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Approximation by Szász Type Operators Including Sheffer Polynomials

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ABSTRACT: In present article, we discuss voronowskaya type theorem, weighted approximation in terms of weighted modulus of continuity for Szász type operators using Sheffer polynomials. Lastly, we investigate statistical approximation for these sequences.

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1. Introduction

First, we recall n^{th} Bernstein operators due to Bernstein [1] defined as follows

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$ and $0 \le x \le 1$. The purpose of this probabilistic method was to prove Weierstass approximation theorem more elegantly. In 1950, Szász [6] generalized operators given by (1.1) for unbounded interval on the space of continuous functions defined on $(0,\infty)$ as

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad \forall x \in (0,\infty), \qquad n \in \mathbb{N}. \tag{1.2}$$

A new type of generalization of Szász-Mirakjan operators which involves Appell polynomials was given by Jakimovski and Leviatan [4] as follows

$$P_n(f;x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right).$$

In above relation p_k are Appell polynomials defined by the generating functions

$$A(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k,$$

where $A(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 = 0)$ is an analytic function in the disc $|z| < R$ $(R > 1)$ and $A(z) \neq 0$. A more generalized form of Szász operators including Sheffer polynomials was given by Ismail [3]

$$T_n(f;x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{1.3}$$

In above relation p_k are Sheffer polynomials given by the generating functions

$$A(u)e^{xH(u)} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1.4}$$

where

$$\begin{aligned} A(z) &= \sum_{k=0}^{\infty} a_k z^k \qquad (a_0 \neq 0) \\ H(z) &= \sum_{k=0}^{\infty} h_k z^k \qquad (h_1 \neq 0) \end{aligned} \tag{1.5}$$

be analytic functions in the disc $|\, z \,| < R$ $(R > 1)$. Under the following restrictions:

(i) for $x \in [0, \infty)$ and $k \in N \cup 0$, $p_k(x) \geq 0$,

(ii) $A(1) \neq 0$ and $H'(1) = 1$,

(iii) relation (1.4) is valid for $|\, u \,| < R$ and the power series given by (1.5) converges for $|\, z \,| < R$, $R > 1$. Moreover, Ismail introduced the Kantorovich form of the operator (1.3) as

$$T_n^*(f;x) = n\frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)ds. \tag{1.6}$$

Recently, Sucu and Ertan Ibikli [7] proved results on rate of convergence using modulus of continuity for (1.3) and (1.6). Motivated by the above development, we prove weighted approximation, statistical approximation and Voronovskaya type result for T_n in the present paper.
Various investigators such as Gairola et al. [9], Singh et al. [10], Mishra et al. [16-21], Gandhi et al. [22] and the references therein, have discussed the approximation properties of various linear positive operators in this direction.

2. Some properties of the operator T_n

We recall following lemmas due to Sezgin et al. [7]:

Lemma 2.1. *Let* $e_i = t^i$, $i = 0, 1, 2$, $x \in [0, \infty)$, *we have*

$$\begin{aligned} T_n(e_0; x) &= 1, \\ T_n(e_1; x) &= x + \frac{A'(1)}{nA(1)}, \\ T_n(e_2; x) &= x^2 + \left(\frac{2A'(1)}{A(1)} + H^{''}(1) + 1 \right) \frac{x}{n} + \frac{A'(1) + A^{''}(1)}{n^2 A(1)}. \end{aligned}$$

Lemma 2.2. *Let* $\psi_x^i(t) = (t - x)^i, i = 0, 1, 2$, *for* $x \geq 0$ *and* $n \in N$ *we have*

$$\begin{aligned} T_n(\psi_x^0(t); x) &= 1, \\ T_n(\psi_x^1(t); x) &= \frac{A'(1)}{nA(1)}, \\ T_n(\psi_x^2(t); x) &= \left(\frac{H''(1) + 1}{n} \right) + \frac{A'(1) + A''(1)}{n^2 A(1)}. \end{aligned}$$

Next we prove

Lemma 2.3. *For* $x \geq 0$, *we have*

$$\begin{aligned} T_n(e_3; x) &= x^3 + \left(3 + \frac{3A'(1)}{A(1)} + 3H^{''}(1) \right) \frac{x^2}{n} \\ &+ \left(\frac{2 + 3A^{''}(1)}{A(1)} + \frac{6A'(1)}{A(1)} + \frac{3A'(1)H^{''}(1)}{A(1)} + H''(1) + H'''(1) \right) \frac{x}{n^2} \\ &+ \frac{2A'(1) + 3A^{''}(1) + A^{'''}(1)}{n^3 A(1)}, \\ T_n(e_4; x) &= x^4 + \left(6 + \frac{4A'(1)}{A(1)} + 6H^{''}(1) \right) \frac{x^3}{n} \\ &+ \left(11 + \frac{6A^{''}(1)}{A(1)} + \frac{18A'(1)}{A(1)} + 18H''(1) + \frac{9A'(1)H^{''}(1)}{A(1)} + 3(H^{''}(1))^2 \right. \\ &+ \left. 4H'''(1) \right) \frac{x^2}{n^2} + \left(6 + \frac{4A'''(1)}{A(1)} + \frac{18A''(1)}{A(1)} + \frac{22A'(1)}{A(1)} + \frac{6A''(1)H''(1)}{A(1)} \right. \\ &+ \left. \frac{18A'(1)H''(1)}{A(1)} + \frac{4A'(1)H'''(1)}{A(1)} + 6H'''(1) + 11H''(1) + H''''(1) \right) \frac{x}{n^3} \\ &+ \frac{6A'(1) + 11A''(1) + A''''(1)}{A(1)}. \end{aligned}$$

Proof. From the generating functions of Sheffer polynomials, we obtain

$$\begin{aligned}\sum_{K=0}^{\infty} K^3 P_K(nx) &= [(2A'(1)+3A''(1)+A'''(1))\\ &+ nx(3A''(1)+6A'(1)+3A'(1)H''(1)\\ &+ 3A(1)H''(1)+2A(1)+A(1)H'''(1))+n^2 2x^2(3A(1)+3A'(1)\\ &+ 3A(1)H''(1))+n^3x^3A(1)]e^n xH(1),\\ \sum_{K=0}^{\infty} K^4 P_K(nx) &= [(6A'(1)+11A''(1)+6A'''(1)\\ &+ A''''(1))+nx(4A'''(1)+18A''(1)\\ &+ 22A'(1)+6A''(1)H''(1)+18A'(1)H''(1)+4A'(1)H'''(1)\\ &+ 6A(1)H'''(1)+\ 1A(1)H''(1)+6A(1)+A(1)H'''')\\ &+ n^2x^2(11A(1)+18A'(1)\\ &+ 8A(1)H''(1)+6A''(1)+9A'(1)H''(1)\\ &+ 3A(1)(H''(1))^2+4A(1)H'''(1))+n^3x^3(6A(1)+4A'(1)\\ &+ 6A(1)H''(1))+n^4x^4A(1)]e^n xH(1).\end{aligned}$$

The proof of Lemma 2.3 is obvious using these relation. □

Lemma 2.4. *The operator (1.3) satisfies the following relation:*

$$\begin{aligned}T_n(\psi_x^4(t);x) &= \left(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H'')^2+4H''(1)\right)\frac{x^2}{n^2}\\ &+ \left(6+\frac{6A''(1)}{A(1)}+\frac{14A'(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}+\frac{18A'(1)H''(1)}{A(1)}\right.\\ &+ \frac{4A''(1)H''(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}\\ &+ \left.6H'''(1)+11H''(1)+H''''(1)\right)\frac{x}{n^3}\\ &+ \frac{6A'(1)+11A''(1)+A''''(1)}{n^4A(1)}.\end{aligned}$$

Proof. Proof of this relation can be obtained using Lemma 2.1 and linearity property of the operators

$$T_n((t-x)^4;x) = T_n(t^4;x) - 4xT_n(t^3;x) + 6x^2T_n(t^2;x) - 4x^3T_n(t;x) + T_n(1;x).$$

□

3. The Voronovskaya type theorem for T_n

Theorem 3.1. *Let $f \in C^2[0,b]$. Then $\forall\ x \in [0,b]$, we have*

$$\lim_{n\to\infty} n\{T_n(f;x) - f(x)\} = \frac{A'(1)}{A(1)}f'(x) + (H''(1)+1)x\frac{f''(x)}{2!}.$$

Proof. Let $x_0 \in [0,b]$ be a fixed point. Then for $f \in C^2[0,b]$ and $t \in [0,b]$ we have by Taylor's formula

$$f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2}f''(x_0)(t-x)^2 + \varphi(t;x_0)(t-x_0)^2\ ,$$

where $\varphi(t;x_0) \in C[0,b]$ and $\lim_{t\to x_0} g(t;x_0) = 0$. Now, applying the operator on both the side and in the light of linearity property, we have

$$\begin{aligned} T_n(f;x) &= f(x_0)T_n(1;x_0) + f'(x_0)T_n((t-x_0);x_0) + \frac{1}{2}f''(x_0)T_n((t-x)^2;x_0) \\ &\quad + T_n(\varphi(t;x_0)(t-x_0)^2;x_0)). \end{aligned}$$

Subtract $f(x_0)$ and then on multiplying by n both side, we obtain

$$\begin{aligned} n\{T_n(f;x_0) - f(x_0)\} &= f'(x_0)nT_n((t-x_0);x_0) + \frac{f''(x_0)}{2}nT_n((t-x_0)^2;x_0) \\ &\quad + nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big). \end{aligned}$$

We have

$$\begin{aligned} \lim_{n\to\infty} n\{T_n(f;x) - f(x)\} &= \frac{A'(1)}{A(1)}f'(x) + (H''(1)+1)x\frac{f''(x)}{2!} \\ &\quad + \lim_{n\to\infty} nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big). \end{aligned}$$

Using Holder's inequality. The last term can be given by

$$nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big) \le n^2T_n\Big((t-x)^4;x_0\Big)T_n\Big(\varphi(t;x_0)^2;x_0\Big).$$

Let $\eta(t;x_0)=\varphi^2(t;x_0)$. Then $\lim\eta(t;x_0)=\lim\varphi^2(t;x_0)=0$ as $n\to\infty$. By using

$$\lim_{n\to\infty} n^2T_n(\psi_x^4(t);x)=\left(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H''(1)^2)+4H''(1)\right)x^2,$$

we get

$$\lim_{n\to\infty} nT_n\left(\varphi(t;x_0)(t-x)^2;x_0\right)=0,$$

which proves the Theorem 3.1. □

4. Weighted approximation

Here, we recall some notation from [11] to prove next result. Let $B_{1+x^2}[0,\infty)=\{f(x):|f(x)|\leq M_f(1+x^2), 1+x^2$ is weight function, M_f is a constant depending on f and $x\in[0,\infty)\}$, $C_{1+x^2}[0,\infty)$ is the space of continuous function in $B_{1+x^2}[0,\infty)$ with the norm $\|f(x)\|_{1+x^2}=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$ and $C^k_{1+x^2}[0,\infty)=\{f\in C_{1+x^2}:\lim\limits_{|x|\to\infty}\frac{f(x)}{1+x^2}=k$, where k is a constant depending on $f\}$.

Modulus of continuity for the function f defined on closed interval $[0,a]$ with $a>0$ is denoted as follows

$$\omega_a(f,\delta)=\sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|. \tag{4.1}$$

Theorem 4.1. *Let $f\in C_{1+x^2}[0,\infty)$ and $\omega_{b+1}(f;\delta)$ be its modulus of continuity defined on $[0,b+1]\subset[0,\infty)$. Then, we have*

$$\|T_n(f;x)-f(x)\|_{C[0,b]}\leq 6M_f(1+b^2)\delta_n(b)+2\omega_{b+1}(f;\sqrt{\delta_n(b)}),$$

where $\delta_n(b)=T_n(\psi_b^2;b)$.

Proof. From ([12], p. 378), for $x\in[0,b]$ and $t\in[0,\infty)$, we have

$$|f(t)-f(x)|\leq 6M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta).$$

This implies that

$$|T_n(f;x)-f(x)|\leq 6M_f(1+b^2)T_n((t-x)^2;x)+\left(1+\frac{T_n(|t-x|;x)}{\delta}\right)\omega_{b+1}(f;\delta).$$

Thus, using Lemma 2.4, for $x\in[0,b]$, we have

$$|T_n(f;x) - f(x)| \leq 6M_f(1+b^2)\delta_n(b) + \left(1 + \frac{\sqrt{\delta_n(b)}}{\delta}\right)\omega_{b+1}(f;\delta).$$

Choosing $\delta = \delta_n(b)$, we arrive at the desired result. □

Theorem 4.2. *If the operators* T_n *defined by (1.3) from* $C^k_{1+x^2}[0,\infty)$ *to* $B_{1+x^2}[0,\infty)$ *satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(e_i;x) - x^i\|_{1+x^2} = 0, \quad i = 0,1,2,$$

then for each $C^k_{1+x^2}[0,\infty)$

$$\lim_{n\to\infty} \|T_n(f;x) - f\|_{1+x^2} = 0.$$

Proof. To prove this Theorem, it is enough to show that

$$\lim_{n\to\infty} \|T_n(e_i;x) - x^i\|_{1+x^2} = 0, \quad i = 0,1,2.$$

From Lemma 2.2, we have

$$\|T_n(e_0;x) - x^0\|_{1+x^2} = \sup_{x\in[0,\infty)} \frac{|T_n(1;x) - 1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$

$$\begin{aligned} \|T_n(e_1;x) - x^1\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\frac{A'(1)}{nA(1)}}{1+x^2} \\ &= \frac{A'(1)}{nA(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

This implies that $\|T_n(e_1;x) - x^1\|_{1+x^2} \to 0$ an $n \to \infty$. For $i = 2$

$$\begin{aligned} \|T_n(e_2;x) - x^2\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left|\left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)\frac{x}{n} + \frac{A'(1)+A''(1)}{n^2A(1)}\right|}{1+x^2} \\ &\leq \frac{\left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)}{n} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \frac{A'(1) + A''(1)}{n^2A(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

Which shows that $\|T_n(e_2;x) - x^2\|_{1+x^2} \to 0$ an $n \to \infty$. □

Let $f \in C_{\rho}^{k}[0,\infty)$, Yüksel and Ispir [13] introduced weighted modulus of continuity as follows

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Theorem 4.3. *Let* $f \in C_{1+x^2}^{k}[0,\infty)$. *Then*

(i) $\Omega(f;\delta)$ *is a monotone increasing function of* δ*;*

(ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;

(iii) for each $m \in \mathbb{N}$, $\Omega(f;m\delta) \leq m\Omega(f;\delta)$*;*

(iv) for each $\lambda \in [0,\infty)$, $\Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$

and for $t,x \in [0,\infty)$, *one get*

$$|f(t)-f(x)| \leq 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega(f;\delta). \tag{4.2}$$

Theorem 4.4. *Let* $f \in C_{1+x^2}^{k}[0,\infty)$. *Then, we have*

$$\sup_{x\in[0,\infty)} \frac{|T_n(f;x)-f(x)|}{(1+x^2)^3} \leq C\left(1+\frac{1}{n}\right)\Omega\left(f;\frac{1}{\sqrt{n}}\right),$$

where $C > 0$ *is a constant.*

Proof. Using (4.2) and $x,t \in (0,\infty)$, we have

$$\begin{aligned} |T_n(f;x)-f(x)| &\leq 2\left(1+\tfrac{T_n(|t-x|;x)}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right) \\ &\times \left(1+T_n((t-x)^2;x)\right)\Omega(f;\delta). \end{aligned} \tag{4.3}$$

Applying Cauchy-Schwarz inequality for (4.2), we get

$$\begin{aligned} |T_n(f;x)-f(x)| \quad &\leq \quad 2\left(1+\delta^2\right)(1+x^2)\Omega(f;\delta)\Bigg(1+T_n((t-x)^2;x) \\ &+ \quad \tfrac{\sqrt{T_n((t-x)^2;x)}}{\delta} + \tfrac{\sqrt{T_n((t-x)^2;x)T_n((t-x)^4;x)}}{\delta}\Bigg). \end{aligned} \tag{4.4}$$

Using Lemma 2.2 and Lemma 2.4, we get

$$T_n((t-x)^2;x) \leq C_1\frac{(1+x)}{n} \text{ and } T_n((t-x)^4;x) \leq C_2\frac{(1+x+x^2+x^3)}{n}. \tag{4.5}$$

From and (4.3), we have

$$\begin{aligned} |T_n(f;x)-f(x)| &\leq 2\left(1+\delta^2\right)(1+x^2)\Omega(f;\delta)\Bigg(1+C_1\frac{(1+x)}{n} \\ &+ \frac{\sqrt{C_1\frac{(1+x)}{n}}}{\delta}+\frac{\sqrt{C_1\frac{(1+x)}{n}C_2\frac{(1+x+x^2+x^3)}{n}}}{\delta}\Bigg). \end{aligned}$$

On choosing $\delta=\frac{1}{\sqrt{n}}$ and $C=\{1+C_1+\sqrt{C_1}+\sqrt{C_1C_2}\}$, we get the required result. □

Theorem 4.5. *For $f \in C^k_{1+x^2}[0,\infty)$ and $\theta > 0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}}=0.$$

Proof. For any fixed real number $x_0>0$, one has say

$$\begin{aligned} \sup_{x\in[0,\infty)}\tfrac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}} &\leq \sup_{x\leq x_0}\tfrac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}}+\sup_{x\geq x_0}\tfrac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}} \\ &\leq \||T_n(f;x)|-f(x)\|_{C[0,x_0]} \\ &+ \|f\|_{1+x^2}\sup_{x\geq x_0}\tfrac{|T_n(1+t^2;x)|}{(1+x^2)^{1+\theta}}+\sup_{x\geq x_0}\tfrac{|f(x)|}{(1+x^2)^{1+\theta}} \\ &= I_1+I_2+I_3. \end{aligned} \tag{4.6}$$

Since $|f(x)| \leq \|f\|_{1+x^2}(1+x^2)$, we have

$$\begin{aligned} I_3 &= \sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\theta}} \\ &\leq \sup_{x\geq x_0}\frac{\|f\|_{1+x^2}(1+x^2)}{(1+x^2)^{1+\theta}} \leq \frac{\|f\|_{1+x^2}}{(1+x^2)^{\theta}}. \end{aligned}$$

Let $\epsilon>0$ be arbitrary real number. Then, from Theorem 4.2 there exists $n_1\in\mathbb{N}$ such that

$$\begin{aligned} I_2 &< \frac{1}{(1+x^2)^{\theta}}\|f\|_{1+x^2}\Big(1+x^2+\frac{\epsilon}{3\|f\|_{1+x^2}}\Big) \text{ for all } n_1\geq n, \\ &< \frac{\|f\|_{1+x^2}}{(1+x^2)^{\theta}}+\frac{\epsilon}{3} \text{ for all } n_1\geq n. \end{aligned}$$

This implies that

$$I_2+I_3<2\frac{\|f\|_{1+x^2}}{(1+x^2)^{\theta}}+\frac{\epsilon}{3}.$$

Next, let for a large value of x_0, we have $\frac{\|f\|_{1+x^2}}{(1+x^2)^\theta} < \frac{\epsilon}{6}$.

$$I_2 + I_3 < \frac{2\epsilon}{3} \text{ for all } n_1 \geq n. \tag{4.7}$$

From Theorem 4.2, there exists $n_2 > n$ in such a way

$$I_1 = \|T_n(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3} \text{ for all } n_2 \geq n. \tag{4.8}$$

Let $n_3 = max(n_1, n_2)$. Then, combining (4.6), (4.7) and (4.8), we have

$$\sup_{x\in[0,\infty)} \frac{|T_n(f;x)| - f(x)}{(1+x^2)^{1+\theta}} < \epsilon.$$

Hence, the proof of Theorem 4.5 is completed. □

5. A-statistical approximation

Gadjiev et al. [14] was the first who introduced Statistical approximation theorems in operators theory. Here, we recall same notation from [14], let $A = (a_{nk})$ be a non-negative infinite suitability matrix. For a given sequence $x := (x_k)$, the A-transform of x denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each n. A is said to be regular if $\lim (Ax)_n = L$ whenever $\lim x = L$. Then $x = (x_n)$ is said to be a A-statistically convergent to L i.e. $st_A - \lim\ x = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \epsilon} a_{nk} = 0$.

Theorem 5.1. *Let $A = (a_{nk})$ be a non-negative regular suitability matrix and $x \geq 0$. Then, we have*

$$st_A - \lim_n \|T_n(f;x) - f\|_{1+x^{2+\lambda}} = 0, \text{ for all } f \in C^k_{1+x^{2+\lambda}}[0,\infty) \text{ and } \lambda > 0.$$

Proof. From ([15], p. 191, Th. 3), it is sufficient to show that for $\lambda = 0$

$$st_A - \lim_n \|T_n(e_i;x) - e_i\|_{1+x^2} = 0, \text{ for } i \in \{0,1,2\}. \tag{5.1}$$

Using Lemma 2.2, we have

$$\begin{aligned} \|T_n(e_1;x) - x\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \left| \frac{A'(1)}{nA(1)} \right| \\ &= \frac{A'(1)}{nA(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$\begin{aligned} M_1 : &= \left\{ n : \|T_n(e_1;x) - x\| \geq \epsilon \right\}, \\ M_2 : &= \left\{ n : \frac{A'(1)}{nA(1)} \geq \epsilon \right\}. \end{aligned}$$

This implies that $M_1 \subseteq M_2$, which shows that $\sum_{k \in M_1} a_{nk} \leq \sum_{k \in M_2} a_{nk}$. Hence, we have

$$st_A - \lim_n \|T_n(e_1;x) - x\|_{1+x^2} = 0. \tag{5.2}$$

For $i = 2$ and using Lemma 2.2, we have

$$\begin{aligned} \|T_n(e_2;x) - x^2\|_{1+x^2} &= \sup_{x \in [0,\infty)} \frac{x}{1+x^2} \left| \left(\frac{2A'(1)}{A(1)} + H^{''}(1) + 1 \right) \frac{1}{n} \right. \\ &+ \frac{A'(1) + A^{''}(1)}{n^2 A(1)} \sup_{x \in [0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

For a given $\varepsilon > 0$, we have the following sets

$$\begin{aligned} T_1 : &= \left\{ n : \left\| T_n(e_2;x) - x^2 \right\| \geq \epsilon \right\}, \\ T_2 : &= \left\{ n : \left(\frac{2A'(1)}{A(1)} + H^{''}(1) + 1 \right) \frac{1}{n} \geq \frac{\epsilon}{2} \right\}, \\ T_3 : &= \left\{ n : \frac{A'(1) + A^{''}(1)}{n^2 A(1)} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

This implies that $T_1 \subseteq T_2 \bigcup T_3$. By which, we get

$$\sum_{k \in T_1} a_{nk} \leq \sum_{k \in T_2} a_{nk} + \sum_{k \in T_3} a_{nk}.$$

As $n \to \infty$, we have

$$st_A - \lim_n \|T_n(e_2;x) - x^2\|_{1+x^2} = 0. \tag{5.3}$$

This completes the proof of Theorem 5.1. □

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