can see the formulae (116′) and (116) are identical.

• Numerical interpretation of solution (116) for equation (83)

Let us make a procedure 4 for the equation (83) and graphical illustrate of the solution (116) or (116′) for constant C=10 [2], [9], [12].

Program 4 *(Mathematica 4.1)*	Commentary
In[1]:= M5[x_,y_]=(y/x)+3*x^2	*Function $M_5(x,y)$ of Eq.(83)*
N5[x_,y_]=1+(x^3)/y	*Function $N_5(x,y)$ of Eq.(83)*
D[M5[x,y],y]=D[N5[x,y],x]//Simplify	*Checking derivatives: False*
A1[x_]=(D[M5[x,y],y]-D[N5[x,y],x])/ N5[x,y]//Simplify	*Defining function $A_1(x)$*
B1[y_]=(D[N5[x,y],x]-D[M5[x,y],y])/ N5[x,y]//Simplify	*Defining function B1(y)*
{expectation method of determination coefficients A_1 and B_1};	
w1={{1,–1},{1,–3}}	*Matrix w1*
wa1={{1,–3},{-1,–3}}	*Matrix wa1*
wb1={{1,1},{1,–3}}	*Matrix wb1*
W1=Det[w1]	*Determinant w1*
WA1=Det[wa1]	*Determinant wa1*
WB1=Det[wb1]	*Determinant wb1*
A_1=WA1/W1	*Coefficient A_1*
B_1=WB1/W1	*Coefficient B_1*
{the end of expectation method};	
f1[x_]=A_1/x	*Definition of a function $f_1(x)$*
g1[y_]=B_1/y	*Definition of a function $g_1(y)$*
calka1=Integarte[f1[x],x]	*Integrating of function $f_1(x)$*
calka2=Integarte[g1[x],y]	*Integrating of function $g_1(y)$*
F1=Exp[calka1,x]	*Determine of a function $\varphi(x)$*
F2=Exp[calka2,y]	*Determine of a function $\psi(x)$*

`F3=F1*F2`	*Determine of a function* $\mu(x,y)$
`M6[x_,y_]=F3*M5[x,y]//Simplify`	*Function* $M_6(x,y)$ *of Eq.(65)*
`N6[x_,y_]=F3*N5[x,y]//Simplify`	*Function* $N_6(x,y)$ *of Eq.(65)*
`D[M6[x,y],y]=D[N6[x,y],x]//Simplify`	*Checking derivatives: True*
`step1=Integrate[M6[x,y],x]`	*Integrating of function* M_6
`step2=D[step1+ϕ[y],y]`	*Differentiating of step1*
`step3=Solve[step2==N6[x,y]+ϕ′[y]]//Simplify`	*Determine of a* $\phi'[y]$ *factor*
`step4=Integrate[ϕ′[y]/.step3[[1]],y]`	*Integrating of factor* $\phi'[y]$
`solution=step1+step4+C`	*General solution of Eq.(83)*
`CORR=solution/.C->10`	*Particular solution of Eq.(83)*
Out[1]= $3x^2+\frac{y}{x}$	*Function* $M_5(x,y)$ *of Eq.(83)*
Out[2]= $1+\frac{x^3}{y}$	*Function* $N_5(x,y)$ *of Eq.(83)*
Out[3]= $\frac{1}{2}==\frac{3x^2}{y}$	*Checking derivatives: False*
Out[4]= $\frac{-3x^3+y}{x(x^3+y)}$	*Function* $A_1(x)$
Out[5]= $\frac{3x^3-y}{3x^3y+y^2}$	*Function* $B_1(y)$

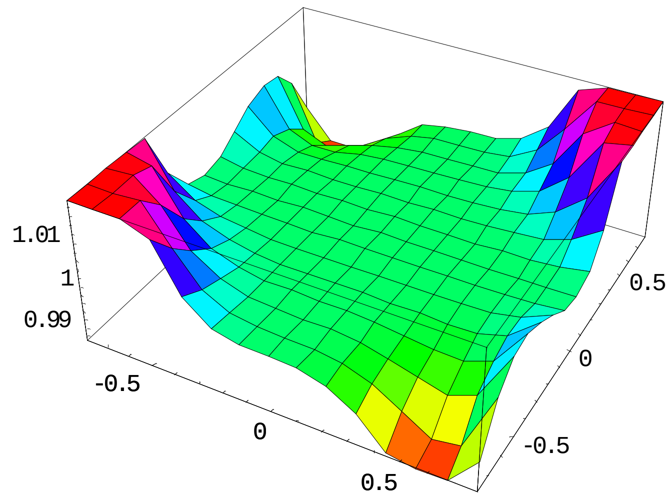
{expectation method of determination coefficients A_1 and B_1}

Out[7]= {{1,–1},{1,–3}}	*Matrix w1*
Out[8]= {{1,–3},{-1,–3}}	*Matrix wa1*
Out[9]= {{1,1},{1,–3}}	*Matrix wb1*
Out[10]= -2	*Determinant of matrix w1*
Out[11]= -6	*Determinant of matrix wa1*
Out[12]= -4	*Determinant of matrix wb1*
Out[13]= 3	*Coefficient* A_1
Out[14]= 2	*Coefficient* B_1

{the end of expectation method}

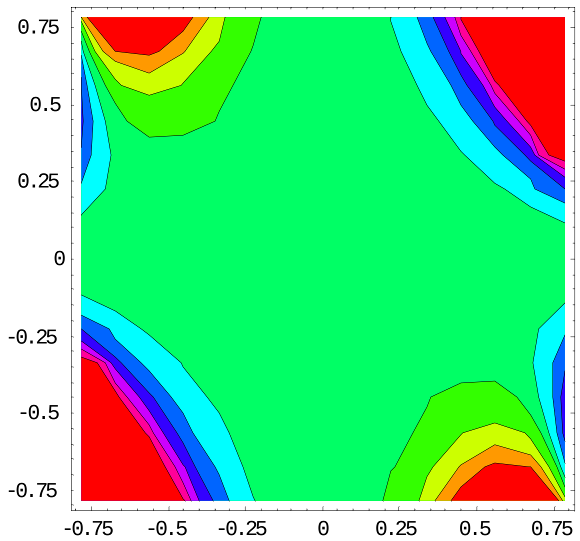
Out[16]= $\frac{3}{x}$	*Obtained function* $f_1(x)$
Out[17]= $\frac{2}{y}$	*Obtained function* $g_1(y)$
Out[18]= 3Log[x]	*Integration of a function* $f_1(x)$
Out[19]= 2Log[y]	*Integration of a function* $g_1(y)$
Out[20]= x^3	*Function* $\varphi(x)$
Out[21]= y^2	*Function* $\psi(y)$
Out[22]= x^3y^2	*Function* $\mu(x,y)=\varphi(x)\,\psi(y)$
Out[23]= $x^2y^2(3x^3+y)$	*Function* $M_6(x,y)$ *of Eq.(102)*
Out[24]= $x^3y(x^3+y)$	*Function* $N_6(x,y)$ *of Eq.(102)*
Out[25]= True	*Checking if* $M6'_y=N6'_x$
Out[26]= $\frac{x^6y^2}{2}+\frac{x^3y^3}{3}$	*Integration* $M_6(x,y)$

`Out[27]=` $x^6y+x^3y^2+\phi'[y]$	*Result of differentiation*
`Out[28]=` $\{\{\phi'[y]\to 0\}\}$	*Obtained function* $\phi'[y]$
`Out[29]=` 0	*Obtained function* $\phi[y]$
`Out[30]=` $C+\frac{x^6y^2}{2}+\frac{x^3y^3}{3}$	*General solution of Eq. (83)*
`Out[1]=` $10+\frac{x^6y^2}{2}+\frac{x^3y^3}{3}$	*Particular solution of Eq. (83)*
`Plot3D[CSRR,{x,-0.25,0.25Pi}, {y,-0.05,0.25Pi},ColorFunction → Hue]`	*Creating* `Plot3D`
`ContourPlot[CSRR,{x,-0.25,0.25Pi}, {y,-0.05,0.25Pi},ColorFunction → Hue]`	*Creating* `ContourPlot`
`DensityPlot[CSRR,{x,-0.25,0.25Pi}, {y,-0.05,0.25Pi},ColorFunction → Hue]`	*Creating* `DensityPlot`



Out[32]=

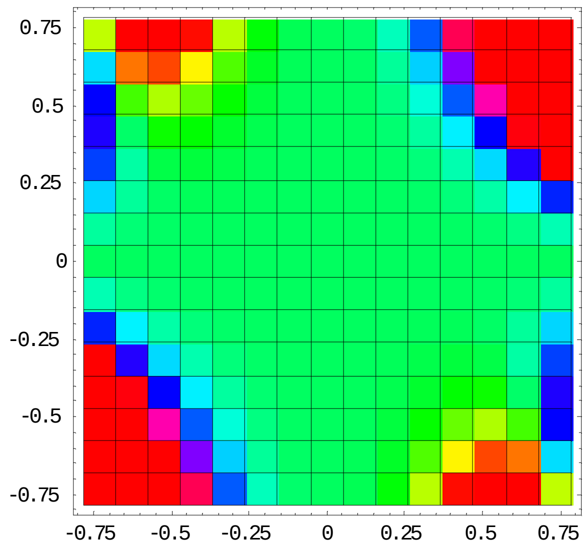
Fig. 4a. `Plot3D` of solution (116) for C=10, *Mathematica (program 4)*



Out[33]=

Fig. 4b. `ContourPlot` of solution (116) for C=10 *Mathematica (program 4)*

Source: Elaboration by the Authors



Out[34]=

Fig. 4c. `DensityPlot` of solution (116) for C=10 *Mathematica (program 4)*

Source: Elaboration by the Authors

Example 5

Let us analyse the following not exact differential equation [1], [3], [4], [10]:

$$(3y-2xy-3y^2)dx+(2x^2+3xy-4x)\,dy=0 \tag{123}$$

where

$$M_7(x,y)=3y-2xy-3y^2, \tag{124}$$

$$N_7(x,y)=2x^2+3xy-4x. \tag{125}$$

- **Analytical solution**

Let us verify if the equation (123) is not exact differential equation:

$$\frac{\partial M_7}{\partial y}=3-2x-6y, \tag{126}$$

$$\frac{\partial N_7}{\partial x}=4x+3y-4. \tag{127}$$

So that

$$\frac{\partial M_7}{\partial y}\neq\frac{\partial N_7}{\partial x} \tag{128}$$

i.e. condition (3) is not fulfilled. Thus the equation (123) is not exact differential equation.

Let us define the formula (24)

$$A(x)=\frac{3-2x-6y-4x-3y+4}{3y-2xy-3y^2}=\frac{7-6x-9y}{y(3-2x-3y)}. \tag{129}$$

The function A(x) depends on x and y. Let us define the formula (26)

$$B(y)=\frac{4x+3y-4-3+2x+6y}{2x^2+3xy-4x}=-\frac{7-6x-9y}{x(2x+3y-4)}. \tag{130}$$

The function B(y) also depends on x and y.
Let us see to the third case. Thus in accordance with the formula (28) we have:

$$-6x-9y+7\equiv(2x^2+3xy-4x)\cdot f_2(x)-(3y-2xy-3y^2)\cdot g_2(y). \tag{131}$$

Using *expectation method for* the functions $f_2(x)$ and $g_2(y)$ we propose that:

$$f_2(x)\equiv\frac{A_2}{x}, \tag{132}$$

$$g_2(y)\equiv\frac{B_2}{y}. \tag{133}$$

Inserting functions (132) and (133) into (131) we obtain:

$$-6x-9y+7\equiv(2x^2+3xy-4x)\cdot\frac{A_2}{x}-(3y-2xy-3y^2)\cdot\frac{B_2}{y}. \tag{134}$$

After transformation in (134) we have:

$$-6x-9y-7\equiv(2A_2-2B_2)x+(3A_2+3B_2)7+(-4A_2-3B_2). \tag{135}$$

If we compare the right functions in above equation (135) we have the following system of unknown coefficients:

$$\begin{cases} A_2 + B_2 = -3 \\ -4A_2 - 3B_2 = 7 \end{cases} \Leftrightarrow \begin{cases} A_2 = 2 \\ B_2 = -5. \end{cases} \tag{136}$$

The functions $f_2(x)$ and $g_2(y)$ have the following forms:

$$f_2(x) \equiv \frac{2}{x}, \tag{137}$$

$$g_2(y) \equiv \frac{-5}{y}. \tag{138}$$

So the functions $\varphi(x)$ and $\psi(y)$, in accordance with (29)-(30), are defined by the formulae:

$$\varphi(x) = \exp\left(2\int \frac{dx}{x}\right) = \exp(2\ln x) = \exp(\ln x^2) = x^2, \tag{139}$$

$$\psi(y) = \exp\left(-5\int \frac{dy}{y}\right) = \exp(-5\ln y) = \exp\left(\ln \frac{1}{y^5}\right) = \frac{1}{y^5}. \tag{140}$$

The integrating factor $\mu(x, y) = \varphi(x)\cdot\psi(y)$ has the following form:

$$\mu(x, y) = \varphi(x)\cdot\psi(y) = \frac{x^2}{y^5}. \tag{141}$$

Multiplying both side of equation (123) by the expression x^2/y^5 we obtain the following exact differential equation:

$$\left(\frac{3x^2}{y^4} - \frac{2x^3}{y^4} - \frac{3x^2}{y^3}\right)dx + \left(\frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5}\right)dy = 0 \tag{142}$$

where

$$M_8(x, y) = \frac{3x^2}{y^4} - \frac{2x^3}{y^4} - \frac{3x^2}{y^3}, \tag{143}$$

$$N_8(x, y) = \frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5}. \tag{144}$$

Let us see if the equation (142) is a total differential equation:

$$\frac{\partial M_8}{\partial y} = \frac{\partial}{\partial y}\left(\frac{3x^2}{y^4} - \frac{2x^3}{y^4} - \frac{3x^2}{y^3}\right) = -\frac{12x^2}{y^5} + \frac{8x^3}{y^5} + \frac{9x^2}{y^4} = \frac{x^2(-12 + 8x + 9y)}{y^5}, \tag{145}$$

$$\frac{\partial N_8}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5}\right) = \frac{8x^3}{y^5} + \frac{9x^2}{y^4} - \frac{12x^2}{y^5} = \frac{x^2(-12 + 8x + 9y)}{y^5}. \tag{146}$$

what means

$$\frac{\partial M_8}{\partial y} = \frac{\partial N_8}{\partial x}. \tag{147}$$

Thus the given equation (142) is exact differential equation. So exists a function u(x,y) that

$$\frac{\partial u}{\partial x} = M_8(x,y) \quad \Leftrightarrow \quad \frac{\partial u}{\partial x} = \frac{3x^2}{y^4} - \frac{2x^3}{y^4} - \frac{3x^2}{y^3}, \tag{148}$$

$$\frac{\partial u}{\partial y} = N_8(x,y) \quad \Leftrightarrow \quad \frac{\partial u}{\partial y} = \frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5}. \tag{149}$$

➢ **The first way to obtain the general solution of differential equation (83)**

Let us integrate the equation (148) with respect to x where y is constant and add φ(y):

$$u(x,y) = \int \left(\frac{3x^2}{y^4} - \frac{2x^3}{y^4} - \frac{3x^2}{y^3} \right) dx + \varphi(y), \tag{150}$$

$$u(x,y) = \frac{x^3}{y^4} - \frac{x^4}{2y^4} - \frac{x^3}{y^3} + \varphi(y), \tag{151}$$

where φ(y) is some differentiable function.

Now let us differentiate the equation (151) with respect to y, where x is constant:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^3}{y^4} - \frac{x^4}{2y^4} - \frac{x^3}{y^3} + \varphi(y) \right] = -\frac{4x^3}{y^5} + \frac{2x^4}{y^5} + \frac{3x^3}{y^4} + \frac{d\varphi}{dy}. \tag{152}$$

Next let us compare the equations (149) and (152), thus we obtain:

$$\frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5} = -\frac{4x^3}{y^5} + \frac{2x^4}{y^5} + \frac{3x^3}{y^4} + \frac{d\varphi}{dy}, \tag{153}$$

$$\frac{d\varphi}{dy} = 0. \tag{154}$$

After integration (154) with respect to y, we have:

$$\varphi(y) = C, \quad C \in \Re. \tag{155}$$

The obtained function φ(y) we put into the formula (151). Thus the general solution of the exact differential equation (142) and also not exact equation (123) has the following form:

$$u(x,y) = \frac{x^3}{y^4} - \frac{x^4}{2y^4} - \frac{x^3}{y^3} + C, \quad C \in \Re. \tag{156}$$

➢ **The second way to obtain the general solution of differential equation (83)**

The first, let us integrate the equation (149) with respect to y, where x is constant:

$$u(x,y) = \int \left(\frac{2x^4}{y^5} + \frac{3x^3}{y^4} - \frac{4x^3}{y^5} \right) dy + \zeta(x), \tag{157}$$

$$u(x,y) = -\frac{x^4}{2y^4} - \frac{x^3}{y^3} + \frac{x^3}{y^4} + \zeta(x) \tag{158}$$

where ζ(x) is some differentiable function.

Now let us differentiate the equation (118) with respect to x, where y is constant:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[-\frac{x^4}{2y^4}-\frac{x^3}{y^3}+\frac{x^3}{y^4}+\zeta(x)\right]=-\frac{2x^3}{y^4}-\frac{3x^2}{y^3}+\frac{3x^2}{y^4}+\frac{d\zeta}{dx}. \tag{159}$$

Next let us compare the equations (148) and (159), thus we obtain:

$$\frac{3x^2}{y^4}-\frac{2x^3}{y^4}-\frac{3x^2}{y^3}=-\frac{2x^3}{y^4}-\frac{3x^2}{y^3}+\frac{3x^2}{y^4}+\frac{d\zeta}{dx}, \tag{160}$$

$$\frac{d\zeta}{dx}=0. \tag{161}$$

After integration (161) with respect to x, we have:

$$\zeta(x)=C,\quad C\in\Re. \tag{162}$$

The obtained function ζ(x) we put into the formula (158). Thus the general solution of the exact differential equation (142) and also not exact equation (123) has the following form:

$$u(x,y)=-\frac{x^4}{2y^4}-\frac{x^3}{y^3}+\frac{x^3}{y^4}+C,\quad C\in\Re. \tag{156′}$$

As we can see the formulae (156′) and (156) are identical.

• Numerical interpretation of solution (116) for the equation (83)

Let us make a procedure 5 for the equation (123) and graphical illustrate of the solution (156) or (156′) for constant C=10 [2], [9], [12].

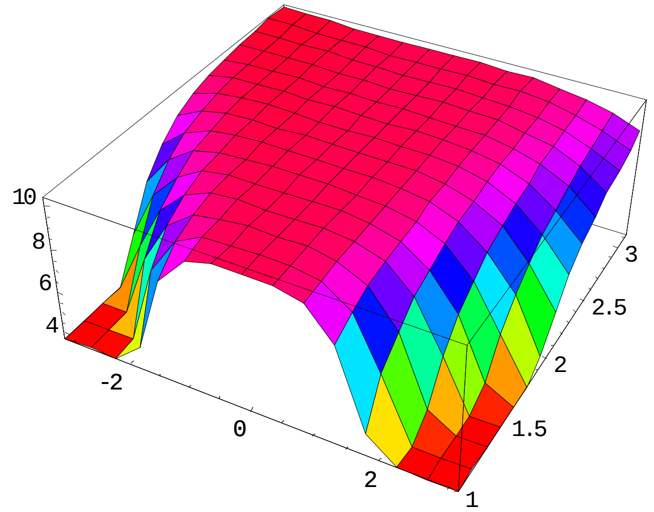
Program 5 *(Mathematica 4.1)*	Commentary
`In[1]:= M7[x_,y_]=3y-2xy-3y^2`	*Function $M_7(x,y)$ of Eq.(123)*
`N7[x_,y_]=2(x^2)+3y-4x`	*Function $N_7(x,y)$ of Eq.(123)*
`D[M7[x,y],y]=D[N7[x,y],x]//Simplify`	*Checking derivatives: False*
`A2[x_]=(D[M7[x,y],y]-D[N7[x,y],x])/ N7[x,y]//Simplify`	*Definition of a function $A_2(x)$*
`B2[y_]=(D[N7[x,y],x]-D[M7[x,y],y])/ N7[x,y]//Simplify`	*Definition of a function $B_2(y)$*
`{expectation method of determination coefficients A2 and B2};`	
`w2={{1,1},{-4,–3}}`	*Matrix w2*
`wa2={{-3,1},{7,–3}}`	*Matrix wa2*
`wb2={{1,-3},{-4,7}}`	*Matrix wb2*
`W2=Det[w2]`	*Determinant w2*
`WA2=Det[wa2]`	*Determinant wa2*
`WB2=Det[wb2]`	*Determinant wb2*
`A2=WA2/W2`	*Coefficient A_2*
`B2=WB2/W2`	*Coefficient B_2*
`{the end of expectation method};`	
`f2[x_]=A2/x`	*Definition of function $f_2(x)$*
`g2[y_]=B2/y`	*Definition of function $g_2(y)$*
`calka1=Integarte[f2[x],x]`	*Integration of function $f_2(x)$*

`calka2=Integarte[g2[x],y]`	*Integration of function $g_2(y)$*
`F1=Exp[calka1,x]`	*Determinition function $\varphi(x)$*
`F2=Exp[calka2,y]`	*Determinition function $\psi(x)$*
`F3=F1*F2`	*Determinition function $\mu(x,y)$*
`M8[x_,y_]=F3*M7[x,y]//Simplify`	*Function $M_8(x,y)$ of Eq.(65)*
`N8[x_,y_]=F3*N7[x,y]//Simplify`	*Function $N_8(x,y)$ of Eq.(65)*
`D[M8[x,y],y]=D[N8[x,y],x]//Simplify`	*Checking derivatives: True*
`step1=Integrate[M8[x,y],x]`	*Integrating of function M_8*
`step2=D[step1+ϕ[y],y]`	*Differentiating of step1*
`step3=Solve[step2==N8[x,y]+ϕ′[y]]//Simplify`	*Determine of ϕ′[y] factor*
`step4=Integrate[ϕ′[y]/.step3[[1]],y]`	*Integrating of factor ϕ′[y]*
`solution=step1+step4+C`	*General solution of Eq.(123)*
`CORR=solution/.C->10`	*Particular solution of Eq.(123)*
Out[1]= $-2xy+3y-3y^2$	*Function $M_7(x,y)$ of Eq.(123)*
Out[2]= $-4+2x^2+3xy$	*Function $N_7(x,y)$ of Eq.(123)*
Out[3]= $3-6y == -4+4x$	*Checking derivatives: False*
Out[4]= $\dfrac{-3x^3+y}{x(x^3+y)}$	*Function $A_2(x)$*
Out[5]= $\dfrac{3x^3-y}{3x^3y+y^2}$	*Function $B_2(y)$*

{expectation method of determination coefficients A_2 and B_2}

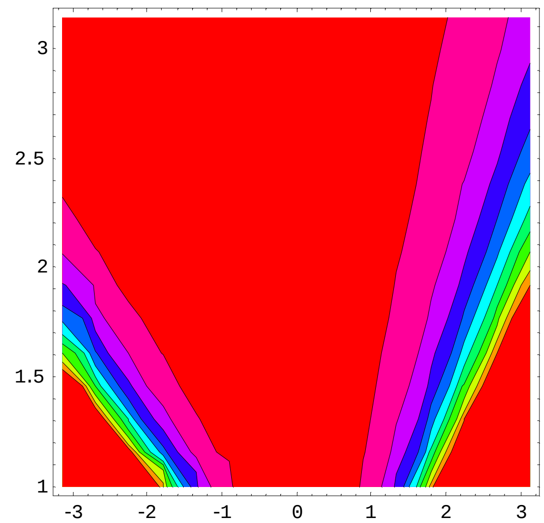
Out[7]= {{1,1},{-4,–3}}	*Matrix w2*
Out[8]= {{-3,1},{7,–3}}	*Matrix wa2*
Out[9]= {{1,-3},{-4,7}}	*Matrix wb2*
Out[10]= 1	*Determinant of matrix w2*
Out[11]= 2	*Determinant of matrix wa2*
Out[12]= -5	*Determinant of matrix wb2*
Out[13]= 2	*Coefficient A_2*
Out[14]= -5	*Coefficient B_2*
{the end of expectation method}	
Out[16]= 2/x	*Obtained function $f_2(x)$*
Out[17]= -5/y	*Obtained function $g_2(y)$*
Out[18]= 2Log[x]	*Integration of function $f_2(x)$*
Out[19]= -5Log[y]	*Integration of function $g_2(y)$*
Out[20]= x^2	*Function $\varphi(x)$*
Out[21]= $1/y^5$	*Function $\psi(y)$*
Out[22]= x^2/y^5	*Function $\mu(x,y)=\varphi(x)\psi(y)$*
Out[23]= $-x^2(-3+2x+3y)/y^4$	*Function $M_8(x,y)$ of Eq.(142)*
Out[24]= $x^3(-4+2x+3y)/y^5$	*Function $N_8(x,y)$ of Eq.(142)*
Out[25]= True	*Checking if $M8'_y = 6N8'_x$*
Out[26]= $\dfrac{x^3}{y^4}-\dfrac{x^4}{2y^4}-\dfrac{x^3}{y^3}$	*Integration $M_8(x,y)$*

Out[27]= $-\frac{4x^3}{y^5}+\frac{2x^4}{y^5}+\frac{3x^3}{y^4}+\phi'[y]$	*Result of differentiation*
Out[28]= $\{\{\phi'[y]\to 0\}\}$	*Obtained function $\phi'[y]$*
Out[29]= 0	*Obtained function $\phi[y]$*
Out[30]= $C+\frac{x^3}{y^4}-\frac{x^4}{2y^4}-\frac{x^3}{y^3}$	*General solution of Eq. (123)*
Out[1]= $10+\frac{x^3}{y^4}-\frac{x^4}{2y^4}-\frac{x^3}{y^3}$	*Particular solution of Eq. (123)*
`Plot3D[CSRR,{x,-Pi,Pi},{y,1,Pi}, ColorFunction → Hue]`	*Creating* `Plot3D`
`ContourPlot[CSRR,{x,-Pi,Pi},{y,1,Pi}, ColorFunction → Hue]`	*Creating* `ContourPlot`
`DensityPlot[CSRR,{x,-Pi,Pi},{y,1,Pi}, ColorFunction → Hue]`	*Creating* `DensityPlot`



Out[32]=

Fig. 5a. `Plot3D` of solution (156) for C=10, *Mathematica (program 5)*

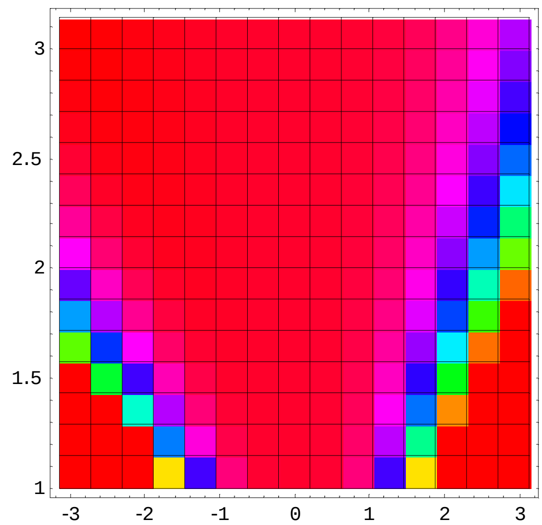


Out[33]=

Fig. 5b. `ContourPlot` of solution (156) for C=10

Mathematica (program 5)

Source: Elaboration by the Authors



Out[34]=

Fig. 5c. `DensityPlot` of solution (156) for C=10

Mathematica (program 5)

Source: Elaboration by the Authors

Conclusions

- To solve a not total differential equations you need to find a function called integrating factor. Integrating factor may depend only on the variable x, only the variable y, or of both variables x and y. After determination a suitable integrating factor it is necessary to multiply both sides of the not total differential equation by this factor. Then it is obtained a complete differential equation that we can solve.
- In order to find an integrating factor and to carry out the entire procedure of solving some not total differential equations can be used one of known numerical programs. One of them is a *Mathematica* program.
- Program *Mathematica* allows us to look, for more difficult cases, some integrating factor dependent on two variables x and y by using *some expectation method*. Let us notice that search an integral factor, which depends only on the variables x and y, in generally is not easy way.

References

[1] Demidowicz B.: *Problems in mathematical analysis,* Mir Publishers, Moscow 1976.

[2] Drwal G., Grzymkowski R., Kapusta A., Słota D.: *Mathematica 4,* Wyd. Pracowni Komputerowej Jacka Skalmierskiego, Gliwice 2000.

[3] Krasnov M.L., Kiselyov A.I., Makarenko G.I.: *A book of problems in ordinary differential equations,* Mir Publishers, Moscow 1981.

[4] Krysicki W., Włodarski L.: *Analiza matematyczna w zadaniach, Tom 2,* PWN, Warszawa 1994, w. XIX.

[5] Morchało J., Ratajczak Z., Werbowski J.: *Matematyka. Równania różniczkowe zwyczajne w zastosowaniach,* Wyd. Pol. Poznańskiej, Poznań 1987, w. III rozszerzone.

[6] Leja F.: *Rachunek różniczkowy i całkowy ze wstępem do równań różniczkowych,* Tom 2 Biblioteka Matematyczna, PWN, Warszawa 1969, w. X, rozszerzone.

[7] Palczewski A.: *Równania różniczkowe zwyczajne. Teoria i metody numeryczne z wykorzystaniem komputerowego systemu obliczeń symbolicznych,* WNT, Warszawa 1999.

[8] Ross C.C.: *Differential equations, An introduction with Mathematica,* Springer-Verlag, New York Berlin Heidelberg London Paris Tokyo Hong Kong Barcelona Budapest 1995.

[9] Trott M.: *The Mathematica guide book for symbolics,* Springer, New York 2006.

[10] Siewierski L. (pod red.): *Ćwiczenia z analizy matematycznej z zastosowaniami, Tom II,* PWN Warszawa 1981.

[11] Tymowski Sławomir: *Wykłady z równań różniczkowych zwyczajnych,* Wyd. WSP w Olsztynie, Olsztyn 1990.

[12] Wolfram S.: *The Mathematica Book, 4th edition,* Wolfram Media and Cambridge University Press, Champaign and Cambridge 1999.