Group Decision Making Problem by General Convexity or Concavity and Aggregation Process

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Abstract:
The averaging aggregation operators are defined and some interesting properties are derived. Moreover, we have extended concave and convex property. The main results concerning aggregation of generalized quasiconcave and quasiconvex functions are presented and some their properties are derived and discussed. The class of concavity and convexity of two variable aggregation operators that preserve these properties are studied.

Keywords: *aggregation functions, interval-valued fuzzy sets, S-convexity, T-concavity*

1. Introduction

In this paper is focused on aggregation of a finite number of real numbers into a single number and its use in designing new classes of generalized convex functions that may be useful in optimization theory and decision analysis. In decision making, aggregated values are typically preference or satisfaction degrees restricted to the unit interval [0, 1]. Here, a decision problem in X is considered, i.e., the problem to find the "best" decision in the set of feasible decisions X with respect to several criteria functions, where preferences via interval-valued fuzzy values are represented.

Interval-valued fuzzy relations are a tool that make it possible to model in an effective way imperfect information. In this paper also the transitivity problem of interval-valued fuzzy relations is discussed. Transitivity property reflects the consistency of a preference relation. Therefore transitivity is important from the point of view of real problems appearing, e.g., in group decision making, choice and utility theories widely making use of the preference procedures. In economics, the utility function measures welfare or satisfaction of a consumer as a function of consumption of real goods, such as food, clothing and composite goods rather than nominal goods measured in nominal terms. Utility function is widely used in the rational choice theory to analyze human behavior. Thus interval-valued fuzzy relations can be applied in group decision making problems in a situation when a solution from the individual preferences over some set of options should be derived. We will consider here group decision making while each option fulfills a set of criteria to some extent and, on the other hand, it does not fulfill this set of criteria to some extent (the alternatives can be conveniently expressed via interval-valued fuzzy sets). Group decision making and the notion of the interval fuzzy alternatives can be applied in social choice theory. Social choice theory is the study of group decision processes and procedures, concerning the aggregation of individual inputs (e.g., votes, preferences, judgments, welfare) into collective outputs (e.g., group decisions, preferences, judgments, welfare). Social choices can be made by: voting or the market mechanism, typically used to make political decisions or economic decisions, respectively. Social choice theory took off in the 20th century with the Kenneth Arrow works [2] and Amartya Sen [25]. It is influence extends across economics, political science, philosophy, mathematics, and recently computer science and biology. Arrow's impossibility theorem arises by adding additional condition, transitive rationality: the social choice function is derived from a social welfare function in the sense that the winners are the elements which are ranked highest by the social welfare function. Apart from contributing to our understanding of collective decision procedures, social choice theory has applications in the areas of institutional design, welfare economics, and social epistemology. There, preference relations appear, for example, in choice and utility theories. Preference relations are of great interest nowadays because of their applications [31], [33] or [13]. Among others, transitivity property of interval-valued fuzzy relations is examined. This property is important because of its possible applications in the preference procedures [32]. The accuracy of the final ranking of the alternatives must be based on consistent judgments as an inconsistent preference relation may lead to wrong conclusions. Traditionally, the consistency of a preference relation is characterized by transitivity. Convexity of utility functions is one of the most important aspects connected with the study of geometric properties of not only crisp, but also fuzzy sets and interval-valued fuzzy sets. Some generalization of convexity in multi-expert decision problems by penalty function is used (see [5], [7]). Various definitions of convexity and its generalized versions are widely used, mostly in optimization problems (see [23]). An important property of convex sets is that convexity is preserved under aggregations. We will study a similar question for the case of quasiconvex, quasiconcave and T-concave, S-convex interval-valued fuzzy sets. Together with the aggregation of lattice elements we will consider also an aggregation of interval-valued fuzzy sets.

Aggregation is a fundamental process in group decision making and in other scientific disciplines where the fusion of different pieces of information for obtaining the final result is important. In the group decision making a finite set of alternatives $X = \{x_1, ..., x_n\}$ and a finite set of criteria on the base of which the alternatives are evaluated $Y = \{y_1, ..., y_k\}$ may be considered. Interval-valued fuzzy relations $R_1, ...R_k$ on a set X corresponding to each criterion are provided. With the use of a aggregation function the aggregated fuzzy relation is obtained and it is supposed to help decision makers to make up their minds. There are several works contributing to the problem of preservation of properties of interval-valued fuzzy relations during aggregation process, such as transitivity, convexity or concavity during group decision making. In the next part of this paper we will explore reciprocal properties. Considering decision making problems in the interval fuzzy environment [8, 18, 21, 33] we deal with the finite set of alternatives X and an expert providing his/her preference information over alternatives. In the sequel, we will consider a preference relation on a finite set X which makes it possible to represent interval-valued fuzzy relations by matrices $R = [\underline{R}, \overline{R}]$. Concerning both $\underline{R}(i,j)$ and $\overline{R}(i,j)$, as follows:

$R(i,j) = [0.5, 0.5]$ *indicates indifference between*

$$x_i \; and \; x_j \; (x_i \sim x_j),$$

$R(i,j) > [0.5, 0.5]$ *represents an uncertain*

$$preference \; of \; x_i \; over \; x_j$$

$$(x_i > x_j \; (x_i \geq x_j \; for \; \rho_{ij} \geq [0.5, 0.5])),$$

$R(i,j) = [1,1]$ *when x_i is definitely (certainly)*

$$preferred \; to \; x_j,$$

$R(i,j) = [0,0]$ *when x_j is definitely (certainly)*

$$preferred \; to \; x_i.$$

In this case, the preference matrix, R, is usually assumed additive reciprocal, i.e., $\underline{R}(i,j), \overline{R}(i,j)$ satisfy the following characteristics for all $i, j = 1, ..., n$:

$$\underline{R}(i,j) = 1 - \overline{R}(j,i), \;\; \underline{R}(i,i) = \overline{R}(i,i) = 0.5.$$

The aim of this paper is also to present algorithm with different comparability relation on interval values, which follow from the theoretical results presented here. These algorithms theoretical results are compared in order to obtain the most useful practical result.

The paper is structured as follows. In Section 2 equivalence relation on lattice is introduced and suitable definitions and properties are mentioned. Then, aggregation functions and their basic properties are defined, some interesting properties are derived. In Section 3 T-concave and S-convex functions in some lattice are considered, i.e. in family of interval-valued fuzzy relations. Finally the results concerning aggregation of generalized concave and convex functions are presented and discussed and example of application is presented.

2. Basic Definitions

The idea of a lattice was defined by Birkhoff in 1967.

Definition 2.1 ([4]) • *A poset $(P, \leq)$ is a set P with a relation $\leq$ which is reflexive, antisymmetric and transitive. A chain in a poset is a totally ordered set.*
• *A lattice $L = (L, \leq, \wedge, \vee)$ is a poset with the partial ordering $\leq$ in L and operations $\wedge$ and $\vee$ satisfying the properties of absorption, idempotency, commutativity and associativity. That is a poset such that any two elements have a unique minimal upper bound and a unique maximal lower bound in L.*
• *A lattice is called a bounded lattice if $1_L = \max L \in L$, $0_L = \min L \in L$.*
• *A lattice is called a complete lattice if for arbitrary set $A \in L$ there exist $\inf A \in L$ and $\sup A \in L$.*

2.1. Equivalent Relation and Quasiconcavity, Quasiconvexity

Now the notion of equivalent relation is recalled. Using a modified relation equivalence from Murali 2002 we put:

Definition 2.2 ([19], [20]) *Functions $f, g : X \to L$ are equivalent ($f \sim g$), if*

$$f(x) \leq f(y) \Leftrightarrow g(x) \leq g(y) \; for \; x, y \in X. \quad (1)$$

We can easily see

Lemma 2.3 *Let L be a lattice, $f, g : X \to L$ are equivalent if and only if, there exists bijection $\varphi : V_f \to V_g$ isotonic with φ^{-1} such that*

$$\forall_{x \in X} g(x) = \varphi(f(x)).$$

More generally to [10] we have the following results:

Lemma 2.4 *Let L be a lattice, $f, g : X \to L$ be equivalent. Then*

$$\forall_{x,y \in X}(f(x) = f(y) \Leftrightarrow g(x) = g(y)), \quad (2)$$

$$\forall_{x,y \in X}(f(x) \neq f(y) \Leftrightarrow g(x) \neq g(y)), \quad (3)$$

$$\forall_{x,y \in X}(f(x) > f(y) \Leftrightarrow g(x) > g(y)), \quad (4)$$

$$\forall_{x,y \in X}(f(x) \,||\, f(y) \Leftrightarrow g(x) \,||\, g(y)). \quad (5)$$

Proof. We consider (2). Let $f \sim g$ and $x, y \in P \subset X$. Then by Definition 2.2 we can write
$f(x) = f(y) \Leftrightarrow (f(x) \geq f(y) \; and \; f(x) \leq f(y)) \Leftrightarrow (g(x) \geq g(y) \; and \; g(x) \leq g(y)) \Leftrightarrow g(x) = g(y)$,
which prove (2), a condition (3) we obtain by complement.
By Definition 2.2 and (2), (3) we have
$f(x) > f(y) \Leftrightarrow (f(x) \geq f(y) \; and \; f(x) \neq f(y)) \Leftrightarrow (g(x) \geq g(y) \; and \; g(x) \neq g(y)) \Leftrightarrow g(x) > g(y)$.
Thus (4) is true.
By (4) we have:
$f(x) \,||\, f(y) \Leftrightarrow$
$NOT(f(x) \leq f(y) \; or \; f(x) > f(y)) \Leftrightarrow$

$(NOT(f(x) \leq f(y))$ and $NOT(f(x) > f(y))) \Leftrightarrow$
$(NOT(g(x) \leq g(y))$ and $NOT(g(x) > g(y))) \Leftrightarrow$
$NOT(g(x) \leq g(y)$ or $g(x) > g(y)) \Leftrightarrow g(x) \,||\, g(y)$,
what end the proof.

Proposition 2.5 $f, g : X \to L$ *are equivalent if and only if f and g fulfill (4) and (5).*

Proof. By Lemma 2.4 we see, that $(1) \Rightarrow (4)$ i (5). By (4) and (5) we have
$NOT(f(x) \leq f(y)) \Leftrightarrow$
$f(x) > f(y)$ or $f(x) \,||\, f(y) \Leftrightarrow$
$g(x) > g(y)$ or $g(x) \,||\, g(y) \Leftrightarrow NOT(g(x) \leq g(y))$
the negation of both sides give (1).
   By above theorem and Lemma 2.4 we obtain

Corollary 2.6 *Let $f, g : X \to L$. If f and g fulfill (4) and (5), then also fulfill (2) and (3).*

Proposition 2.7 *Let $f, g : X \to L$. If f and g fulfills (2) and (4), then they fulfill (5).*

Proof. By Lemma 2.4 we have $NOT(f(x) \,||\, f(y)) \Leftrightarrow$
$f(x) > f(y)$ or $f(x) < f(y)$ or $f(x) = f(y) \Leftrightarrow$
$g(x) > g(y)$ or $g(x) < g(y)$ or $g(x) = g(y) \Leftrightarrow$
$NOT(g(x) \,||\, g(y))$, so we prove (5).

Lemma 2.8 *Let L be a complete lattice, $f, g : X \to L$. If $f \sim g$, then*

$$\forall_{x \in X}(f(x) = \bigvee_{y \in P \subset X} f(y) \Leftrightarrow g(x) = \bigvee_{i \in \mathbb{N}} g(y)), \quad (6)$$

$$\forall_{x \in X}(f(x) = \bigwedge_{y \in P \subset X} f(y) \Leftrightarrow g(x) = \bigwedge_{i \in \mathbb{N}} g(y)). \quad (7)$$

Proof. Let $P \subset X$, $x, z \in X$ and $f(x) = \bigwedge_{y \in P \subset X} f(y)$. Then by definition of infimum we have

$$f(x) = \bigwedge_{y \in P \subset X} f(y) \Leftrightarrow \forall_{y \in P}(f(x) \leq f(y)) \text{ and}$$
$$(\forall_{y \in P}(f(z) \leq f(y)) \Rightarrow (f(z) \leq f(x))). \quad (8)$$

By (1) we have

$$\forall_{y \in P}(f(x) \leq f(y)) \Leftrightarrow \forall_{y \in P}(g(x) \leq g(y)),$$

$$\forall_{y \in P}(f(z) \leq f(y)) \Leftrightarrow \forall_{y \in P}(g(z) \leq g(y)),$$

$$f(z) \leq f(x) \Leftrightarrow g(z) \leq g(x),$$

then by (8) we obtain

$$\forall_{y \in P}(g(z) \leq g(y)) \Rightarrow (g(z) \leq g(x)),$$

thus

$$g(x) = \bigwedge_{y \in P} g(y).$$

Similarly we can prove the converse implication in (7) and by the notion of supremum we may prove (6).

   From now on, the set $X \neq \emptyset$, $X \subseteq R^n$.

Definition 2.9 ([1]) *Let $X \subseteq R^n$. A function $f : X \to L$ is called*
• *quasiconcave on X if $f(\lambda x + (1 - \lambda)y) \geq f(x) \wedge f(y)$,*
• *quasiconvex on X if $f(\lambda x + (1 - \lambda)y) \leq f(x) \vee f(y)$,*
for every $x, y \in X$ and every $\lambda \in [0, 1]$ with $\lambda x + (1 - \lambda)y \in X$.

   We know that if a function is concave (convex), then it is quasiconcave (quasiconvex), but not vice-versa.
By lemmas 2.4 and 2.8 we observe

Corollary 2.10 *Let L be a linear ordered lattice. Let $f, g : X \to L$. If f is quasiconcave (quasiconvex) and $f \sim g$, then g is also quasiconcave (quasiconvex).*

Moreover, we will consider transitivity property and dually transitive property.

Definition 2.11 *Let $f, g : L^2 \to L$.*
• *f is transitive if*

$$f(a, b) \wedge f(b, c) \leq f(a, c),$$

• *f is dually transitive if*

$$f(a, b) \vee f(b, c) \geq f(a, c).$$

For equivalence binary relations we have

Theorem 2.12 *Let L be a linear ordered lattice. If $f \sim g$, then f is transitive (dually transitive) if and only if g is transitive (dually transitive).*

Proof. Let $f \sim g$, f be transitive and $x, y, z \in L$. We consider:
If $f(x, y) \leq f(y, z)$, then

$$f(x, y) = f(x, y) \wedge f(y, z) \Leftrightarrow g(x, y) = g(x, y) \wedge g(y, z),$$

$$f(x, y) \leq f(x, z) \Leftrightarrow g(x, y) \leq g(x, z),$$

so

$$f(x, y) \wedge f(y, z) \leq f(x, z) \Leftrightarrow g(x, y) \wedge g(y, z) \leq g(x, z). \quad (9)$$

If $f(x, y) > f(y, z)$, then

$$f(y, z) = f(x, y) \wedge f(y, z) \Leftrightarrow g(y, z) = g(x, y) \wedge g(y, z),$$

$$f(y, z) \leq f(x, z) \Leftrightarrow g(y, z) \leq g(x, z),$$

so f and g are transitive. Similarly we may prove dually transitivity.

3. T-concavity and S-convexity for Binary Relations

   We recall the concept of an aggregation function on L^I, which is a crucial definition for this paper and

$$L^I = \{[x_1, x_2] : x_1, x_2 \in [0, 1] : x_1 \leq x_2\}$$

$[x_1, x_2] \leq_{L^I} [y_1, y_2] \Leftrightarrow x_1 \leq y_1 \wedge x_2 \leq y_2$,
with operations
$[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)]$,
$[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)]$. $(L^I, \leq_L)$ is a complete lattice with the units $1_{L^I} = [1, 1]$ and $0_{L^I} = [0, 0]$.

Definition 3.1 ([3], [14]) *An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an aggregation function if it is increasing with respect to the order $\leq$ and*

$$\mathcal{A}(\underbrace{0_{L^I}, ..., 0_{L^I}}_{n\times}) = 0_{L^I}, \quad \mathcal{A}(\underbrace{1_{L^I}, ..., 1_{L^I}}_{n\times}) = 1_{L^I} .$$

A relevant class of aggregation functions is that of representable aggregation functions.

Definition 3.2 ([9]) *Let $\mathcal{A} : (L^I)^2 \to L^I$ be an aggregation function. $\mathcal{A}$ is said to be a representable aggregation function if there exist two (real) aggregation functions $A_1, A_2 : [0,1]^2 \to [0,1]$ such that, for every $[x_1, x_2], [y_1, y_2] \in L^I$, $A_1 \leq A_2$ it holds that*

$$\mathcal{A}([x_1, x_2], [y_1, y_2]) = [A_1(x_1, y_1), A_2(x_2, y_2)].$$

Observe that both $\wedge$ and $\vee$ on L^I define representable aggregation functions on L^I, with $A_1 = A_2 =$ min in the first case and $A_1 = A_2 =$ max in the second. Moreover, many other examples of representable aggregation functions may be considered, such as:
- the representable product

$$\mathcal{A}_P([x_1, x_2], [y_1, y_2]) = [x_1 y_1, x_2 y_2],$$

- the representable arithmetic mean

$$\mathcal{A}_{mean}([x_1, x_2], [y_1, y_2]) = [\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}],$$

- the representable geometric mean

$$\mathcal{A}_g([x_1, x_2], [y_1, y_2]) = [\sqrt{x_1 y_1}, \sqrt{x_2 y_2}],$$

- the representable product-mean

$$\mathcal{A}_{P,mean}([x_1, x_2], [y_1, y_2]) = [x_1 y_1, \frac{x_2 + y_2}{2}].$$

Example 3.3 *Let $A : [0,1]^2 \to [0,1]$ be an aggregation function. The function $\mathcal{A} : (L^I)^2 \to L^I$, where*

$$\mathcal{A}(x,y) = \begin{cases} [1,1], & (x,y) = ([1,1],[1,1]) \\ [0, A(x_1, y_2)], & otherwise \end{cases}$$

is a non-representable aggregation function on L^I.

Special classes of aggregation functions are t-norms and t-conorms (introduced by De Baets and Mesiar in 1999)

Definition 3.4 (cf. [17]) *A triangular norm T on a bounded lattice L is an increasing, commutative, associative operation $T : L^2 \to L$ with the neutral element 1_L.*
A triangular conorm S on L is an increasing, commutative, associative operation $S : L^2 \to L$ with the neutral element 0_L.

Especially we have representable T-norm or S-conorm, i.e.

$$T(x,y) = [T_1(x_1, y_1), T_2(x_2, y_2)],$$

where T_1, T_2 are fuzzy t-norms and $T_1 \leq T_2$ or

$$S(x,y) = [S_1(x_1, y_1), S_2(x_2, y_2)],$$

where S_1, S_2 are fuzzy t-conorms and $S_1 \leq S_2$.

Now we will consider T-concavity (S-convexity) for relations, thus more generally than in [15] we have following definition.

Definition 3.5 *Let $X \subseteq R^n$, L be a bounded lattice. A function $f : X^2 \to L$ is called*
• *T-concave on X^2 if*

$$f(\lambda x + (1 - \lambda)y, \lambda z + (1 - \lambda)t) \geq T(f(x,z), f(y,t)),$$

• *S-convex on X^2 if*

$$f(\lambda x + (1 - \lambda)y, \lambda z + (1 - \lambda)t) \leq S(f(x,z), f(y,t)),$$

for every $x, y, z, t \in X$ and $\lambda \in [0,1]$ with $\lambda x + (1 - \lambda)y$, $\lambda z + (1 - \lambda)t \in X$.

Some extension of fuzzy relations theory [34] is interval-valued fuzzy relations theory introduced independently by Sambuc in 1975 and Gorzałczany in 1987.

Definition 3.6 ([24]) *An interval-valued fuzzy relation R on X is a mapping*

$$R : X^2 \to L^I.$$

In this presentation we will use following notation for the interval-valued fuzzy relation $f = [f_1, f_2]$, where f_1, f_2 are fuzzy relations.

We can observe the following condition for T-concavity (S-convexity) of interval-valued fuzzy relations (more generally than in [16]).

Lemma 3.7 *Let $T : (L^I)^2 \to L^I$ be a representable triangular norm. The interval-valued fuzzy relation $f = [f_1, f_2] : X^2 \to L^I$ is T-concave (S-convex) if and only if f_1 is T_1-concave and f_2 T_2-concave (S_1, S_2-convex, respectively).*

Then we observe following result.

Proposition 3.8 *Let $X \subseteq R^n$, $\mathcal{A} : (L^I)^2 \to L^I$ be representable aggregation, $f, g : X^2 \to L^I$ be T-concave (S-convex) interval-valued fuzzy relations, where $T(S) : (L^I)^2 \to L^I$ be representable triangular norm (conorm). $\mathcal{A}$ preserves T-concavity (S-convexity) if and only if its representatives A_1, A_2 preserve T_1-concavity, T_2-concavity (S_1-convexity, S_2-convexity), respectively.*

We may observe an interesting connection between T-concavity (S-convexity) and T-transitivity (dually S-transitivity) in following theorems (cf. [22]). Thus we will use following definitions.

Definition 3.9 ([13]) *Let $f, g : L^2 \to L$. For $a, b, c, d \in L$.*
• *f is T-transitive if*

$$T(f(a,b), f(b,c)) \leq f(a,c),$$

• *f is dually S-transitive if*

$$S(f(a,b), f(b,c)) \geq f(a,c).$$

Proposition 3.10 *Let $X \subseteq R^n$, $\mathcal{A} : (L^I)^2 \to L^I$ be a representable aggregation, $f, g : X^2 \to L^I$ be T-concave and T-transitive interval-valued fuzzy relations. If $\mathcal{A}$ preserves T-transitivity, then $\mathcal{A}(f, g)$ is T-concave.*

Proof. Let us assume f, g are T-transitive and $\mathcal{A}$ preserve T-transitivity, i.e. for $z, k, m \in X$
$T(f(z, k), f(k, m)) \leq f(z, m)$,
$T(g(z, k), g(k, m)) \leq g(z, m)$ and

$$T(\mathcal{A}(f(z, k), g(z, k)), \mathcal{A}(f(k, m), g(k, m))) \leq \mathcal{A}(f(z, m), g(z, m)). \tag{10}$$

Especially:
$T(f(z, k), f(k, m)) = f(z, m)$,
$T(g(z, k), g(k, m)) = g(z, m)$, then

$$\mathcal{A}(f(z, m), g(z, m)) = \mathcal{A}(T(f(z, k), f(k, m)), T(g(z, k), g(k, m))). \tag{11}$$

By isotonicity of operation $\mathcal{A}$ we obtain

$$\mathcal{A}(T(f(z, k), f(k, m)), T(g(z, k), g(k, m))) \leq T(\mathcal{A}(f(z, k), g(z, k)), \mathcal{A}(f(k, m), g(k, m))). \tag{12}$$

So by (10)-(12) we have

$$\mathcal{A}(T(f(z, k), f(k, m)), T(g(z, k), g(k, m))) = T(\mathcal{A}(f(z, k), g(z, k)), \mathcal{A}(f(k, m), g(k, m))). \tag{13}$$

Now by T-concavity of f, g and monotonicity of $\mathcal{A}$ and by (13) we obtain

$$\mathcal{A}(f(\lambda z + (1 - \lambda)k, \lambda k + (1 - \lambda)m),$$

$$g(\lambda z + (1 - \lambda)k, \lambda k + (1 - \lambda)m)) \geq$$

$$\mathcal{A}(T(f(z, k), f(k, m)), T(g(z, k), g(k, m))) =$$

$$T(\mathcal{A}(f(z, k), g(z, k)), \mathcal{A}(f(k, m), g(k, m))).$$

So $\mathcal{A}(f, g)$ is T-concave.

For example, representable weight arithmetic mean $\mathcal{A} = [A_1, A_2]$ preserves $\mathcal{T}_L$-transitivity, where $\mathcal{T}_L = [T_L, T_L]$ is t-norm and $T_L(a, b) = max(a + b - 1, 0)$ or representable weight geometric mean $\mathcal{A} = [A_1, A_2]$ preserves $\mathcal{T}_P$-transitivity, where $T_P(a, b) = ab$ and $\mathcal{T}_P = [T_P, T_P]$.
And dually we obtain

Proposition 3.11 *Let $X \subseteq R^n$, $\mathcal{A} : (L^I)^2 \to L^I$ be a representable aggregation, $f, g : X^2 \to L^I$ be S-convex interval-valued fuzzy relations. If $\mathcal{A}$ preserves dually S-transitivity, then $\mathcal{A}(f, g)$ is S-convex.*

By Theorem 2.12 and Lemma 2.3 we can generate family equivalence aggregation functions preserving T-concavity (S-convexity).

4. Application

By imprecise or incomplete information, for example in analysis of social network, we have problem with comparability of interval values. Then we can use one of presented in this paper comparability relation, possible, necessary or classical. Mentioned comparability relations we may use in ranking problem in social networks. Social networks are used to represent the relationships between individuals of a population. Where decision making involves individuals generating problems, providing potential solutions, voting for solutions, and the software aggregating individual votes and ultimately deriving a final decision. Many decision making processes take place in an environment in which the information is not precisely known. As a consequence, experts may feel more comfortable using an interval number rather than an exact crisp numerical value to represent their preference. Therefore, interval-valued fuzzy reciprocal preference relations can be considered an appropriate representation format to capture experts' uncertain preference information. Social Network Analysis (SNA) methodology studies the relationships between social entities like members of a group, corporations or nations and it is a useful methodology to examine structural and location properties such as: centrality, prestige and structural balance. Thus social network analysis can be applied to analysis of the structure and the property of personal relationship. There we have such an important issue as ranking, which as the name suggests deals with ordering the search of the best results, workers, product, etc.. The most basic tool for creating marketing strategies and business plans is SWOT analysis. Its great advantage is versatility and analysis in following aspects: strengths, weaknesses, opportunities and threats ([30], [27] or [26]). We propose to use in these studies presented in this paper comparability relations to interval values representing uncertainty information in our network.

Our results above allow to perform the following applications. Interval-valued fuzzy relations on $X = \{x_1, ..., x_m\}$ (set of users) which represent workers in company are considered. We would like to find ranking of workers by given criterions $K = \{k_1, ..., k_n\}$ and by group of experts $E = \{e_1, ..., e_k\}$. Opinion of our experts we present as interval-valued fuzzy relations (we use interval values because sometimes information are not precisely or incomplete) $R_{k_i}^{e_j}$, where $i = 1, ..., k$ and $j = 1, ..., n$. Because the consistency of a preference relation is characterized by transitivity, then weakly transitive property is considered (transitivity property is more restricted). Relation has weakly transitive property if it fulfills following condition:
$R(i, j) \geq [0.5, 0.5]$, $R(j, k) \geq [0.5, 0.5] \Rightarrow R(i, k) \geq [0.5, 0.5]$. This property can be interpreted as follows: If the alternative x_i is preferred to x_k, and x_k is preferred to x_j, then xi should be preferred to x_j.
The following algorithm gives an alternative (a user) who has the worst/best relationships in the considered group X.

We present algorithm to obtain the final solution from a given set of alternatives. We use here theoretical results presented in the paper.

Algorithm

Inputs: $X = \{x_1, ..., x_m\}$ set of alternatives; $R_{k_i}^{e_j}$ interval-valued fuzzy reciprocal relations with their representation by T-concavity and T-transitive utility functions $f_{k_i}^{e_j} : X^2 \to L^I$ (where $f(i,j) \geq f(i,k)$ means that x_i more prefers x_j then x_k); aggregation function preserves T-transitivity (Proposition 3.10); linear order $\preceq$:

1. Generated by aggregation functions A, B:
$[a,b] \preceq_{A,B} [c,d] \Leftrightarrow A(a,b) < A(c,d)$
or $(A(a,b) = A(c,d)$ and $B(a,b) \leqslant B(c,d))$.
This is an admissible order on L^I (linear order refining $\leqslant_{L^I}$ on L^I, cf. [6])
or

2. To solve a problem of incomparability of the interval-valued fuzzy values we can use the following method (cf. [29]) for ranking of the alternatives Y_i:

$$SK(Y_i) = 0.5(1 + l_{Y_i})d_H(M, Y_i), \qquad (14)$$

where M is ideal positive alternative $[1, 1]$ and $l_{Y_i} = \overline{Y_i} - \underline{Y_i}$. This equation tells us about the "quality" of an alternative Y_i – the lower the value of $SK(Y_i)$, the better the alternative Y_i in the sense of the amount of positive information included, and reliability of information.

In (14) distance between the IVFRs (cf. [28]) is used:

$$d_H(R, S) =$$

$$\frac{1}{2n}\sum_{i,j=1}^{n} |\underline{R}(i,j) - \underline{S}(i,j)| + |\overline{R}(i,j) - \overline{S}(i,j)| + |l_R(i,j) - l_S(i,j)|$$

for $R = [\underline{R}, \overline{R}], S = [\underline{S}, \overline{S}] \in IVFR(X), X = \{x_1, ..., x_n\}$, $n \in \mathbb{N}$.
or

3. $[a,b] \preceq_\pi [c,d] \Leftrightarrow a \leq d$.
In ([12], [11]) we observe that in the structure $(L^I, \preceq_\pi)$, the relation $\preceq_\pi$ is an interval order (complete, Ferrers property).

Output: Solution alternative: $x_{selection}$.

(Step1) Aggregate interval-valued fuzzy relations by given criterion in group of experts:

$$Ag_j(R_{k_i}^{e_j}) = \mathbf{R}_{k_i} \, for \, j = 1, ..., k;$$

(Step2) Aggregate obtained relations with respects to each criterions:

$$Ag_i(\mathbf{R}_{k_i}) = \mathbf{R};$$

(Step3) Verification of the weak transitive **R**.
If weak transitivity holds then we go to (Step5);
If weak transitivity does not holds then we go to (Step4);

(Step4) Create weakly transitive reciprocal interval-valued fuzzy relation by following way:

If $i = k$ then $\mathbf{R}'_{ik} = [0.5, 0.5]$ else
If $i < k$ then

 if $\underline{R}(i,k) \geq 0.5$ or $(\exists_{j \neq k}\underline{R}(i,j) < 0.5$ or $\exists_{j \neq i}\underline{R}(j,k) < 0.5)$

 then $\mathbf{R}'_{ik} = \mathbf{R}_{ik}$

 else $\mathbf{R}'_{ik} = [1 - \underline{R}_{ik}, 1 - \underline{R}_{ik}]$
else $\mathbf{R}'_{ik} = [1 - \overline{R}'_{ki}, 1 - \underline{R}'_{ki}]$;

(Step5) Find:

$$\max_{1 \leq i \leq m} \mathbf{R}(i,j);$$

(Step6) Create ranking of the alternatives:
We find order obtained values by relations 1, 2 and 3. As a consequence we obtain $x_i \geq ... \geq x_k$ for $i, k \in \{1, ..., m\}$.

Analysis of the position in the network - which individuals occupy the best positions in the structure of the network? Who will first get valuable information? Are leaders occupying central positions in the network, or perhaps are located on the outskirts? Social network analysis allows us to answer these questions, providing among other things, reliable basis for an efficient allocation of tasks. In summary, social network analysis provides a new quality in the analysis of phenomena and processes in the organization. Using adequate proposed orders gives another practical interpretation, because 1 based on aggregation on given interval values, 2 based on distance from best value (we can replace the value of [1,1] for other suggested as the best) and 3 gives the broadest concepts of preference, because the smaller the value of intervals requires only a smaller value of the bottom of the first interval of the upper second.

By Lemma 2.3 and Theorem 2.12 we can generate equivalent transitive functions by linear order, so then we may create the same ranking of alternatives.

5. Conclusion

We considered some generalisations of convexity and concavity and the problem of preserving these properties by aggregations. In the future research we may use other generalisations of convexity and concavity. For example, instead of $\lambda x + (1 - \lambda)y$ we can use averaging $A(x, y)$, and we want consider generalisations of convexity and concavity in the Cartesian product of lattices and their aggregations. Moreover we can consider connection of some properties of representable aggregations with some linear order defined in L^I.

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