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STEERING PROJECTIONS IN VON NEUMANN ALGEBRAS

Adam Wegert

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Abstract. A steering projection of an arbitrary von Neumann algebra is introduced. It is shown that a steering projection always exists and is unique (up to Murray-von Neumann equivalence). A general decomposition of arbitrary projections with respect to a steering projection is established.

Keywords: Murray-von Neumann order, central projection, steering projection.

Mathematics Subject Classification: 46L10.

1. INTRODUCTION

The Murray-von Neumann order (in the set of all (equivalence classes of) projections) is a useful tool in studies of W^*-algebras. For example, this order is involved to define types of von Neumann algebras. In most classical textbooks on operator algebras (see e.g. [1, 4, 5, 7, 8, 10]) this order is introduced mainly for this purpose. However, even from a purely set-theoretic point of view the Murray-von Neumann order is an interesting subject of investigation. Although it was carefully studied by Tomiyama ([11]) and Griffin Jr. ([2, 3]) in general von Neumann algebras, none of the books mentioned above discusses this topic so generally. The aim of this paper is to give a new form as well as new proofs of the results by Tomiyama and Griffin Jr.

Our approach concentrates on distinguishing a certain projection, called by us *steering*, in an arbitrary von Neumann algebra, which turns out to be unique up to Murray-von Neumann equivalence. The key property of this projection is the following: it allows decomposing any other projection into parts each of which is a 'cardinal multiple' of a part of the steering projection except one part which is a refinement – see Theorem 4.4 in Section 4. This is the main difference between our approach and those of [2, 3, 11]. Moreover, for a reader familiar with Tomiyama's results, it may be nontrivial that homogenous projections (in the terminology of [11]) of different dimensions actually come from a common (steering) projection – which is an immediate consequence of our main result. From Theorem 4.4 one can derive that the generalized dimension function (see [11]) of the steering projection in a properly infinite von Neumann algebra is equal to $\aleph_0$ on II_∞ and III summands and 1 elsewhere. So, as a consequence – in that case the steering projection turns out to be $\aleph_0$-homogenous which may be not transparent in the II_∞ case, since our definition in that case is in a totally different spirit.

In part of the presentation we involve traces on von Neumann algebras of type II_1. To reduce the size of the paper, the existence and fundamental properties of them are stated without proofs. For details the reader is referred to any of the books cited above.

The presented material mainly comes from [6]. The idea of steering projections was introduced therein. Also the approach to the so-called dimension theory of W^*-algebras by means of steering projections comes from that paper.

2. PRELIMINARIES

Throughout this paper we will use the following notation: $\mathfrak{A}$ will (usually) stand for a von Neumann algebra. The center of $\mathfrak{A}$ will be denoted by $\mathfrak{Z}(\mathfrak{A})$ and by $\mathfrak{E}(\mathfrak{A})$ we denote the lattice of all projections in $\mathfrak{A}$. Typical projections in $\mathfrak{A}$ will be denoted usually by $P, Q, R \ldots$ while projections in $\mathfrak{Z}(\mathfrak{A})$ will be denoted by Z. For a projection $P \in \mathfrak{A}$, C_P will stand for the *central carrier* of P; that is, C_P is the smallest projection $Z \in \mathfrak{Z}(\mathfrak{A})$ such that $P \leq Z$. We briefly recall the basic facts about von Neumann algebras which will be used constantly:

- The center of any von Neumann algebra is a von Neumann algebra as well.
- Let $P \in \mathfrak{A}$ be a projection in a von Neumann algebra. Then $P\mathfrak{A}P$ is also a von Neumann algebra.
- $\mathfrak{Z}(P\mathfrak{A}P)$, the center of $P\mathfrak{A}P$, coincides with $\mathfrak{Z}(\mathfrak{A})P$.
- Let $P \in \mathfrak{A}$ be a (nonzero) projection. Then P is properly infinite if and only if $P = \sum_{n\in\mathbb{N}} P_n$ where each $P_n \sim P$.

Definition 2.1. We say that two projections $P, Q \in \mathfrak{A}$ are *Murray-von Neumann equivalent* if there exists $V \in \mathfrak{A}$ such that:

$$V^*V = P, \quad VV^* = Q. \tag{2.1}$$

Then we will write $P \sim Q$. Given two projections P, Q in $\mathfrak{A}$ we will write $P \preceq Q$ iff $P \sim Q_0$ for a certain subprojection $Q_0 \leq Q$ belonging to $\mathfrak{A}$.

The next theorem is one of the most important tools in dealing with Murray-von Neumann order. Its proof may be found in [5] (Theorem 6.2.7) or [10] (Chapter V, Theorem 1.8).

Theorem 2.2 (Comparison Theorem)**.** *For any pair $P, Q \in \mathfrak{A}$ of projections there exists a central projection $Z \in \mathfrak{A}$ such that*

$$QZ \preceq PZ, \quad P(I - Z) \preceq Q(I - Z).$$

Definition 2.3. For a cardinal number α, a projection $P \in \mathfrak{A}$ is called *α-decomposable* (in $\mathfrak{A}$) if the cardinality of any family of mutually orthogonal nonzero subprojections $Q_i \in \mathfrak{A}$ of P is at most α. For $\alpha = \aleph_0$ we call P *countably decomposable.*

The following result will allow us to define the so-called steering projection in a type III von Neumann algebra. For the proof see [5, Theorem 6.3.4].

Theorem 2.4. *Suppose that $P, Q \in \mathfrak{A}$ are two projections, P is properly infinite and Q is countably decomposable. If $C_Q \leq C_P$, then $Q \preceq P$.*

Recall that a projection P in a von Neumann algebra $\mathfrak{A}$ acting on a Hilbert space H is called *cyclic* when $P(H) = \overline{\mathfrak{A}'x}$ for some vector $x \in H$. Then we have the following result (Lemma 6.3.9 in [5]):

Theorem 2.5. *Suppose that $\{P_i\}_{i \in J_1}$ and $\{Q_j\}_{j \in J_2}$ are two infinite families consisting of mutually orthogonal nonzero projections, each P_i is cyclic and $Q \preceq P$ where $Q := \sum_{j \in J_2} Q_j$, $P := \sum_{i \in J_1} P_i$. Then $|J_1| \leq |J_2|$. If moreover each Q_j is cyclic and $P \sim Q$ then $|J_1| = |J_2|$.*

Remark 2.6. Theorem 2.5 is valid for infinite orthogonal families of *cyclic* projections as well as for families of countably decomposable projections. This follows from the fact that each nonzero projection is the sum of a family of nonzero cyclic projections (see for instance [4, Theorem 5.5.9]) – so, in case of a countably decomposable projection this family is countable. So, an infinite family of nonzero countably decomposable projections may be replaced by a family of nonzero cyclic projections without changing the sum and the cardinality.

For the proof of the next result consult [5, Theorem 6.3.11].

Theorem 2.7 (Generalized Invariance of Dimension)**.** *Suppose that $P, R \in \mathfrak{A}$ are two projections and R is finite and nonzero. Let $\mathfrak{P} = \{P_i\}_{i \in J_1}$ and $\mathfrak{Q} = \{Q_j\}_{j \in J_2}$ be two orthogonal families of subprojections of P maximal with respect to the property:*

$$P_i \sim Q_j \sim R, \quad i \in J_1, \quad j \in J_2.$$

Then $|J_1| = |J_2|$.

Definition 2.8. A projection $P \in \mathfrak{A}$ is called *abelian* if $P\mathfrak{A}P$ is a commutative von Neumann algebra.

We briefly recall some basic properties of abelian projections (see [5] for a detailed presentation).

Let $P, Q, R \in \mathfrak{A}$ be three projections with P and R abelian. Then:

- if $Q \leq P$, then $Q = C_Q P$,
- P is finite,
- if $Q \sim P$, then Q is also abelian,
- if $C_R = C_P$, then $R \sim P$.

The next result is of key importance. For the proof (of a slightly more general version than that presented below) see [5, Theorem 6.5.2].

Theorem 2.9 (Type Decomposition)**.** *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space H. There are mutually orthogonal central projections*

$$Z^{\mathrm{I}},\ Z^{\mathrm{II}_1},\ Z^{\mathrm{II}_\infty},\ Z^{\mathrm{III}}$$

with sum I and such that either $Z^\varepsilon = 0$ or $\mathfrak{A}Z^\varepsilon$ is a type ε von Neumann algebra where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$.

3. STEERING PROJECTION

The aim of this part is to define a *steering* projection in an arbitrary von Neumann algebra, which, in a sense, controls the Murray-von Neumann order in the algebra. Since its definition depends on the type of the algebra, it will be given separately for each type. The simplest cases include types I and II_1, whereas the case of type II_∞ algebras is most involved.

Definition 3.1.

(a) If $\mathfrak{A}$ is a type II_1 von Neumann algebra, we will simply call the identity of $\mathfrak{A}$ the *steering projection.*
(b) If $\mathfrak{A}$ is a type I von Neumann algebra, a projection $P \in \mathfrak{A}$ is called a *steering projection* if P is abelian and $C_P = I$.

Definition 3.2 ([6])**.** A von Neumann algebra $\mathfrak{A}$ is called *quasi-abelian* if for every projection $P \in \mathfrak{A}$ the relation $P \sim C_P$ holds. A projection P is called *quasi-abelian* if $P = 0$ or $P\mathfrak{A}P$ is a quasi-abelian von Neumann algebra.

Definition 3.3. Assume $\mathfrak{A}$ is a type III von Neumann algebra. A projection $P \in \mathfrak{A}$ is called a *steering projection* if P is quasi-abelian and $C_P = I$.

The existence as well as uniqueness (up to equivalence) of steering projections in type I and II_1 von Neumann algebras can easily be established. It appears that similar properties hold for type III algebras, but they are not so obvious. To establish them, we need the following auxiliary result.

Lemma 3.4. *If $P \in \mathfrak{A}$ is a projection, then the following conditions are equivalent:*

(i) *P is quasi-abelian,*
(ii) *$Q \sim PC_Q$ for every subprojection $Q \leq P$,*
(iii) *for a projection Q we have that $P \preceq Q \iff P \leq C_Q$.*

Proof. Suppose that P is quasi-abelian and $Q \leq P$. Since $P\mathfrak{A}P$ is quasi-abelian, $Q \sim_{P\mathfrak{A}P} C'_Q$ where the latter projection is the central carrier of Q computed with respect to $P\mathfrak{A}P$. But $C'_Q = PC_Q$, so $Q \sim PC_Q$. This shows that (ii) follows from (i). The reverse implication is proved similarly.

To show that (iii) follows from (ii), suppose that

$$P \leq C_Q. \tag{3.1}$$

If $P = 0$, we are done; so, let $P \neq 0$. Let $\{P_i\}_{i\in\mathcal{I}}$ be a maximal family of projections with the following properties:

a) each P_i is a nonzero subprojection of P,
b) $P_i \preceq Q$ for every $i \in \mathcal{I}$,
c) for $i \neq j$ we have $PC_{P_i}C_{P_j} = 0$ (in particular, P_i's are mutually orthogonal).

It follows from the maximality and (3.1) that $P = \sum_{i\in\mathcal{I}} PC_{P_i}$. Note also that $C_{C_{P_i}C_{P_j}P} = C_{P_i}C_{P_j}C_P = 0$ (because $C_{P_i}C_{P_j}P = 0$) which means that $\{C_{P_i}C_P\}_{i\in\mathcal{I}}$ is orthogonal. From our assumption, condition $P_i \leq P$ implies $P_i \sim C_{P_i}P$; thus $C_{P_i}P \preceq Q$ and $(C_{P_i}C_P)C_{P_i}P = C_{P_i}P \preceq C_{P_i}C_PQ$. This yields

$$P = \sum_{i\in\mathcal{I}} C_{P_i}P \preceq \sum_{i\in\mathcal{I}} (C_{P_i}C_P)Q \leq Q.$$

The reverse implication in (iii) needs no additional assumptions: $P \preceq Q$ implies $P \preceq C_Q$ and thus $P \leq C_Q$.

It remains to prove (ii) under the assumption of (iii): let $Q \leq P$ and consider the projection $R := Q + (I - C_Q)P$. Then

$$\begin{aligned} C_R &= C_{Q+(I-C_Q)P} = C_Q \vee C_{(I-C_Q)P} = C_QC_P \vee (I - C_Q)C_P \\ &= \Big(C_Q \vee (I - C_Q)\Big)C_P = C_P \geq P. \end{aligned}$$

This yields, from (iii), that $P \preceq R$, thus $C_QP \preceq C_QR = C_QQ + C_Q(I - C_Q)P = Q \leq C_QP$, i.e. $Q \sim C_QP$. □

Theorem 3.5. *Suppose $\mathfrak{A}$ is a type III von Neumann algebra. Then $\mathfrak{A}$ has a steering projection. Any two steering projections are equivalent.*

Proof. Let $\{P_i\}_{i\in\mathcal{I}}$ be a maximal family of nonzero projections with properties:

a) $C_{P_i}C_{P_j} = 0$ for $i \neq j$,
b) each P_i is countably decomposable.

Put $P := \sum_{i\in\mathcal{I}} P_i$ and observe that $C_P = I$. This follows from the maximality of the above family and the fact that every nonzero projection contains a nonzero subprojection which is countably decomposable (recall that any cyclic projection is such). Now observe that each P_i is quasi-abelian: taking into account the previous lemma, it is enough to show that for a projection $Q_i \leq P_i$ we have $Q_i \sim C_{Q_i}P_i$. Since we are working in a type III von Neumann algebra, all projections are properly infinite, so, by Theorem 2.4, it is enough to show that $C_{Q_i} = C_{C_{Q_i}P_i}$. But the latter is equal to $C_{Q_i}C_{P_i} = C_{Q_i}$. It remains to note that P is also quasi-abelian. In fact, for $Q \in P\mathfrak{A}P$ of the form $Q = \sum_{i\in\mathcal{I}} P_iQ_0P_i$ we have that:

$$C_Q = C_{\sum_{i\in\mathcal{I}} P_iQ_0P_i} = \sum_{i\in\mathcal{I}} C_{P_iQ_0P_i} \sim \sum_{i\in\mathcal{I}} P_iQ_0P_i = Q.$$

For uniqueness, note that if P, Q are steering, then both are quasi-abelian and $P \leq C_Q = I = C_P \geq Q$, thus, by the previous lemma, $P \preceq Q$ and $Q \preceq P$ and consequently $P \sim Q$. □

We turn to deal with the last case – when $\mathfrak{A}$ is a type II_∞ von Neumann algebra. First, let us introduce the following notation: for a projection $Q \in \mathfrak{A}$ and $n \in \{1, 2, \ldots\} \cup \{\omega\}$, by $n \odot Q$ we denote any projection of the form $\sum_{k=1}^{n} Q_k$ where each $Q_k \sim Q$ (when $n = \omega$, we think of the sum $\sum_{k=1}^{\infty} Q_k$). If $n = 0$ we put simply $n \odot P := 0$. For example, any properly infinite projection P could be written as $P = \omega \odot P$.

We shall first establish some properties of the trace concerning the Murray-von Neumann order. They will be applied later. (Confront Lemma 3.7 with Theorem 8.4.4 in [5].)

Theorem 3.6. *Let $\mathfrak{A}$ be a finite von Neumann algebra and $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ denote its (unique) trace. If $P, Q \in \mathfrak{E}(\mathfrak{A})$, then:*

(a) $P \preceq Q \iff \mathrm{tr}(P) \leq \mathrm{tr}(Q)$,
(b) $P \sim Q \iff \mathrm{tr}(P) = \mathrm{tr}(Q)$.

Lemma 3.7. *If $\mathfrak{A}$ is a type II_1 von Neumann algebra, $\mathrm{tr} : \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ is its trace, $P \in \mathfrak{A}$ is a projection and $Z \in \mathfrak{A}$ is a central element (not necessarily a projection) with $0 \leq Z \leq \mathrm{tr}(P)$, then there is a projection $Q \leq P$ such that $\mathrm{tr}(Q) = Z$.*

Proposition 3.8. *Suppose that $\mathfrak{A}$ is a type II_∞ von Neumann algebra and E is a finite projection in $\mathfrak{A}$. Let $\mathrm{tr} : E\mathfrak{A}E \to \mathfrak{Z}(E\mathfrak{A}E)$ be the trace on $E\mathfrak{A}E$. If P and Q are two projections in $E\mathfrak{A}E$ and $\mathrm{tr}(P) \leq n \cdot \mathrm{tr}(Q)$ (where n is a positive integer), then $P \preceq n \odot Q$ in $\mathfrak{A}$.*

Proof. We will show the existence of a sequence of mutually orthogonal projections $P_1, P_2, \ldots, P_n$ such that $\sum_{i=1}^{n} P_i = P$ and $\mathrm{tr}(P_i) = \frac{1}{n}\mathrm{tr}(P)$ for $i = 1, 2, \ldots, n$. Using Lemma 3.7 we find a projection $P_1 \leq P$ with $\mathrm{tr}(P_1) = \frac{1}{n}\mathrm{tr}(P)$. Suppose that we have already constructed mutually orthogonal projections $P_1, \ldots, P_k$, $(k < n)$ with the properties: $\sum_{i=1}^{k} P_i \leq P$ and $\mathrm{tr}(P_i) = \frac{1}{n}\mathrm{tr}(P)$. Then $P - \sum_{i=1}^{k} P_i$ is a projection with:

$$\mathrm{tr}\Big(P - \sum_{i=1}^{k} P_i\Big) = \frac{n-k}{n}\mathrm{tr}(P) \geq \frac{1}{n}\mathrm{tr}(P).$$

Again, by Lemma 3.7 we can find a projection $P_{k+1} \leq P - \sum_{i=1}^{k} P_i$ with $\mathrm{tr}(P_{k+1}) = \frac{1}{n}\mathrm{tr}(P)$. After the construction we know that $P_1 + P_2 + \ldots + P_n \leq P$ and

$$\mathrm{tr}(P_1 + P_2 + \ldots + P_n) = \mathrm{tr}(P_1) + \mathrm{tr}(P_2) + \ldots + \mathrm{tr}(P_n) = n \cdot \frac{1}{n}\mathrm{tr}(P) = \mathrm{tr}(P).$$

From the faithfulness of the trace we conclude that

$$P_1 + P_2 + \ldots + P_n = P.$$

By virtue of Theorem 3.6 we have:

(i) $P_j \sim P_k$ for $j, k \in \{1, \ldots, n\}$ and consequently $P \sim n \odot P_1$,
(ii) $\mathrm{tr}(P_1) \leq \mathrm{tr}(Q)$, thus $P_1 \preceq Q$.

From (i) and (ii) we infer that $P \sim n \odot P_1 \preceq n \odot Q$. □

Lemma 3.9. *Let $\mathfrak{A}$ be a type II_∞ von Neumann algebra. For a projection $P \in \mathfrak{A}$ the following conditions are equivalent:*

(i) *P is finite,*
(ii) *for any projection $Q \in \mathfrak{A}$, $P \leq C_Q$ iff there is a sequence of central projections $\{Z_n\}_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} Z_n = I$ and $PZ_n \leq n \odot Q$ for each n.*

Proof. Assume (ii) holds for a projection $P \in \mathfrak{A}$. Choose $Q_0 \in \mathfrak{A}$ to be a finite projection with $C_{Q_0} = I$. Since then $P \leq C_{Q_0}$, (ii) allows us to find central projections $Z_1, Z_2, \dots$ such that $PZ_n \leq n \odot Q_0$. Since $n \odot Q_0$ is finite, thus PZ_n is finite as well and hence $P = \sum_{n\in\mathbb{N}} PZ_n$ is also finite. Conversely, if P is finite and $P \leq C_Q$, we can find a family $\{Q_i\}_{i\in\mathcal{I}}$ of mutually orthogonal projections such that $P = \sum_{i\in\mathcal{I}} Q_i$ and for all $i \in \mathcal{I}$ relation $Q_i \preceq Q$ holds. Since P is finite, there is a (unique) trace $\mathrm{tr} : P\mathfrak{A}P \to \mathfrak{Z}(P\mathfrak{A}P)(= \mathfrak{Z}(\mathfrak{A})P)$. For each Q_i we find central (in $\mathfrak{A}$) projections $Z_{i,n,k}$, $n \in \mathbb{N}$, $k = 1, \dots, 2^n$ such that:

$$\mathrm{tr}(Q_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k} P.$$

(This can simply be deduced from the commutative Gelfand-Naimark theorem and the total disconnectedness of the Gelfand spectrum of $\mathfrak{Z}(\mathfrak{A})$.) From the above equality we obtain $\mathrm{tr}(Z_{i,n,k}P) \leq 2^n \mathrm{tr}(Q_i)$. We infer from Proposition 3.8 that

$$Z_{i,n,k}P \preceq 2^n \odot Q_i \preceq 2^n \odot Q.$$

Moreover, by

$$P = \mathrm{tr}(P) = \sum_{i\in\mathcal{I}} \mathrm{tr}(Q_i) = \sum_{i\in\mathcal{I}} \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k} P$$

we have $PZ = P$ and thus $Z \geq P$ where $Z := \bigvee_{i,n,k} Z_{i,n,k}$ (these projections need not be mutually orthogonal!). Reindexing the family $\{Z_{i,n,k}\}_{i,n,k}$ we obtain a collection $\{C_j\}_{j\in\mathcal{J}}$ of central projections with $PC_j \preceq n_j \odot Q$ and $C := \bigvee_{j\in\mathcal{J}} C_j \geq P$. Now equip $\mathcal{J}$ with a well ordering and denote by j_0 its first element. We put $C'_{j_0} := C_{j_0}$ and if C'_j's are already defined for all $j < l$, then we put $C'_l := C_l(I - \sum_{j<l} C_j)$. Then $C'_l \leq C_l$, the family $\{C'_j\}_{j\in\mathcal{J}}$ is orthogonal and $\sum_{j\in\mathcal{J}} C'_j = C$. We define $C'_\infty := I - C$. Then the family $\{C_k\}_{k\in\mathcal{K}}$ where $\mathcal{K} := \mathcal{J} \cup \{\infty\}$ is still orthogonal, $\sum_{k\in\mathcal{K}} C'_k = I$ and $PC'_k \preceq n_k \odot Q$ for each $k \in \mathcal{K}$ (since $PC'_\infty = P - PC = 0$, we may take any number for n_∞, e.g. $n_\infty := 1$). Putting $Z_n := \sum\{C'_k : n_k = n\}$, we get the desired sequence. □

Proposition 3.10. *Let $\mathfrak{A}$ be a type II_∞ and*

$$\mathfrak{E}_\omega(\mathfrak{A}) := \{Q \in \mathfrak{E}(\mathfrak{A}) : Q \sim \omega \odot P \text{ for some finite projection } P\}.$$

Then:

(i) *if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and Z is a central projection, then $PZ \in \mathfrak{E}_\omega(\mathfrak{A})$,*
(ii) *for $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and a properly infinite projection Q we have $P \preceq Q \iff P \leq C_Q$,*
(iii) *if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ satisfies $C_P = I$, then $Q \sim C_Q P$ for every $Q \in \mathfrak{E}_\omega(\mathfrak{A})$.*

Proof. (i) Let $P \sim \omega \odot P_0$ where P_0 is finite. This allows us to write $P = \sum_{n=1}^{\infty} P_n$ where $P_n \sim P_0$ for $n \in \mathbb{N}$. Then $PZ = \sum_{n=1}^{\infty} P_n Z$ and $P_n Z \sim P_0 Z$ for $n \in \mathbb{N}$ (obviously $P_0 Z$ is still finite).

(ii) Suppose that $P \leq C_Q$. Take a finite projection P_0 such that $P \sim \omega \odot P_0$ and find, using Lemma 3.9, a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections such that $P_0 Z_n \preceq n \odot Q$. Since Q is properly infinite, then $Q \sim \omega \odot Q$ and thus we get:

$$P_0 Z_n \preceq n \odot Q \preceq \omega \odot Q \sim Q.$$

The above holds for any $n \in \mathbb{N}$, thus $\sum_{n=1}^{\infty} P_0 Z_n = P_0 \preceq Q$. Consequently

$$P \sim \omega \odot P_0 \preceq \omega \odot Q \sim Q.$$

(iii) If $P, Q \in \mathfrak{E}_\omega(\mathfrak{A})$, then (in particular) P, Q are properly infinite and $Q \leq C_P (= I)$. So, we can use (ii) to get $Q \preceq P$ and consequently $Q \preceq C_Q P$. As $C_Q P \in \mathfrak{E}_\omega(\mathfrak{A})$ by (i), in order to prove $C_Q P \preceq Q$ it is enough to show that $C_Q \geq C_{C_Q P}$. But $C_{C_Q P}$ is equal to $C_Q C_P = C_Q$ and the assertion follows. □

Definition 3.11. Suppose that $\mathfrak{A}$ is a type II_∞ von Neumann algebra. Then $P \in \mathfrak{A}$ is called a *steering projection* if $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and $C_P = I$.

Theorem 3.12. *Let $\mathfrak{A}$ be a type II_∞ von Neumann algebra. Then $\mathfrak{A}$ has a steering projection and any two steering projections are equivalent.*

Proof. Since I is properly infinite, for any $Q \in \mathfrak{E}(\mathfrak{A})$, $\omega \odot Q$ makes sense. Now take a finite projection $P_0 \in \mathfrak{A}$ such that $C_{P_0} = I$ and form $P := \omega \odot P_0$. Then $P \in \mathfrak{E}_\omega(\mathfrak{A})$ and still $C_P = I$. This establishes the existence. To deal with the uniqueness assume that P, Q are both steering. Then they both belong to $\mathfrak{E}_\omega(\mathfrak{A})$ and $P \leq C_Q (= I)$ and $Q \leq C_P (= I)$, so we can use Proposition 3.10 to get $P \preceq Q$ and $Q \preceq P$, i.e. $P \sim Q$. □

We have defined a steering projection in a von Neumann algebra of any of types: $\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$. Now, if $\mathfrak{A}$ is an arbitrary von Neumann algebra, then we define a steering projection to be the sum of steering projections of $\mathfrak{A}Z^{\mathrm{I}}$, $\mathfrak{A}Z^{\mathrm{II}_1}$, $\mathfrak{A}Z^{\mathrm{II}_\infty}$, $\mathfrak{A}Z^{\mathrm{III}}$ where Z^ε is defined as in Theorem 2.9 for $\varepsilon \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$.

As a consequence of the results of this section, we get

Theorem 3.13. *An arbitrary von Neumann $\mathfrak{A}$ algebra has a steering projection. Any two steering projections in $\mathfrak{A}$ are equivalent. If P is a steering projection in $\mathfrak{A}$, then $C_P = I$ and for any nonzero central $Z \in \mathfrak{E}(\mathfrak{A})$ the projection PZ is steering in $\mathfrak{A}Z$.*

The proof is left to the reader.

4. GENERAL DECOMPOSITION

Before formulating the main theorem, we need some auxiliary results, which are however interesting in themselves. By Card we mean the *class* of all cardinal numbers, while Card_∞ is its subclass of infinite cardinals. For $\alpha \in \mathrm{Card}$ by α^+ we denote the immediate successor of α. The notation of the form '$Q \sim \alpha \odot P$' (where P and Q are projections in a common von Neumann algebra and $\alpha \in \mathrm{Card}$) means that Q may be written in the form $Q = \sum_{s \in S} Q_s$ where $Q_s \sim P$ for all $s \in S$ and $|S| = \alpha$. In particular, $Q \sim 0 \odot P$ is equivalent to $Q = 0$. We start with the following result (which may be seen as a variation of Lemma 1 in [11], Theorem 2 in [2] and Lemma 1.2 in [3]).

Lemma 4.1. *Let $\mathfrak{A}$ be a von Neumann algebra of type ε where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_\infty, \mathrm{III}\}$ and P be a steering projection in $\mathfrak{A}$. Suppose that the projections Q, Q' satisfy: $Q \sim \alpha \odot P$ and $Q' \sim \beta \odot P$ where $\alpha, \beta \in \mathrm{Card}_\infty \cup \{0\}$. Then $Q \sim Q' \iff \alpha = \beta$. If $\mathfrak{A}$ is of type* I*, then the statement is valid for any $\alpha, \beta \in \mathrm{Card}$.*

Proof. The 'if' part is obvious. Further, if, for example, $\alpha = 0$, then $0 = Q \sim Q'$, so $Q' = 0$ and $\beta = 0$. We therefore assume that $\alpha, \beta > 0$ and consider three cases, namely when $\mathfrak{A}$ is of type ε with:

- $\varepsilon = \mathrm{I}$; then a steering projection is defined to be abelian, hence finite. Since
$$\alpha \odot P \sim Q \sim Q' \sim \beta \odot P,$$
so, by Theorem 2.7 we get $\alpha = \beta$. (Note that we have not used the fact that α, β are infinite.)
- $\varepsilon = \mathrm{II}_\infty$; in this case $P \sim \omega \odot P_0$ for some finite projection P_0. We have then, using the fact that α, β are infinite:
$$Q \sim \alpha \odot P \sim \alpha \odot (\omega \odot P_0) = (\alpha \cdot \aleph_0) \odot P_0 = \alpha \odot P_0,$$
and similarly $Q' \sim \beta \odot P_0$. As P_0 is finite, we are in the same situation as from the previous step and we obtain $\alpha = \beta$ (by Theorem 2.7).
- $\varepsilon = \mathrm{III}$; from the proof of Theorem 3.5 we know that $P = \sum_{i \in \mathcal{I}} P_i$ where each P_i is countably decomposable and $\{P_i\}$'s are centrally orthogonal, i.e. $C_{P_i} C_{P_j} = 0$ for $i \neq j$. Choose $i_0 \in \mathcal{I}$ and denote for simplicity $Z := C_{P_{i_0}}$. Then
$$PZ = \sum_{i \in \mathcal{I}} P_i Z = P_{i_0} Z + \sum_{i \neq i_0} \underbrace{P_i Z}_{0} = P_{i_0}$$
and thus we obtain
$$QZ \sim (\alpha \odot P)Z \sim \alpha \odot PZ = \alpha \odot P_{i_0} \sim$$
$$\sim Q'Z \sim (\beta \odot P)Z \sim \beta \odot PZ = \beta \odot P_{i_0}.$$

Now, using Remark 2.6 (see also Theorem 2.5), we obtain $\alpha = \beta$. □

Proposition 4.2. *Suppose that $\mathfrak{A}$ is a von Neumann algebra of type ε where $\varepsilon \in \{\mathrm{I}, \mathrm{II}_\infty, \mathrm{III}\}$ with a steering projection P, Q is a properly infinite projection and $\alpha \in \mathrm{Card}_\infty \cup \{0\}$. Assume that the following condition is satisfied: $\alpha \odot P \preceq Q$ and for any central $0 \neq Z \in \mathfrak{E}(\mathfrak{A})$, QZ does not contain α^+ copies of PZ, i.e. $\alpha^+ \odot PZ \npreceq QZ$. Then $Q \sim \alpha \odot P$. If $\mathfrak{A}$ is of type* I*, the assumption for Q being properly infinite can be dropped and α may be taken arbitrary.*

Proof. Step 1. First we will show that there is a central projection $Z \neq 0$ such that:

$$QZ \sim \alpha \odot PZ. \tag{4.1}$$

Case $\alpha = 0$. With our assumptions we have that

$$P \preceq Q \iff C_Q = I. \tag{4.2}$$

Indeed, if $\varepsilon \in \{\mathrm{I}, \mathrm{III}\}$, then P is quasi-abelian (even abelian for $\varepsilon = \mathrm{I}$), thus $P \preceq Q \iff P \leq C_Q$ (by Lemma 3.4), but the latter inequality is equivalent to $C_P \leq C_Q$. Since P is steering, so $C_P = I$ which yields (4.2). In case $\varepsilon = \mathrm{II}_\infty$ we are assuming that Q is properly infinite and P, being a steering projection, belongs to $\mathfrak{E}_\omega(\mathfrak{A})$, so again we have $P \preceq Q \iff P \leq C_Q$, this time by Proposition 3.10. The argument as above gives (4.2). As we assumed that $\alpha = 0$, i.e. Q contains no copy of P, we have $P \not\preceq Q$; so, from (4.2) we obtain $C_Q \neq I$. Taking $Z := I - C_Q \neq 0$ we have

$$QZ = Q - QC_Q = 0 \sim 0 \odot (PZ).$$

Case $\alpha > 0$. From our assumption, there exists an orthogonal family $\{P_i\}_{i \in \mathcal{I}_0}$ with $|\mathcal{I}_0| = \alpha$ such that each $P_i \sim P$, $P_i \leq Q$. We extend this family to a maximal family $\{P_i\}_{i \in \mathcal{I}}$ with all the above properties – the fact that we still have $|\mathcal{I}| = \alpha$ follows from the assumption about Q (in the case when $\varepsilon = \mathrm{I}$ and α is finite the first family *is already* maximal). If it happens that $Q = \sum_{i \in \mathcal{I}} P_i (= \alpha \odot P)$ then putting $Z := I$ we get (4.1). Suppose now that $Q' := Q - \sum_{i \in \mathcal{I}} P_i \neq 0$. By the maximality of the family $\{P_i\}_{i \in \mathcal{I}}$ we have $P \not\preceq Q'$ and by the Comparison Theorem, we find a nonzero central projection Z such that $Q'Z \prec PZ$. It turns out that this Z can be taken to fulfill (4.1) for $\varepsilon \in \{\mathrm{II}_\infty, \mathrm{III}\}$. Indeed, for those ε the steering projection P is properly infinite, so PZ is also properly infinite and the same is true for $P_{i_0}Z$ (being equivalent to PZ; here $i_0 \in \mathcal{I}$ is fixed). Moreover, $Q'Z \preceq P_{i_0}Z$ and since $Q' \perp P_{i_0}$, also $Q'Z \perp P_{i_0}Z$. Putting these facts together we obtain

$$P_{i_0}Z \preceq Q'Z + P_{i_0}Z \preceq 2 \odot P_{i_0}Z \sim P_{i_0}Z \sim PZ,$$

hence all of the above components are equivalent. In particular,

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z = Q'Z + P_{i_0}Z + \sum_{i \neq i_0} P_i Z \sim \alpha \odot P.$$

It remains to investigate the case when $\varepsilon = \mathrm{I}$. Now P is no longer properly infinite. As before, we have $Q'Z \preceq PZ$ and

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z.$$

If it happens that $Q'Z = 0$ then we are already done, because

$$QZ = \sum_{i \in \mathcal{I}} P_i Z \sim \alpha \odot P.$$

So, assume that $Q'Z \neq 0$ and form $0 \neq Z' := C_{Q'Z} = ZC_{Q'} \leq Z$. This implies that

$$Q'Z' = Q'ZZ' \preceq PZZ' = PZ' \sim P_{i_0}Z'$$

for each $i_0 \in \mathcal{I}$ meaning that there is P' with

$$Q'Z' \sim P' \leq P_{i_0}Z' \leq P_{i_0}.$$

Recall that now P is abelian (in particular quasi-abelian) so by Lemma 3.4 we have

$$P' = C_{P'}P_{i_0}. \tag{4.3}$$

But observe that since $P' \sim Q'Z'$, thus we have

$$C_{P'} = C_{Q'Z'} = Z'C_{Q'} = ZC_{Q'}C_{Q'} = ZC_{Q'} = Z',$$

so (4.3) turns out to be $P' = P_{i_0}Z'$. This means that

$$Q'Z' \sim P' = P_{i_0}Z' \sim PZ',$$

so we have that

$$QZ' = \underbrace{Q'Z'}_{\sim PZ'} + \sum_{i \in \mathcal{I}} \underbrace{P_iZ'}_{\sim PZ'} \sim (\alpha + 1) \odot PZ'.$$

In the case when α is infinite this yields $QZ' \sim \alpha \odot PZ'$, while in case of finite α we obtain a contradiction with the fact that $Q'Z \neq 0$. In that case we conclude that $Q'Z = 0$, which was discussed before.
Step 2. Let $\{Z_j\}_{j \in \mathcal{J}}$ be a maximal family of nonzero central projections such that for each $j \in \mathcal{J}$ we have $QZ_j \sim \alpha \odot PZ_j$. Then we obtain

$$Q\left(\sum_{j \in \mathcal{J}} Z_j\right) = \sum_{j \in \mathcal{J}} QZ_j \sim \sum_{j \in \mathcal{J}} \alpha \odot PZ_j \sim \alpha \odot P\left(\sum_{j \in \mathcal{J}} Z_j\right).$$

We will show that $\sum_{j \in \mathcal{J}} Z_j = I$. Assume the contrary, i.e. $Z' := I - \sum_{j \in \mathcal{J}} Z_j \neq 0$. Form $\mathfrak{A}_0 := \mathfrak{A}Z'$. Note that $\mathfrak{A}_0$ is again a von Neumann algebra of type ε and denote by $P' := PZ'$ a steering projection for $\mathfrak{A}_0$ (see Theorem 3.13) and $Q' := QZ'$. Take $0 \neq Z \in \mathfrak{Z}(\mathfrak{A}_0)$. In particular, $Z \leq Z'$ and

$$Q'Z = QZ'Z = QZ \quad \text{and} \quad P'Z = PZ'Z = PZ. \tag{4.4}$$

It follows from our assumption that QZ does not contain α^+ copies of PZ so, by (4.4), $Q'Z$ does not contain α^+ copies of $P'Z$. Moreover, Q contains α copies of P and consequently $Q' = QZ'$ contains α copies of $PZ' = P'$. So, all assumptions of the theorem are satisfied for $\mathfrak{A}_0$ and Q', thus we can apply Step 1 for $\mathfrak{A}_0$ and find $0 \neq Z_0 \in \mathfrak{Z}(\mathfrak{A}_0)$ such that

$$QZ_0 = QZ'Z_0 = Q'Z_0 \sim \alpha \odot P'Z_0 = \alpha \odot PZ'Z_0 = \alpha \odot PZ_0.$$

Since $Z_0 \leq Z' = I - \sum_{j \in \mathcal{J}} Z_j$, thus Z_0 is orthogonal to all of Z_j's and this contradicts the maximality of the taken family. As a consequence, $\sum_{j \in \mathcal{J}} Z_j = I$. □

Lemma 4.3. *Let $\mathfrak{A}$ be a von Neumann algebra, $(X, \leq_X)$ be a well ordered set with maximal and minimal elements x_{max}, x_{min} and $\{Z_x\}_{x\in X} \subset \mathfrak{Z}(\mathfrak{A})$ be a family of central projections. Suppose that this family satisfies the following conditions:*

a) *if $x \leq_X y$, then $Z_x \geq Z_y$,*
b) *for any limit element $x \in X \setminus \{x_{min}\}$, $Z_x = \bigwedge_{y<x} Z_y$.*

Denote $W_x := Z_x - Z_{x^+}$ for $x \neq x_{max}$ (where x^+ is the immediate successor of x). Then $\sum_{x\neq x_{max}} W_x = Z_{x_{min}} - Z_{x_{max}}$.

Proof. The proof is by transfinite induction: we claim that for $x \in X$ we have

$$\sum_{y<_X x} W_y = Z_{x_{min}} - Z_x. \tag{4.5}$$

For $x = x_{min}$ both sides of (4.5) are equal to 0 (we use the convention that the summation over the empty set is 0).

For a successor element $x = y^+$ we are assuming that $\sum_{t<_X y} W_t = Z_{x_{min}} - Z_y$, so:

$$\sum_{t<_X x} W_t = \sum_{t<_X y} W_t + W_y = Z_{x_{min}} - Z_y + Z_y - Z_{y^+} = Z_{x_{min}} - Z_x.$$

Finally, let $x \neq x_{min}$ be a limit element. From the transfinite induction hypothesis we have $\sum_{t<_X y} W_t = Z_{x_{min}} - Z_y$ for every $y < x$. We claim that

$$\sum_{t<_X x} W_t = \bigvee_{y<_X x} \sum_{t<_X y} W_t. \tag{4.6}$$

To prove (4.6), observe that obviously $\sum_{t<_X y} W_t \leq \sum_{t<_X x} W_t$ and if for each $y <_X x$ we have $\sum_{t<_X y} W_t \leq W$, then in particular $W_y \leq W$ (just put y^+ in place of y), so $W_y W = W_y$ and thus $(\sum_{y<_X x} W_y)W = \sum_{y<_X x} W_y W = \sum_{y<_X x} W_y$ which means that $\sum_{y<_X x} W_y \leq W$ and proves (4.6). Now we compute:

$$\sum_{t<_X x} W_t = \bigvee_{y<_X x} (Z_{x_{min}} - Z_y) = Z_{x_{min}} - \bigwedge_{y<_X x} Z_y = Z_{x_{min}} - Z_x.$$

Thus, we have established that (4.5) is valid for any $x \in X$. Put $x := x_{max}$ to get the assertion. □

For the purpose of the next result, let $\Lambda_{\mathrm{I}} = \mathrm{Card}$, $\Lambda_{\mathrm{II}} = \{0, 1\} \cup \mathrm{Card}_\infty$ and $\Lambda_{\mathrm{III}} = \{0\} \cup \mathrm{Card}_\infty$.

The next result comes from the treatise [6], where it is one of the most important tools, but not the main aim and therefore it was stated with no proof. However, this result is interesting in its own right. Thus, in the opinion of both the authors – of [6] and of the present paper, the proof of this result should be found in the literature.

Theorem 4.4. *Let P be a steering projection in a von Neumann algebra $\mathfrak{A}$ and denote by $Z^{\mathrm{I}}, Z^{\mathrm{II}}, Z^{\mathrm{III}}$ the central projections such that $Z^{\mathrm{I}} + Z^{\mathrm{II}} + Z^{\mathrm{III}} = I$ and $\mathfrak{A}Z^i$ is of type i for $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. Additionally, let $Z^{\mathrm{II}_1} \in \mathfrak{Z}(\mathfrak{A})$ be such that $Z^{\mathrm{II}_1} \leq Z^{\mathrm{II}}$, $\mathfrak{A}Z^{\mathrm{II}_1}$ is of* *type* II_1 *and* $Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ *is properly infinite. Then for each* $Q \in \mathfrak{E}(\mathfrak{A})$ *there is a unique system of central projections*

$$\{Z_\alpha^{\mathrm{I}}(Q)\}_{\alpha\in\Lambda_{\mathrm{I}}} \cup \{Z_\alpha^{\mathrm{II}}(Q)\}_{\alpha\in\Lambda_{\mathrm{II}}} \cup \{Z_\alpha^{\mathrm{III}}(Q)\}_{\alpha\in\Lambda_{\mathrm{III}}}$$

with the following properties:

a) $Z^i = \sum_{\alpha\in\Lambda_i} Z_\alpha^i(Q)$ *for* $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$*,*
b) *for* $\alpha \in \Lambda_i$ *with* $(i, \alpha) \neq (\mathrm{II}, 1)$*:* $Z_\alpha^i(Q)Q \sim \alpha \odot Z_\alpha^i(Q)P$*,*
c) (i) $Z_1^{\mathrm{II}}(Q)Q$ *is finite,*
(ii) *if* W *is a nonzero central projection and* $W \leq Z_1^{\mathrm{II}}(Q)$*, then* $WQ \neq 0$*,*
(iii) $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)Q$*.*

Proof. **Existence.** *Step 1.* First we deal with the II_1 part. Define

$$Z_1^{\mathrm{II}}(Q) := \bigvee\{W \leq Z^{\mathrm{II}}C_Q : WQ \text{ is finite}\}.$$

Then $Z_1^{\mathrm{II}}(Q)Q$ is finite. Obviously, $Z_1^{\mathrm{II}}(Q) \leq C_Q$ and hence if $W \leq Z_1^{\mathrm{II}}(Q) \leq C_Q$ is central and nonzero, then $C_{WQ} = WC_Q = W \neq 0$, which yields $WQ \neq 0$. For the last property of point (c), observe that $Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ is properly infinite (or 0), so the same is true for $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)$, thus $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)P$ is a steering projection (if nonzero!) in some type II_∞ von Neumann algebra (call it $\tilde{\mathfrak{A}}$). From the fact that $Z_1^{\mathrm{II}}(Q)Q$ is finite, we conclude that $(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)Q$ is finite and its central carrier is equal to

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)C_Q = (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)$$

being the identity of $\tilde{\mathfrak{A}}$. Hence from the properties of steering projections in type II_∞ von Neumann algebras we have that

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})Z_1^{\mathrm{II}}(Q)Q.$$

We also conclude that

$$(Z^{\mathrm{II}} - Z_1^{\mathrm{II}}(Q))C_Q \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_Q. \tag{4.7}$$

Indeed, (4.7) is equivalent to

$$Z^{\mathrm{II}_1}C_Q \leq Z_1^{\mathrm{II}}(Q)C_Q = Z_1^{\mathrm{II}}(Q) \tag{4.8}$$

and this follows from the definition of $Z_1^{\mathrm{II}}(Q)$ since $Z^{\mathrm{II}_1}Q = Z^{\mathrm{II}_1}C_Q Q$ is finite ($\leq Z^{\mathrm{II}_1}$) and $Z^{\mathrm{II}_1}C_Q \leq Z^{\mathrm{II}}C_Q$.
Step 2. Now we set $Z_0^i(Q) := Z^i(I - C_Q)$. It is then immediate that $Z_0^i(Q)Q = 0$ i.e.

$$Z_0^i(Q)Q \sim 0 \odot Z_0^i(Q)P.$$

Define

$$E^i := Z^i - Z_0^i(Q), \quad i = \mathrm{I}, \mathrm{III},$$
$$E^{\mathrm{II}} := Z^{\mathrm{II}} - Z_0^{\mathrm{II}}(Q) - Z_1^{\mathrm{II}}(Q),$$

(note that E^{II} is indeed a projection, since $Z_0^{\mathrm{II}}(Q) \le I - C_Q$ and $Z_1^{\mathrm{II}}(Q) \le C_Q$). Direct calculations show that $E^i = Z^i C_Q$ for $i = \mathrm{I}, \mathrm{III}$. For $i = \mathrm{II}$ we have that

$$E^{\mathrm{II}} \le (Z^{\mathrm{II}} - Z^{\mathrm{II}_1}) C_Q.$$

In fact, since $Z^{\mathrm{II}_1} C_Q \le Z_1^{\mathrm{II}}(Q) C_Q$, we have:

$$E^{\mathrm{II}} = Z^{\mathrm{II}} C_Q - Z_1^{\mathrm{II}}(Q) = \Big(Z^{\mathrm{II}} - Z_1^{\mathrm{II}}(Q)\Big) C_Q \le (Z^{\mathrm{II}} - Z^{\mathrm{II}_1}) C_Q.$$

Finally, $E^{\mathrm{II}}Q$ is properly infinite (or 0): to see this, take a nonzero central projection $W \le E^{\mathrm{II}}$. Obviously $W \le Z^{\mathrm{II}}$ and also (by the above argument) $W \le C_Q$, thus $W \le Z^{\mathrm{II}} C_Q$. S, if WQ is finite, then $W \le Z_1^{\mathrm{II}}(Q)$ and simultaneously $W \le E^{\mathrm{II}}$ yielding a contradiction. Thus WQ is infinite and this means that $E^{\mathrm{II}}Q$ is properly infinite.

For $0 \ne \alpha \in \Lambda_i$ such that $(i, \alpha) \ne (\mathrm{II}, 1)$ let

$$E^i_\alpha := \bigvee \{ W \le E^i : W \in \mathfrak{Z}(\mathfrak{A}),\ \alpha \odot WP \preceq WQ \}.$$

The projections E^i_α have the following properties:

(i) if $\alpha \ge \beta$, then $E^i_\alpha \le E^i_\beta$ (indeed, if $\alpha \odot WP \preceq WQ$, then of course $\beta \odot WP \preceq WQ$);

(ii) $\alpha \odot E^i_\alpha P \preceq E^i_\alpha$. To see this, write E^i_α as $E^i_\alpha = \bigvee_{s \in S} W_s$ where for each $s \in S$ it holds

$$\alpha \odot W_s P \preceq W_s Q. \tag{4.9}$$

We can find an orthogonal family $\{V_s\}_{s \in S} \subset \mathfrak{Z}(\mathfrak{A}) \cap \mathfrak{E}(\mathfrak{A})$ such that $V_s \le W_s$ $(s \in S)$, and $E^i_\alpha = \sum_{s \in S} V_s$. From (4.9) we conclude that

$$V_s(\alpha \odot W_s P) \sim \alpha \odot V_s P \preceq V_s W_s Q = V_s Q$$

and therefore summing up gives

$$\sum_{s \in S} (\alpha \odot V_s P) \sim \alpha \odot \sum_{s \in S} V_s P = \alpha \odot E^i_\alpha P \preceq \sum_{s \in S} V_s Q = E^i_\alpha Q.$$

(iii) If α is a limit (and not the first) element[1)] in $\Lambda_i \setminus \{0\}$, $i = \mathrm{I}, \mathrm{III}$ (in $\Lambda_i \setminus \{0, 1\}$ for $i = \mathrm{II}$), then $E^i_\alpha = \bigwedge \{ E^i_\beta :\ \beta < \alpha,\ \beta \ne 0,\ (i, \beta) \ne (\mathrm{II}, 1) \}$. For the proof denote by Z the right-hand side of the above relation and note that $E^i_\alpha \le Z$: indeed, from (i) we have $E^i_\alpha \le E^i_\beta$ for $\beta < \alpha$, so $E^i_\alpha \le Z$. For the converse, we will show that Z contains $\alpha \odot PZ$. If $Z = 0$, then it is trivial. For $Z \ne 0$ we have $PZ \ne 0$ since $C_P = I$ and

$$C_{PZ} = Z C_P = ZI = Z \ne 0.$$

Clearly (when $\beta < \alpha$) $Z \le E^i_\beta$ and, as $\beta \odot E^i_\beta P \preceq E^i_\beta$ therefore we get:

$$Z(\beta \odot E^i_\beta P) = \beta \odot PZ \preceq E^i_\beta Z = Z, \tag{4.10}$$

1) For example, $\aleph_0$ is a limit cardinal, but it is not a limit element in Λ_{III}.

in other words, Z contains $\beta \odot PZ$. But $\alpha = \sum_{0 \neq \beta < \alpha} \beta$ (in case $i = \mathrm{II}$ we exclude $\beta = 1$, all considered β will be called *admissible*). Denote by $\{P_t\}_{t \in T}$ a maximal orthogonal family of subprojections of Z with the property that $P_t \sim PZ$ for $t \in T$ (in particular $P_t \neq 0$). We claim that $|T| \geq \alpha$. To show this, it is enough to show that $|T| \geq \beta$ for each admissible $\beta < \alpha$ (because α is limit). From (4.10), there exists a maximal orthogonal family $\{Q_s\}_{s \in S}$ of subprojections of Z such that $Q_s \sim PZ$ for each $s \in S$ and $|S| \geq \beta$. Arguing as in the proof of Lemma 4.1 we conclude $|T| = |S| \geq \beta$ and we are done.

We have showed that the assumptions of Lemma 4.3 are satisfied. Therefore we can form $Z^i_\alpha(Q) := E^i_\alpha - E^i_{\alpha^+}$ and with the help of this lemma conclude that:

$$\sum_{\alpha \in \mathrm{Card}_\infty} Z^i_\alpha(Q) = E^i_{\aleph_0}, \ i = \mathrm{II}, \mathrm{III} \quad \text{and} \quad \sum_{\alpha > 0} Z^{\mathrm{I}}_\alpha(Q) = E^{\mathrm{I}}_1 \tag{4.11}$$

(note that the second term is missing, because $E^i_\alpha = 0$ for sufficently large α, for example $\alpha := |\mathfrak{E}(\mathfrak{A})|^+$).

We claim that

$$Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P. \tag{4.12}$$

If $Z^i_\alpha(Q) = 0$ then (4.12) is valid, so we assume that $Z^i_\alpha(Q) \neq 0$. Define

$$\tilde{Q} := Z^i_\alpha(Q)Q \quad \text{and} \quad \tilde{P} := Z^i_\alpha(Q)P \neq 0.$$

Then $\tilde{P}$ becomes a steering projection in $\tilde{\mathfrak{A}} := \mathfrak{A}Z^i_\alpha(Q)$. As $Z^i_\alpha(Q) \leq E^i_\alpha$ and $\alpha \odot E^i_\alpha P \preceq E^i_\alpha Q$, we conclude that:

$$Z^i_\alpha(Q)(\alpha \odot E^i_\alpha P) = \alpha \odot Z^i_\alpha(Q)P = \alpha \odot \tilde{P} \preceq Z^i_\alpha(Q)Q = \tilde{Q}.$$

Now take $W \in \mathfrak{Z}(\tilde{\mathfrak{A}})$. Then in particular $W \in \mathfrak{Z}(\mathfrak{A})$ and $W \leq Z^i_\alpha(Q)$. The latter gives $WE^i_{\alpha^+} = 0$. So, if $W \neq 0$, we cannot have $W \leq E^i_{\alpha^+}$ and thus $WQ = W\tilde{Q}$ does not contain $\alpha^+ \odot WP = \alpha^+ \odot W\tilde{P}$. Thus we have shown that all assumptions of Proposition 4.2 are satisfied and therefore

$$\tilde{Q} \sim \alpha \odot \tilde{P}.$$

To complete the proof of the existence, it remains to show that

$$E^{\mathrm{I}}_1 = E^{\mathrm{I}} \quad \text{and} \quad E^i_{\aleph_0} = E^i, \ i = \mathrm{II}, \mathrm{III}. \tag{4.13}$$

(Once we have (4.13) we conclude that:

$$\sum_{\alpha \in \Lambda_1} Z^{\mathrm{I}}_\alpha(Q) = Z^{\mathrm{I}}_0(Q) + \sum_{\alpha > 0} Z^{\mathrm{I}}_\alpha(Q) = Z^{\mathrm{I}}_0(Q) + E^{\mathrm{I}}_1 = Z^{\mathrm{I}}_0 + E^{\mathrm{I}} = Z^{\mathrm{I}},$$

$$\begin{aligned} \sum_{\alpha \in \Lambda_2} Z^{\mathrm{I}}_\alpha(Q) &= Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + \sum_{\alpha \geq \aleph_0} Z^{\mathrm{II}}_\alpha(Q) = \\ &= Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + E^{\mathrm{II}}_{\aleph_0} = Z^{\mathrm{II}}_0(Q) + Z^{\mathrm{II}}_1(Q) + E^{\mathrm{II}} = Z^{\mathrm{II}}, \end{aligned}$$

$$\sum_{\alpha\in\Lambda_3} Z_\alpha^{\mathrm{III}}(Q) = Z_0^{\mathrm{III}}(Q) + \sum_{\alpha\geq\aleph_0} Z_\alpha^{\mathrm{III}}(Q) = Z_0^{\mathrm{III}}(Q) + E_{\aleph_0}^{\mathrm{III}} = Z_0^{\mathrm{III}}(Q) + E^{\mathrm{III}} = Z^{\mathrm{III}}.)$$

For the proof of (4.13) note that $E_1^{\mathrm{I}} \leq E^I$ and $E_{\aleph_0}^i \leq E^i$ as a straightforward consequence of the definition of E_α^i. For the reverse inequality it is sufficient to show that E^i appears in the family defining E_α^i with suitable i and α, i.e.[2)]

$$E^{\mathrm{I}}P \preceq E^{\mathrm{I}}Q, \quad \omega \odot E_{\aleph_0}^i P \preceq E_{\aleph_0}^i Q. \tag{4.14}$$

Note that, as E^i are central, $E^i \leq C_Q$ and $C_P = I$, then

$$C_{E^iQ} = E^iC_Q = E^i = E^iC_P = C_{E^iP}$$

and thus $E^iP \leq C_{E^iP} = C_{E^iQ}$. For $i = \mathrm{I}$ we have that PE^{I} is a steering projection in some von Neumann algebra of type I, in particular it is quasi-abelian, so Lemma 3.4 ensures us that $E^{\mathrm{I}}P \preceq E^{\mathrm{I}}Q$, giving the first part of (4.14).

For $i = \mathrm{III}$ we have that E^{III} is a steering projection in some von Neumann algebra of type III and again, from Lemma 3.4 we get $E^{\mathrm{III}}P \preceq E^{\mathrm{III}}Q$. But $E^{\mathrm{III}}P$ is properly infinite, hence $E^{\mathrm{III}}P \sim \omega \odot E^{\mathrm{III}}P$ and thus $\omega \odot E^{\mathrm{III}}P \preceq E^{\mathrm{III}}Q$.

Finally, for $i = \mathrm{II}$ recall that $E^{\mathrm{II}} \leq Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ thus $E^{\mathrm{II}}P$ is a steering projection in some von Neumann algebra of type II_∞. Recall also that $E^{\mathrm{II}}Q$ is properly infinite. This again allows us to conclude that $E^{\mathrm{II}}P \preceq E^{\mathrm{II}}Q$, this time from Proposition 3.10 (and the definition of a steering projection in a type II_∞ von Neumann algebra). Moreover, $E^{\mathrm{II}}P \sim \omega \odot E^{\mathrm{II}}P$ which gives the remaining part of (4.14).

Uniqueness. Suppose that we have two such systems $\{Z_\alpha^i\}$ and $\{W_\alpha^i\}$ (with α and i varying as before). Our task is to show

$$Z_\alpha^i = W_\alpha^i. \tag{4.15}$$

For the proof of (4.15) with $\alpha = 0$ we will show that

$$Z_0^i = (I - C_Q)Z^i. \tag{4.16}$$

Once (4.16) is proved, it gives (4.15) with $\alpha = 0$ since (4.16) gives an explicit formula for Z_0^i and the roles of Z_α^i and W_α^i are symmetric.

We have that

$$Z_0^iQ \sim 0 \odot Z_0^iP = 0$$

and therefore $Z_0^iQ = 0$, hence[3)] $Z_0^i \leq I - C_Q$. Since obviously $Z_0^i \leq Z^i$, we have

$$Z_0^i \leq (I - C_Q)Z^i.$$

We will now show the reverse inequality.

For $(i, \alpha) \neq (\mathrm{II}, 1)$ we have $Z_\alpha^iQ \sim \alpha \odot Z_\alpha^iP$ and if $W \leq Z_\alpha^i$ is central then

$$WQ = W(Z_\alpha^iQ) \sim W(\alpha \odot Z_\alpha^iP) = \alpha \odot WP$$

[2)] Here $\alpha = 1$ for $i = \mathrm{I}$ and $\alpha = \aleph_0$ for $i = \mathrm{II}, \mathrm{III}$.

[3)] From the definition of the central carrier, or from the fact that $C_{Z_0^iQ} = Z_0^iC_Q = 0$.

and this means that $WQ = 0$ implies $WP = 0$, and as $C_P = I$, it also implies that $W = 0$. In particular, for $W := (I - C_Q)Z^i_\alpha$ (being central) we have $WQ = 0$, thus $W = 0$ so $Z^i_\alpha = Z^i_\alpha C_Q$ and obviously $Z^i_\alpha \leq Z^i$ which together yields $Z^i_\alpha \leq C_Q Z^i$.

For $(i, \alpha) = (\mathrm{II}, 1)$ the projection Z^{II}_1 has the property that for every central projection $0 \neq W \leq Z^{\mathrm{II}}_1$, $WQ \neq 0$. Thus for $W := (I - C_Q)Z^{\mathrm{II}}_1$ we obtain as before $W = 0$ and finally $Z^{\mathrm{II}}_1 \leq C_Q Z^{\mathrm{II}}$. Thus we have proved that $Z^i_\alpha \leq C_Q Z^i$ for $i = \mathrm{I}, \mathrm{III}, \mathrm{III}$ and $\alpha \neq 0$. Summing up over all $\alpha \neq 0$ yields:

$$\sum_{\alpha > 0} Z^i_\alpha = Z^i - Z^i_0 \leq C_Q Z^i$$

which means that $Z^i(I - C_Q) \leq Z^i_0$ and proves (4.16).

Now we will prove (4.15) for $(i, \alpha) = (\mathrm{II}, 1)$. For $\alpha \geq \aleph_0$ we have $Z^{\mathrm{II}}_\alpha \sim \alpha \odot Z^{\mathrm{II}}_\alpha P$ and the latter projection is properly infinite (or 0)[4)], so $Z^{\mathrm{II}}_\alpha Q$ is properly infinite or 0 and the same is true for $\sum_{\alpha \geq \aleph_0} Z^{\mathrm{II}}_\alpha Q$. But the sum appearing in this expression is equal to $Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1$ – this means that $(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1)Q$ is properly infinite or 0. Since $W^{\mathrm{II}}_1 Q$ is finite, then $W^{\mathrm{II}}_1(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1)Q$ is also finite. But at the same time, since W^{II}_1 is central, we have that $W^{\mathrm{II}}_1(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1)Q$ is properly infinite or 0. Therefore this second possibility takes place:

$$W^{\mathrm{II}}_1(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1)Q = 0.$$

By taking the central carrier we get

$$W^{\mathrm{II}}_1(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1)C_Q = 0$$

but from the above discussion we know that $W^{\mathrm{II}}_1 \leq C_Q Z^{\mathrm{II}} \leq C_Q$, hence we get:

$$W^{\mathrm{II}}_1(Z^{\mathrm{II}} - Z^{\mathrm{II}}_0 - Z^{\mathrm{II}}_1) = 0$$

therefore

$$\underbrace{W^{\mathrm{II}}_1 Z^{\mathrm{II}}}_{=W^{\mathrm{II}}_1} - \underbrace{W^{\mathrm{II}}_1(I - C_Q)}_{=0} Z^{\mathrm{II}} - W^{\mathrm{II}}_1 Z^{\mathrm{II}}_1 = 0.$$

This means that $W^{\mathrm{II}}_1 = W^{\mathrm{II}}_1 Z^{\mathrm{II}}_1 \leq Z^{\mathrm{II}}_1$, and by repeating this argument (the roles of Z^{II}_1 and W^{II}_1 are symmetric) we obtain $W^{\mathrm{II}}_1 = Z^{\mathrm{II}}_1$.
Now we deal with the case $\alpha \neq 0$ and $(i, \alpha) \neq (\mathrm{II}, 1)$. We claim that

$$Z^i_\alpha W^i_\beta = 0 \text{ whenever } \beta \in \Lambda_i, \beta \neq \alpha. \tag{4.17}$$

For $\beta = 0$ or $(i, \beta) = (\mathrm{II}, 1)$ it has been already proved, as in these cases $Z^i_\beta = W^i_\beta$ and Z^i_α's are all mutually orthogonal. For $\beta > 0$ (in the case $i = \mathrm{I}$) and $\beta \in \mathrm{Card}_\infty$ (in the case $i \neq \mathrm{I}$) as $Z^i_\alpha Q \sim \alpha \odot Z^i_\alpha P$ we have

$$W^i_\beta Z^i_\alpha Q \sim \alpha \odot W^i_\beta Z^i_\alpha P.$$

[4)] Later we will not always underline this fact, although it should be kept in mind.

But at the same time, from $W^i_\beta Q \sim \beta \odot W^i_\beta P$ we get

$$W^i_\beta Z^i_\alpha Q \sim \beta \odot W^i_\beta Z^i_\alpha P.$$

These together yield

$$\alpha \odot W^i_\beta Z^i_\alpha P \sim \beta \odot W^i_\beta Z^i_\alpha P.$$

Now $W^i_\beta Z^i_\alpha \neq 0$ implies $W^i_\beta Z^i_\alpha P \neq 0$ (recall that $C_P = I$) and then $W^i_\beta Z^i_\alpha P$ is a steering projection in some von Neumann algebra of an appropriate type. By virtue of Proposition 4.1 we obtain $\alpha = \beta$, thus proving (4.17). From (4.17) we have

$$Z^i_\alpha \sum_{\beta \in \Lambda_i \setminus \{\alpha\}} W^i_\alpha = Z^i_\alpha (Z^i - W^i_\alpha) = Z^i_\alpha - Z^i_\alpha W^i_\alpha = 0$$

and thus $Z^i_\alpha \leq W^i_\alpha$. Changing the role of Z^i_α and W^i_α we get (4.15). The whole proof is complete. □

The above theorem may be seen as a variation of a result of Tomiyama (Theorem 1 in [11]). A careful reader may notice that our construction is essentially in the same spirit as there.

Corollary 4.5. *Let $Q, Q' \in \mathfrak{A}$ be two projections. Then the following conditions are equivalent:*

a) $Q \preceq Q'$,
b) $Z^i_\alpha(Q) Z^i_\beta(Q') = 0$ *whenever* $\alpha > \beta$; *and* $Z^{\mathrm{II}}_1(Q) Z^{\mathrm{II}}_1(Q') Q \preceq Z^{\mathrm{II}}_1(Q) Z^{\mathrm{II}}_1(Q') Q'$.

Proof. Step 1. First we show the following criterion for the Murray-von Neumann equivalence: for two projections $Q, Q' \in \mathfrak{A}$ we have

$$Q \sim Q' \iff Z^i_\alpha(Q) = Z^i_\alpha(Q') \ (i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\},\ \alpha \in \Lambda_i) \text{ and } Z^{\mathrm{II}}_1(Q) Q \sim Z^{\mathrm{II}}_1(Q') Q'.$$

For the proof assume first that $Q \sim Q'$. Then $Z^i_\alpha(Q') Q \sim Z^i_\alpha(Q') Q' \sim \alpha \odot Z^i_\alpha(Q') P$ for $(i, \alpha) \neq (\mathrm{II}, 1)$. For $(i, \alpha) = (\mathrm{II}, 1)$ we have $Z^{\mathrm{II}}_1(Q') Q \sim Z^{\mathrm{II}}_1(Q') Q'$; since the latter projection is finite, so is $Z^{\mathrm{II}}_1(Q') Q$. Further, $Q \sim Q'$ implies $C_Q = C_{Q'}$ and thus if $0 \neq W \leq Z^{\mathrm{II}}_1(Q') C_Q$ is a central projection, then $0 \neq W \leq Z^{\mathrm{II}}_1(Q') C_{Q'}$ and thus $WQ \neq 0$. Finally, observe that

$$(Z^{\mathrm{II}} - Z^{\mathrm{II}_1}) Z^{\mathrm{II}}_1(Q') P \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1}) Z^{\mathrm{II}}_1(Q') Q' \sim \omega \odot (Z^{\mathrm{II}} - Z^{\mathrm{II}_1}) Z^{\mathrm{II}}_1(Q') Q.$$

So, we have proved that the system $\{Z^i_\alpha\}_{i,\alpha}(Q')$ have all the desired properties (for Q) – from its uniqueness we conclude that $Z^i_\alpha(Q) = Z^i_\alpha(Q')$ for all $i \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and $\alpha \in \Lambda_i$ – then also $Z^{\mathrm{II}}_1(Q) Q \sim Z^{\mathrm{II}}_1(Q') Q'$.

For the proof of the converse implication, denote for simplicity $Z^i_\alpha := Z^i_\alpha(Q) = Z^i_\alpha(Q')$. We have then in particular $Z^{\mathrm{II}}_1 Q \sim Z^{\mathrm{II}}_1 Q'$ and for $(i, \alpha) \neq (\mathrm{II}, 1)$:

$$Z^i_\alpha Q \sim \alpha \odot Z^i_\alpha P \sim Z^i_\alpha Q'.$$

Since $\sum_{i\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}}\sum_{\alpha\in\Lambda_i} Z^i_\alpha = I$, then

$$Q = \sum_{i,\alpha} Z^i_\alpha Q \sim \sum_{i,\alpha} Z^i_\alpha Q' = Q'.$$

Step 2. Now we shall check that (b) is implied by (a). The second part of our statement is immediate. First let $(i,\alpha),(i,\beta) \neq (\mathrm{II},1)$. Then, as $Z^i_\alpha(Q)Q \sim \alpha\odot Z^i_\alpha(Q)P$ and $Z^i_\beta(Q')Q' \sim \beta\odot Z^i_\beta(Q')P$ and $Q \preceq Q'$, we have:

$$\alpha\odot Z^i_\alpha(Q)Z^i_\beta(Q')P \sim Z^i_\alpha(Q)Z^i_\beta(Q')Q \preceq Z^i_\alpha(Q)Z^i_\beta(Q')Q',$$

but, as $Z^i_\beta(Q')Q' \sim \beta\odot Z^i_\beta(Q')P$, we have

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q' \sim \beta\odot Z^i_\alpha(Q)Z^i_\beta(Q')P.$$

Therefore we have (if we denote, to simplify, $Z := Z^i_\alpha(Q)Z^i_\beta(Q')$) that $\alpha\odot ZP \preceq \beta\odot ZP$. Now if we assume $\beta < \alpha$, then the converse inequality is also true, namely $\beta\odot ZP \preceq \alpha\odot ZP$, hence $\alpha\odot ZP \sim \beta\odot ZP$. Now suppose, on the contrary, that $Z \neq 0$. Then $ZP \neq 0$ too and ZP is then a steering projection in some von Neumann algebra, and then from Lemma 4.1 we conclude $\alpha = \beta$ which is a contradiction.

Further, we see that $Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q')Q \preceq Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q')Q' = 0$ and hence, by point (c) of Theorem 4.4 for Q, $Z^{\mathrm{II}}_1(Q)Z^{\mathrm{II}}_0(Q') = 0$. Finally, if $\alpha \in \mathrm{Card}_\infty$, then $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q \preceq Z^{\mathrm{II}}_1(Q')Q'$, which implies that $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q$ is both properly infinite (since $Z^{\mathrm{II}}_\alpha(Q)Q$ is such) and finite (since $Z^{\mathrm{II}}_1(Q')Q'$ is finite). We infer that $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q')Q = 0$ and consequently $Z^{\mathrm{II}}_\alpha(Q)Z^{\mathrm{II}}_1(Q') = 0$, because $Z^{\mathrm{II}}_\alpha(Q) \leq C_Q$. This finishes the proof of (b).

Step 3. For the proof that (a) follows from (b), note that, since

$$\sum_i \sum_{\alpha,\beta\in\Lambda_i} Z^i_\alpha(Q)Z^i_\beta(Q') = I,$$

so it is enough to show that

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q \preceq Z^i_\alpha(Q)Z^i_\beta(Q')Q', \quad \alpha,\beta\in\Lambda_i. \tag{4.18}$$

Fix $i \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}$ and $\alpha,\beta \in \Lambda_i$ and consider the cases:

(1°) $\alpha > \beta$; then from our assumptions $Z^i_\alpha(Q)Z^i_\beta(Q') = 0$ and (4.18) is satisfied.
(2°) $(i,\alpha) = (i,\beta) = (\mathrm{II},1)$; then (4.18) is exactly the second part of our assumption.

(3°) $\alpha \leq \beta$ and $(i, \beta) = (\mathrm{II}, 1), (i, \alpha) \neq (\mathrm{II}, 1)$. Then, in particular, $i = \mathrm{II}$ and the only possibility is that $\alpha = 0$. In this case $Z^i_\alpha(Q)Q \sim 0 \odot Z^i_\alpha(Q)P = 0$, so $Z^i_\alpha(Q)Z^i_\beta(Q')Q = 0$ and (4.18) is satisfied.

(4°) Both (i, α) and (i, β) are different from $(\mathrm{II}, 1)$ and $\alpha \leq \beta$. Then, as before, we conclude from the relations $Z^i_\alpha(Q)Q \sim \alpha \odot Z^i_\alpha(Q)P$ and $Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\beta(Q')P$ that

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q \sim \alpha \odot Z^i_\alpha(Q)Z^i_\beta(Q')P,$$

$$Z^i_\alpha(Q)Z^i_\beta(Q')Q' \sim \beta \odot Z^i_\alpha(Q)Z^i_\beta(Q')P.$$

Since $\alpha \leq \beta$ then:

$$\alpha \odot Z^i_\alpha(Q)Z^i_\beta(Q')P \preceq \beta \odot Z^i_\alpha(Q)Z^i_\beta(Q')P$$

and (4.18) is satisfied.

(5°) The remaining case is $\alpha \leq \beta, (i, \alpha) = (\mathrm{II}, 1), (i, \beta) \neq (\mathrm{II}, 1)$. In this case $i = \mathrm{II}$ and hence $\beta > \alpha$ means that $\beta \geq \aleph_0$. As we have $Z^{\mathrm{II}}_\beta(Q')Q' \sim \beta \odot Z^{\mathrm{II}}_\beta(Q')P$ with infinite β, we conclude that $Z^{\mathrm{II}}_\beta(Q')Q'$ is properly infinite (or 0). But Z^{II_1} is central and finite, thus $Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q')Q' = 0$. Taking the central carrier gives

$$Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q')C_{Q'} = 0. \tag{4.19}$$

But recall[5)] that $Z^{\mathrm{II}}_\beta(Q') \leq C_{Q'}$. Thus (4.19) transforms into $Z^{\mathrm{II}_1}Z^{\mathrm{II}}_\beta(Q') = 0$. We conclude that $Z^{\mathrm{II}}_\beta(Q') \leq Z^{\mathrm{II}} - Z^{\mathrm{II}_1}$ and then (by point (c) of Theorem 4.4 for Q) $Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \sim \omega \odot Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q$. So:

$$Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q \preceq Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \preceq \beta \odot Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)P \sim Z^{\mathrm{II}}_\beta(Q')Z^{\mathrm{II}}_1(Q)Q'$$

and we are done.

□

Remark 4.6. Recently Sherman [9] proved (using Lemma 3.7 and a variation of Theorem 4.4) that the Murray-von Neumann order in an arbitrary W^*-algebra is complete. That is, every set of projections has the g.l.b. as well as the l.u.b. with respect to '$\preceq$'.

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5) It follows from the proof of the previous theorem: $Z^{\mathrm{II}}_\beta(Q') \leq E^{\mathrm{II}} \leq (Z^{\mathrm{II}} - Z^{\mathrm{II}_1})C_{Q'} \leq C_{Q'}$.

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Adam Wegert
Adam.Wegert@im.uj.edu.pl

Jagiellonian University
Faculty of Mathematics and Computer Science
ul. Łojasiewicza 6, 30-348 Krakow, Poland

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