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Minimizing the time spent in an interval by a Wiener process with uniform jumps*

by

Mario Lefebvre

Department of Mathematics and Industrial Engineering,
Polytechnique Montréal, C.P. 6079, Succursale Centre-ville,
Montréal, Québec, Canada H3C 3A7.
mlefebvre@polymtl.ca

Abstract: Let $X_u(t)$ be a controlled Wiener process with jumps that are uniformly distributed over the interval $[-c, c]$. The aim is to minimize the time spent by $X_u(t)$ in the interval $[a, b]$. The integro-differential equation, satisfied by the value function, is transformed into an ordinary differential equation and is solved explicitly for a particular case. The approximate solution obtained is precise when c is small.

Keywords: Brownian motion, Poisson process, first-passage time, optimal stochastic control, integro-differential equation

1. Introduction

Let $\{Y_1, Y_2, \ldots\}$ be independent and identically distributed random variables having the density function

$$f_Y(y) = \frac{1}{2c} \quad \text{for } -c \leq x \leq c. \tag{1}$$

That is, Y_i is uniformly distributed over the interval $[-c, c]$ for $i = 1, 2, \ldots$ We consider the controlled jump-diffusion process $\{X_u(t), t \geq 0\}$, defined by

$$X_u(t) = X_u(0) + \mu t + b_0 \int_0^t u[X_u(s)]\, \mathrm{d}s + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{2}$$

where μ, $b_0 > 0$ and $\sigma > 0$ are constants, $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Hence, the continuous part of $\{X_u(t), t \geq 0\}$ is a controlled Wiener process with infinitesimal parameters μ and σ^2. The processes $\{B(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are assumed to be independent.

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Next, we define the first-passage time

$$T(x) = \inf\{t \geq 0 : X_u(t) \notin (a,b) \mid X_u(0) = x \in [a,b]\}. \tag{3}$$

Our aim is to find the control that minimizes the expected value of the cost criterion

$$J(x) := \int_0^{T(x)} \left\{ \frac{1}{2} q_0 \, u^2[X_u(t)] + \theta \right\} \mathrm{d}t, \tag{4}$$

where q_0 and θ are positive constants. Because $\theta > 0$, the optimizer wants the controlled process to leave the interval $[a,b]$ as soon as possible, while taking the quadratic control costs into account. We could try, instead, to maximize the survival time in $[a,b]$ by choosing $\theta < 0$. We could also consider a risk-sensitive version of the cost criterion.

This type of problem is known as *LQG homing*. Whittle (1982) has proven that, when $\lambda = 0$, it is sometimes possible to linearize the differential equation satisfied by the value function. Actually, in the particular problem defined above, one can transform the stochastic control problem into a purely probabilistic problem for the uncontrolled process $X_0(t)$, obtained by setting $u(\cdot) \equiv 0$ in (2). The optimal control $u^*(x)$ can be expressed in terms of the moment-generating function of the random variable $T_0(x)$ that corresponds to $T(x)$.

In the case of jump-diffusion processes with jumps of constant size ϵ, Lefebvre (2014) (see also Theodorou and Todorov, 2012) has shown that, if ϵ is small, the optimal control can again be obtained (approximately, this time) by computing a mathematical expectation for the uncontrolled process $X_0(t)$. He was able to solve a particular problem by making use of the results obtained by Abundo (2000).

In the present paper, instead of linearizing the equation satisfied by the value function, we will transform the appropriate integro-differential equation into an (approximate) non-linear differential equation. The theoretical results will be presented in Section 2, and a particular problem will be solved explicitly in Section 3. We will see that the approximate solution is almost an exact one if the constant c is small.

2. Approximate differential equation

The infinitesimal generator of the uncontrolled process $\{X_0(t), t \geq 0\}$, defined by

$$X_0(t) = x + \mu t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{5}$$

is (see Kou and Wang, 2003)

$$\mathcal{L}g(x) = \frac{1}{2}\sigma^2 g''(x) + \mu g'(x) - \lambda g(x) + \lambda \int_{-c}^{c} g(x+y)\,\frac{1}{2c}\,\mathrm{d}y \tag{6}$$

for $x \in (a, b)$, where $g(x)$ is a twice continuously differentiable function.

Next, we define the value function

$$F(x) = \inf_{u[X_u(t)],\ 0 \le t \le T(x)} E[J(x)]. \tag{7}$$

By making use of dynamic programming, the optimal control can be expressed as follows:

$$u^*(x) = -\frac{b_0}{q_0} F'(x). \tag{8}$$

Moreover, we can prove the following proposition, which generalizes the result obtained in Lefebvre (2014).

PROPOSITION 2.1 *The value function, for the process defined in (2), satisfies the second-order, non-linear integro-differential equation*

$$0 = \theta + \mu F'(x) - \frac{1}{2}\frac{b_0^2}{q_0}\,[F'(x)]^2 + \frac{1}{2}\sigma^2 F''(x) + \lambda \int_{-c}^{c} [F(x+y) - F(x)]\,\frac{1}{2c}\,\mathrm{d}y \tag{9}$$

for $a < x < b$. Moreover, we have the boundary conditions

$$F(x) = 0 \quad \text{if } x \le a \text{ or } x \ge b. \tag{10}$$

Now, using Taylor's formula, we can write that

$$F(x+y) = F(x) + y F'(x) + \frac{1}{2} y^2 F''(x) + o(y^2). \tag{11}$$

It follows that

$$\int_{-c}^{c} [F(x+y) - F(x)]\,\frac{1}{2c}\,\mathrm{d}y \simeq E[Y]\,F'(x) + \frac{1}{2} E[Y^2]\,F''(x). \tag{12}$$

Since $E[Y] = 0$ and $E[Y^2] = c^2/3$, Eq. (9) becomes

$$0 \simeq \theta + \mu F'(x) - \kappa\,[F'(x)]^2 + \frac{1}{2}\left(\sigma^2 + \frac{\lambda c^2}{3}\right) F''(x), \tag{13}$$

where

$$\kappa := \frac{1}{2}\frac{b_0^2}{q_0}. \tag{14}$$

Remarks. (i) Equation (13) is appropriate if $x + c \leq b$ and $x - c \geq a$. If $x + c > b$, we can write more precisely (using the boundary condition $F(x) = 0$ if $x \geq b$) that

$$0 = \theta + \mu F'(x) - \kappa \left[F'(x)\right]^2 + \frac{1}{2}\sigma^2 F''(x) - \lambda F(x) + \frac{\lambda}{2c}\int_{-c}^{b-x} F(x+y)\,\mathrm{d}y. \quad (15)$$

Similarly, if $x - c < a$, we have

$$0 = \theta + \mu F'(x) - \kappa \left[F'(x)\right]^2 + \frac{1}{2}\sigma^2 F''(x) - \lambda F(x) + \frac{\lambda}{2c}\int_{a-x}^{c} F(x+y)\,\mathrm{d}y. \quad (16)$$

For c large enough, we can have both $x + c > b$ and $x - c < a$.

We can still transform Eq. (15) and Eq. (16) into approximate ordinary differential equations, but they will not be with constant coefficients, so that they will be more difficult to solve explicitly.

(ii) Proceeding as in Whittle (1982), we can linearize the second-order differential equation (13). To do so, we define

$$\Phi(x;\alpha) = e^{-\alpha F(x)}, \quad (17)$$

where

$$\alpha := \frac{b_0^2}{q_0}\frac{1}{\sigma^2 + \frac{\lambda c^2}{3}}. \quad (18)$$

However, it is also possible to solve this equation directly. Notice that Eq. (13) is a Riccati equation for $G(x) := F'(x)$.

In the next section, a particular problem will be solved explicitly. We will see that, as expected, the approximate solution is more precise when c is small.

3. A particular case

We assume that $\mu = 0$ and $b_0 = q_0 = \lambda = \theta = \sigma = 1$. Moreover, we choose the interval $[0, 1]$. Equation (13) reduces to

$$0 \simeq 1 - \frac{1}{2}\left[F'(x)\right]^2 + \frac{1}{2}\left(1 + \frac{c^2}{3}\right) F''(x). \quad (19)$$

The solution that satisfies the boundary conditions $F(0) = F(1) = 0$ is

$$F(x) = \sqrt{2}\,x - \frac{1}{\alpha}\ln\left(\frac{e^{\sqrt{2}\alpha} + e^{2\sqrt{2}\alpha x}}{e^{\sqrt{2}\alpha} + 1}\right), \quad (20)$$

where (see Eq. (18))

$$\alpha = \frac{3}{3 + c^2}. \quad (21)$$

If we substitute the above expression for the function $F(x)$ into the integro-differential equation (see Eq. (9))

$$0 = 1 - \frac{1}{2}\left[F'(x)\right]^2 + \frac{1}{2}F''(x) - F(x) + \int_{-x}^{1-x} F(x+y)\,\frac{1}{2}\,\mathrm{d}y \tag{22}$$

(where we have used the fact that $F(x) = 0$ if $x \leq 0$ or $x \geq 1$), we obtain that the right-hand side of the preceding equation varies from a maximum of approximately 0.25 (at $x = 0$ and $x = 1$) to a minimum of about 0.13 (at $x = 1/2$); see Fig. 1. The error is therefore not negligible.

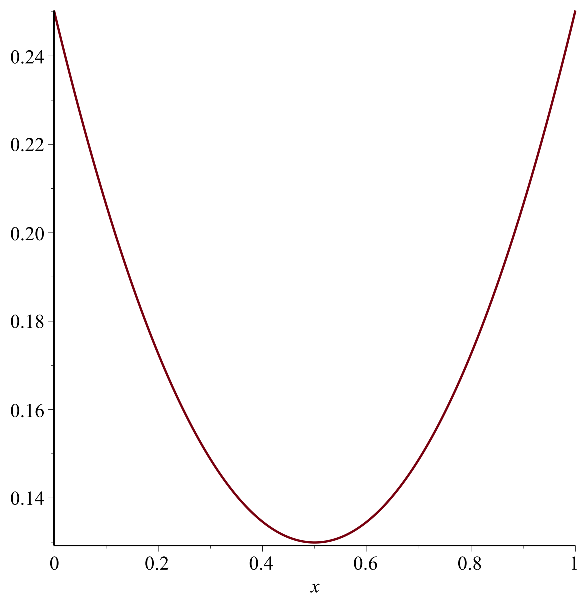

Figure 1. Right-hand side of Eq. (22) in the interval $[0, 1]$

Next, with $c = 1/10$, for $x \in [1/10, 9/10]$ we replace the function $F(x)$ into

$$0 = 1 - \frac{1}{2}\left[F'(x)\right]^2 + \frac{1}{2}F''(x) + 5\int_{-1/10}^{1/10} \left[F(x+y) - F(x)\right]\mathrm{d}y. \tag{23}$$

The right-hand side of the above equation is shown in Fig. 2.

The maximum error is approximately equal to 6.6×10^{-6}. Therefore, the approximate solution is much more precise than when $c = 1$.

For $x \in [0, 1/10)$, the integro-differential equation becomes

$$0 = 1 - \frac{1}{2}\left[F'(x)\right]^2 + \frac{1}{2}F''(x) - F(x) + 5\int_{-x}^{1/10} F(x+y)\,\mathrm{d}y. \tag{24}$$

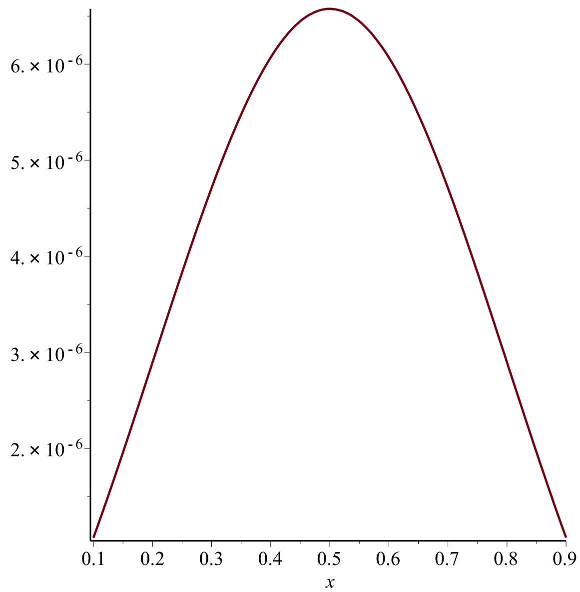

Figure 2. Right-hand side of Eq. (23) in the interval $[1/10, 9/10]$

We present in Fig. 3 the right-hand side of this equation for the function $F(x)$ given in (20), with $\alpha = 300/301$.

The maximum error, at $x = 0$, is approximately 0.022. In the interval $(9/10, 1]$, we obtain the same results, with x replaced by $1 - x$, which actually follows by symmetry. Hence, we can conclude that the expression obtained for the value function $F(x)$ is acceptable when $c = 1/10$. The (approximate) optimal control is shown in Fig. 4.

Remarks. (i) The solution of Eq. (13) depends on λc^2. However, the right-hand side of the integro-differential equation (9) depends on λ/c and c. The approximate value function is much more precise for small values of c.

(ii) As mentioned in the previous section, we should actually use Eq. (13) in the interval $[a + c, b - c]$. For $x \in (b - c, c]$, the appropriate integro-differential equation is Eq. (15) (assuming that $b - 2c \geq a$). When $c = 1/10$, we find that this equation can be transformed into the following approximate ordinary differential equation:

$$\begin{aligned} 0 \quad \simeq \quad & 1 - \frac{1}{2}\left[F'(x)\right]^2 + 5\left(\frac{(1-x)^2}{2} - \frac{1}{200}\right) F'(x) \\ & + \frac{1}{2}\left[1 + \frac{5}{3}\left((1-x)^3 + \frac{1}{1000}\right)\right] F''(x). \end{aligned} \tag{25}$$

In theory, we should first try to solve the above equation subject to the boundary condition $F(1) = 0$, and next use the general solution of Eq. (19) in the interval

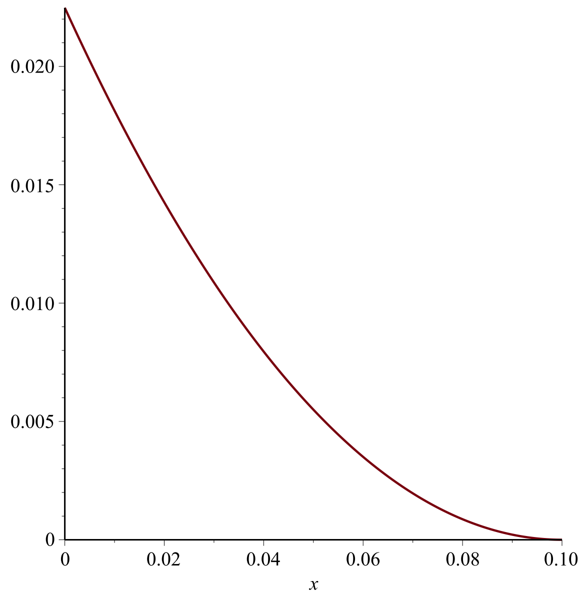

Figure 3. Right-hand side of Eq. (24) in the interval $[0, 1/10)$

$[1/10, 9/10]$. Because the function $F(x)$ must be continuous, we can determine the remaining arbitrary constant at $x = 9/10$. However, since the solution of Eq. (19) is very precise in the interval $[1/10, 9/10]$ when $c = 1/10$, we can more simply try to solve Eq. (25) subject to the conditions $F(9/10) \simeq 0.07925$ (which is the value of the solution obtained above) and $F(1) = 0$. The analytical solution of Eq. (25) is very involved, but we can solve it numerically. It turns out that the numerical solution is almost the same as the value of the solution of Eq. (19) in the interval $(9/10, 1]$. Similarly, by symmetry, for $x \in [0, 1/10)$.

(iii) In the general formulation of the LQG homing problems, there is a final cost given by $K[T(x), X(T(x))]$. Above, we assumed that $K(\cdot, \cdot) \equiv 0$. Suppose, instead, that the function K is of the same form as the value function $F(x)$ given in Eq. (20), so that

$$K = K[X(T(x))] = \sqrt{2}X(T(x)) - \frac{1}{\alpha} \ln \left(\frac{e^{\sqrt{2}\alpha} + e^{2\sqrt{2}\alpha X(T(x))}}{e^{\sqrt{2}\alpha} + 1} \right). \tag{26}$$

Then, the boundary condition in (10) becomes

$$F(x) = K(x) = \sqrt{2}x - \frac{1}{\alpha} \ln \left(\frac{e^{\sqrt{2}\alpha} + e^{2\sqrt{2}\alpha x}}{e^{\sqrt{2}\alpha} + 1} \right) \tag{27}$$

if $x \in (-1/10, 0]$ or $x \in [1, 11/10]$. Notice that the final cost is actually a reward if $X(T(x)) < 0$ or $X(T(x)) > 1$. With the final cost function defined in (26),

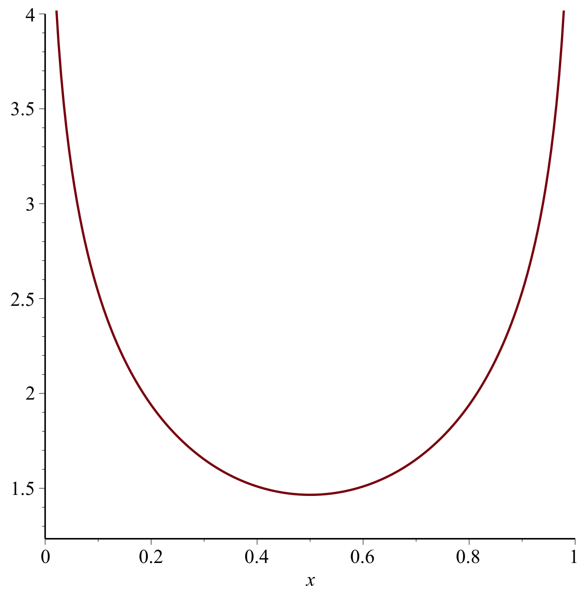

Figure 4. Approximate optimal control in the interval $[0, 1]$ when $c = 1/10$

Eq. (23) is valid for any $x \in [0, 1]$. Moreover, we find that the right-hand side of this equation, with the function $F(x)$ defined in Eq. (20), is of the order of 10^{-6} for $x \in [0, 1]$, so that the approximate control $u^*(x)$ is then practically the exact solution to the optimal control problem.

4. Conclusion

In this note, the LQG homing problem was considered for a controlled Wiener process with random Poissonian jumps. The aim was to leave the interval $[a, b]$ as soon as possible, and the jumps were assumed to be uniformly distributed over the interval $[-c, c]$.

We transformed the integro-differential equation, satisfied by the value function, into an approximate non-linear ordinary differential equation of order two with constant coefficients. This ordinary differential equation (o.d.e.) was solved explicitly in Section 3 in the case of a controlled standard Brownian motion with jumps. As expected, we found that the approximate solution was more precise for small values of the constant c.

In theory, we could extend the results obtained in Section 2 to the case of a general time-homogeneous controlled diffusion process with jumps. However, the o.d.e. that corresponds to Eq. (13) will in general no longer be an equation with constant coefficients. Therefore, it will probably be quite difficult to solve this equation analytically. We could, however, at least use numerical methods to solve it.

Instead of a uniform distribution, we could have used, for instance, a beta distribution, which is also bounded. In some applications, in particular in financial mathematics, it is not realistic to assume that the jumps, especially the negative ones, can be as large as we want (in absolute value).

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