Free Vibrations Of Iso- and Orthotropic Plates Considering Plate Variable Thickness and Interaction With Water

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Abstract

The natural vibrations of thin (Kirchhoff-Love) plates with constant and variable thickness are considered in the paper. Isotropic and orthotropic rectangular plates with different boundary conditions are analysed. The Finite Element Method and the Finite Difference Method are used to describe structural deformation. The elements of stiffness matrix are derived numerically using author's approaches of localization of integration points. The plate inertia forces are expressed by diagonal, lumped mass matrix or consistent mass matrix. The presence of the external medium, which can be a fluid, is described by the fluid velocity potential of double layer and the fundamental solution of Laplace equation which leads to the fully-populated mass matrix. The influence of external additional liquid mass on natural frequencies of plate is analysed, too.

Keywords: thin plates, vibrations, Finite Element Method, Finite Difference Method, adjoined mass

1. Introduction

Structural dynamics is the subject of many studies and the thematic literature is very extensive. Nowacki [1] described in detail natural and forced vibration problems of bar, string, plate and shell structures. Nerantzaki and Katsikadelis [2] carried out the dynamic analysis of thin plates with variable thickness in terms of the Boundary Element Method (BEM) and the Analog Equation Method (AEM). The studies on the influence of the adjoined air mass on natural vibration of thin surfaces were done by e.g. Jones and Moore [3], Sygulski [4] as well as Lee and Lee [5]. Rakowski and Guminiak [6] presented non-linear vibrations of Timoshenko beams in terms of the Finite Element Method (FEM) in combination with the Finite Difference Method (FDM). Plate eigenvibrations, similar to those presented in this paper, were described and solved by Kamiński [7]. Guminiak [8] presented natural and forced vibrations of thin plates including fluid-structure interaction using the BEM in terms of the modified formulation of boundary conditions.

The free vibration analysis of rectangular thin (Kirchhoff-Love) plates with constant and linearly variable thickness are considered in the paper. The FEM and the FDM are used to describe a plate deformation. The fluid-plate interaction is analysed, too. It is assumed that the inviscid and incompressible fluid resting on the plate makes no separate flow resulting from this plate vibration. The presence of the fluid is manifested by means of the fully-populated fluid mass matrix which is adjoined to the structural mass matrix.

2. Description of thin plate deformation in terms of the Finite Element Method

The natural vibration problem of the structure can be described by generalized eigenvalue problem, which in matrix notation has the following form

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{w} = \mathbf{0} \tag{1}$$

Where **K** and **M** are the stiffness and mass matrices of the structure respectively, ω is the natural frequency (eigenvalue) of the structure and **w** is the non-zero vector of dynamic degrees of freedom (eigenvector).

To describe the plate deformation, a rectangular four-node finite element with three degrees of freedom at each node is used [9]. At the node *i* in the cartesian coordinate system there are introduced following degrees of freedom: deflection w_i and two angles of rotation in mutually perpendicular directions – φ_{ix} and φ_{iy} respectively, where $\varphi_{ix} = w_{i,y}$ and $\varphi_{iy} = -w_{i,x}$, thus

$$\mathbf{w}_i^e = \{w_i \quad \varphi_{ix} \quad \varphi_{iy}\}^T, \; i = 1,2,3,4. \tag{2}$$

The function of deflection can be expressed as the polynomial of the fourth order [9]

$$\begin{aligned} w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \\ + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} yx^3 + \alpha_{12} xy^3 \end{aligned} \tag{3}$$

where $\alpha_{11} \neq 0$ or $\alpha_{12} \neq 0$.

The displacement field within single element can be expressed in the following form

$$w^e(x,y) = N_j^{(i)} w_j^{(i)} = \mathbf{N}^e \mathbf{w}^e; \; i = 1,2,3,4; \; j = 1,2,3, \tag{4}$$

where *i* expresses the number of the actual node and *j* expresses the current degree of freedom at i^{th} node. The set of shape functions $N_j^{(i)}$ for each node can be given by formulas derived directly from relation (3). The detailed description of the considered finite element is given by Kuczma [9].

The plate stiffness matrix can be derived analytically e.g. based on [9] or numerically using the Gauss method or another approach, where the integration points are located as in Fig. 1. In this case the integral of the function $f(x,y)$ over the area Ω can be expressed by an approximate formula

$$\int_\Omega f(x,y)\,d\Omega \approx \sum_{i,j} w_{ij} f(\xi_i, \eta_j) \left|J(\xi_i, \eta_j)\right|, \tag{5}$$

where the weights w_{ij} are the areas of the sub-domains into which the finite element has been divided and J is the Jacobian transforming the area Ω into the area $[-1,1] \times [-1,1]$.

The stiffness matrix of the finite element is defined according to FEM methodology

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV = h \int_A \mathbf{B}^T \mathbf{D} \mathbf{B}\, dA, \tag{6}$$

where h is the plate thickness, **B** is the shape functions derivatives matrix and

$$\mathbf{D} = [D_{ij}]; \; i = 1,2,3; j = 1,2,3 \tag{7}$$

is the matrix determining elastic properties of the anisotropic material. For isotropic materials the non-zero elements are $D_{11} = D_{22} = D = Eh^3/\left(12(1-\nu^2)\right)$, $D_{12} = D_{21} =$

$= D\nu$ and $D_{33} = 0.5(1-\nu)D$, where E and ν are the plate Young's modulus and the Poisson's ratio respectively.

The plate mass matrix **M** can be determined as the diagonal, lumped mass matrix or as the consistent matrix using shape functions.

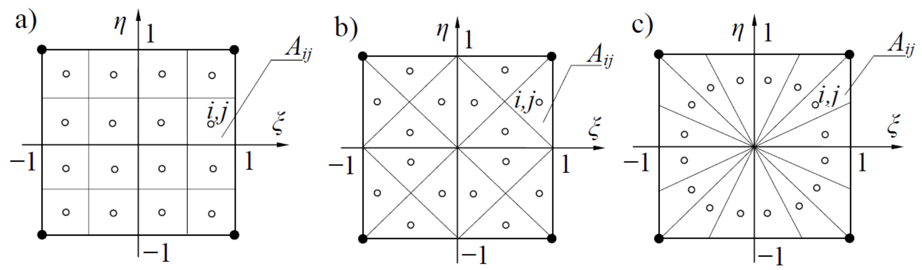

Figure 1. Localization of integration points as geometric centres of created sub-areas

3. Description of natural vibration of isotropic thin plate considering variable thickness in terms of the Finite Difference Method

The natural vibration problem of isotropic thin plate with variable thickness can be described by the following differential equation [2]

$$D\nabla^4 w + 2\frac{\partial D}{\partial x}\nabla^2\frac{\partial w}{\partial x} + 2\frac{\partial D}{\partial y}\nabla^2\frac{\partial w}{\partial y} + \\ -(1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = b(x,y,t) \tag{8}$$

where $D = D(x,y)$ is the thin plate stiffness and $b(x,y,t)$ is the continuous set of inertia forces. It is assumed that the plate thickness $h = h(x,y)$ is a continuous and smooth function of two variables x and y in the plate domain.The differential equation (8) can be replaced by the following difference equation which have the character of an amplitude equation:

$$D\left(\frac{\Delta^4 w}{\Delta x^4} + 2\frac{\Delta^4 w}{\Delta x^2\Delta y^2} + \frac{\Delta^4 w}{\Delta y^4}\right) + 2\left(\frac{\Delta D}{\Delta x}\right)\left(\frac{\Delta^2}{\Delta x^2} + \frac{\Delta^2}{\Delta y^2}\right)\frac{\Delta w}{\Delta x} + 2\left(\frac{\Delta D}{\Delta y}\right)\left(\frac{\Delta^2}{\Delta x^2} + \frac{\Delta^2}{\Delta y^2}\right)\frac{\Delta w}{\Delta y} + \\ -(1-\nu)\left(\left(\frac{\Delta^2 D}{\Delta x^2}\right)\frac{\Delta^2 w}{\Delta y^2} + 2\left(\frac{\Delta^2 D}{\Delta x\Delta y}\right)\frac{\Delta^2 w}{\Delta x\Delta y} + \left(\frac{\Delta^2 D}{\Delta y^2}\right)\frac{\Delta^2 w}{\Delta x^2}\right) = B \tag{9}$$

where the operator $\frac{\Delta^n}{\Delta x^n}(\dots)$ indicates the central difference of the n-th order of the function of two variables x and y with respect to the variable x. In formula (9) the central differences are an approximation of derivatives of functions $w(x,y)$ and $D(x,y)$ at selected points belonging to the plate domain. $B = B(x,y)$ is the set of inertia forces acting in nodes and expressed per unit of sub-domain $\Delta x\Delta y$. The difference operators are created for each internal central point $(i,j) := (x_i, y_j)$ using the set of thirteen points with regular arrangement (Fig. 2) according to [10].

The difference procedures lead to the following matrix equation which has the character of the generalized eigenvalue problem

$$\left(\mathbf{\Delta} - \omega^2 \mathbf{M}\right)\mathbf{w} = \mathbf{0} \tag{10}$$

where $\mathbf{M}$ is a diagonal matrix containing plate masses m_i, concentrated in subsequent FDM mesh nodes, divided by their surface area.

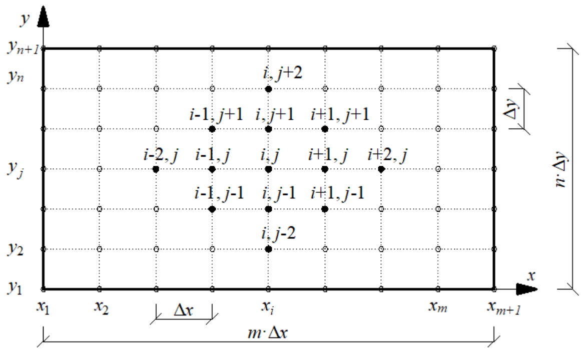

Figure 2. The set of the finite difference points dividing the plate area

4. Liquid medium as the source of additional inertia forces

It is assumed that a plate is fully immersed in a fluid and vibrates at small deflection amplitudes. A fluid is incompressible, inviscid and adheres tightly to the plate. Constructing the velocity potential of a fluid, after discretization of the plate surface into sub-domains with areas S_n, the displacement amplitude in arbitrary point $\tilde{w}(x_m, y_m)$ can be coupled with the hydrodynamic pressure amplitude [8]

$$\omega^2 \tilde{w}(x_m, y_m) = -\frac{1}{4\pi\rho_\mathrm{f}} \sum_{n=1}^{N} \Delta\tilde{p}_n \int_{S_n} \frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right]_{z\to 0} dS_n \tag{11}$$

where ρ_f is the fluid density. The relation (11) can be written in matrix notation

$$-4\pi\rho_\mathrm{f}\,\omega^2 \tilde{\mathbf{w}} = \mathbf{H}\,\tilde{\mathbf{p}} \tag{12}$$

where the element of matrix $\mathbf{H}$ is defined as follows

$$H_{mn} = \int_{S_n} \frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right]_{z\to 0} dS_n \tag{13}$$

which can be evaluated analytically (Fig. 3) [4], [8]

$$H_{mn} = -\frac{1}{y_p}\left(\frac{x_q}{r_2} - \frac{x_p}{r_1}\right) + \frac{1}{x_q}\left(\frac{y_q}{r_3} - \frac{y_p}{r_2}\right) + \frac{1}{y_q}\left(\frac{x_q}{r_3} - \frac{x_p}{r_4}\right) - \frac{1}{x_p}\left(\frac{y_q}{r_4} - \frac{y_p}{r_1}\right) \tag{14}$$

The vector of hydrodynamic forces acting on a plate can be specified by the relations

$$\mathbf{P} = -\mathbf{M}_{\mathrm{f}}\,\omega^2\mathbf{w} \tag{15}$$

and the fluid mass matrix is defined as follows

$$\mathbf{M}_{\mathrm{f}} = 4\pi\rho_{\mathrm{f}}\,\mathbf{S}\mathbf{H}^{-1} \tag{16}$$

where $P_n = \Delta p_n\, S_n$, $\mathbf{S} = \mathrm{diag}(S_1 ... S_N)$ collects values of areas of the individual sub-domains and N is the number of them.

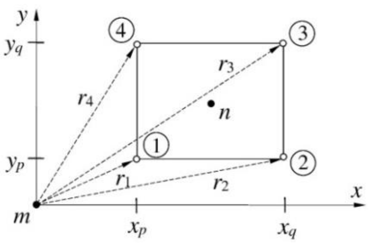

Figure 3. Designations of rectangular sub-domain for the fluid [4], [8]

Finally the mass matrix of the fluid-plate system $\mathbf{M}_{\mathrm{pf}}$ is the sum of the consistent or lumped (diagonal) plate mass matrix and fully-populated fluid mass matrix

$$\mathbf{M}_{\mathrm{pf}} = \mathbf{M}_{\mathrm{p}} + \mathbf{M}_{\mathrm{f}} \tag{17}$$

wherein the components of inertia forces derived from the presence of a fluid should be added to the components of plate inertia forces coupled with translational displacement (deflections).

5. Numerical examples

In the examples presented below, the natural frequencies of selected plate types are determined. The eigenvalue problem (1) or (10) is obtained using the FEM and FDM approaches and received results are compared with the analytical [1], [11] or the BEM solutions [8]. The mass matrix **M** is assumed to be consistent and diagonal in FEM and FDM respectively. The stiffness matrix **K** appearing in the formula (1) is determined using the approximation shown in the formula (5) and in the Fig. 1a – 1c.

<u>Example 1.</u> The square simply-supported isotropic plate in vacuum and fully immersed in water is considered. The material properties are: E = 205 GPa, ν_{p} = 0.3 and $\rho_{\mathrm{p}} = 7850$ kg/m^3. The plate dimensions are $l_x \times l_y \times h = (2.0 \times 2.0 \times 0.01)\,\mathrm{m}$. The FEM mesh and the FDM grid dimension is 20×20. The natural frequencies for the plate are presented in Table 1. The modes of vibrations for both plates are shown in the Fig. 4.

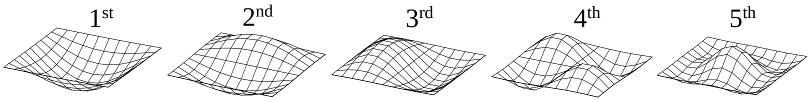

Figure 4. Modes of vibration of the simply-supported isotropic plate [8]

Table 1. Natural frequencies for the plate in vacuum or fully immersed in water

Mode	Solution – natural frequencies ω [rad/s]					
	FEM (a)	FEM (b)	FEM (c)	FDM	[8]	[1]
Plate in vacuum						
1	76.204	76.202	76.197	76.195	76.318	76.313
2, 3	190.322	190.305	190.269	189.549	190.710	190.783
4	303.526	303.492	303.420	302.902	304.918	305.253
5	380.417	380.338	380.168	375.368	380.805	381.567
Plate immersed in water						
1	19.665	19.664	19.662	19.441	19.471	
2,3	62.663	62.657	62.644	61.522	61.940	
4	111.994	111.979	111.947	109.763	110.670	
5	146.542	146.508	146.436	141.257	143.722	

Example 2. The square clamped isotropic plate in vacuum and fully immersed in water is considered. The material properties, the plate dimensions l_x, l_y, the FEM mesh and the FDM grid are the same as in the previous Example 1. The plate thickness h = 0.05 m. Obtained natural frequencies for the plate are presented in Table 2. The modes of vibrations for both plates are shown in the Fig. 5.

Table 2. Natural frequencies for the plate in vacuum or fully immersed in water

Mode	Solution – natural frequencies ω [rad/s]					
	FEM (a)	FEM (b)	FEM (c)	FDM	[8]	[11]
Plate immersed in water						
1	38.716	38.715	38.713	38.146	37.581	34.911
2	108.985	108.980	108.963	117.352	105.379	114.185
3	260.234	260.214	260.167	253.402	256.539	212.750
4	365.034	365.027	365.000	368.981	359.983	376.595
Plate in vacuum						
1	67.087	67.086	67.085	68.018	66.924	
2	164.372	164.369	164.356	183.713	164.302	
3	411.253	411.240	411.207	406.904	415.787	
4	524.961	524.969	524.974	546.831	533.809	

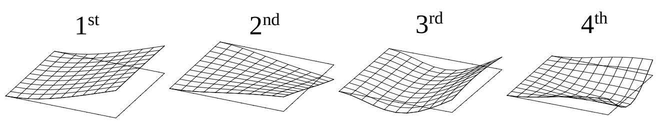

Figure 5. Modes of vibration of the clamped isotropic plate [8]

Example 3. The square clamped isotropic plate with variable thickness is considered (Fig. 6). The material properties are the same as in Example 1. The plate dimensions l_x = = l_y = 2.0 m. The plate thickness is linearly variable in x direction, where the minimum plate thickness h = 0.05 m.

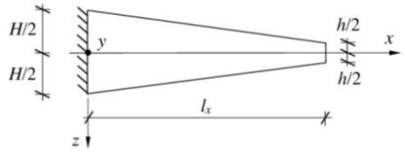

Figure 6. The square clamped plate with linearly variable thickness

Obtained natural frequencies for the plate are presented in Tables 3 and 4 for different values of H and for different FDM grid dimensions respectively. The modes of vibrations are similar to those presented in Example 2.

Table 3. Natural frequencies for the plate obtained using the grid 20×20

Mode	Solution FDM – natural frequencies ω [rad/s]					
	Plate in vacuum [H] = m			Plate immersed in water [H] = m		
	H = 0.1	H = 0.15	H = 0.2	H = 0.1	H = 0.15	H = 0.2
1	134.582	193.945	249.065	81.820	125.228	168.412
2	258.743	334.794	411.831	177.773	241.200	307.357
3	627.409	811.483	986.190	450.866	619.600	776.200
4	790.231	1066.460	1348.691	546.136	763.105	1002.377

Table 4. Natural frequencies for the plate obtained using different grid dimensions

Mode	Solution FDM – natural frequencies ω [rad/s]							
	Plate in vacuum H = 0.2 m				Plate immersed in water H = 0.2 m			
	14×14	20×20	30×30	40×40	14×14	20×20	30×30	40×40
1	246.077	249.065	250.458	250.743	165.495	168.412	171.276	172.882
2	451.284	411.831	389.402	379.177	333.027	307.357	294.169	288.746
3	988.865	986.190	980.990	976.671	781.514	776.200	772.301	770.521
4	1396.401	1348.691	1333.408	1328.042	1031.697	1002.377	998.076	998.865

Example 4. The square simply-supported orthotropic plate in vacuum made of boron/epoxy is considered [12]. The plate dimensions l_x, l_y, h are the same as in Example 1. The plate properties are: $D_{11} = DE_1$, $D_{22} = DE_2$, $D_{12} = D_{21} = D\nu_{12}E_2$, $D_{33} = \frac{h^3 G_{12}}{12}$, where $D = \frac{h^3}{12(1-\nu_{12}\nu_{21})}$, $E_1 = 211$ GPa, $E_2 = 24.1$ GPa, $G_{12} = 6.9$ GPa, $\nu_{12} = 0.36$, $\nu_{21} = E_2\nu_{12}/E_1$. The results are shown in Table 5 for two FEM mesh dimensions. The modes of vibrations are similar to those presented in Example 1.

Table 5. Natural frequencies ω [rad/s] for the orthotropic plate obtained by the FEM

Mode	Mesh 10×10			Mesh 20×20		
	(a)	(b)	(c)	(a)	(b)	(c)
1	84.896	84.882	84.853	85.393	85.390	85.382
2	140.227	140.173	140.055	141.849	141.835	141.805
3	255.029	254.821	254.351	257.968	257.913	257.795
4	303.225	303.086	302.781	305.323	305.287	305.212

6. Conclusions

The linear theory of natural vibrations of thin rectangular iso- and orthotropic plates has been presented in the paper. The plate deformation was described by the Finite Element Method (FEM) and the Finite Difference Method (FDM) for plates with constant and variable thickness respectively. The author's approach of numerical integration was applied to establish the stiffness matrix in terms of the FEM and this approximation gave quite good results close to analytical. The numerical tests included in the paper confirmed that a plate immersed in liquid is characterized by lower natural frequencies than a plate in a vacuum. In turn, a change of a plate’s material or the greater thickness of the fixed edge of the cantilever plate can increase its natural frequencies. Numerical convergence studies carried out for a plate with variable thickness showed that starting from a $\mathbf{30} \times \mathbf{30}$ grid, the results obtained begin to approach each other.

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