State-of-the-art in Modeling Nonlinear Dependence Among Many Random Variables with Copulas and Application to Financial Indexes

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Abstract:

In this paper, we focus our attention on multi–dimensional copula models for returns of the indexes of selected prominent international financial markets. Our modeling results, based on elliptic copulas, 7-dimensional hierarchical Archimedean copulas, vine copulas and factor copulas demonstrate a dominant role of the SPX index among the considered major stock indexes (mainly at the first tree of the optimal vine copulas). Some interesting weaker conditional dependencies can be detected at it's highest trees. Interestingly, while global optimal model (for the whole period of 277 months) belong to the Factor FDG copulas class, the optimal local models can be found (with very minor differences in the values of GoF test statistic) in the classes of Factor FDG and hierarchical Archimedean copulas. The dominance of these models is most striking over the interval of the financial market crisis, where the quality of the best Student class model was providing a substantially poorer fit.

Keywords: *dependence, copula, elliptically contoured distribution, vine copula, factor copula, hierarchical Archimedean copula, international financial market indexes*

1. Introduction

In this paper we apply multi–dimensional copula to model dependence among returns of selected prominent indexes of international financial markets. The following indexes were considered (with months' values from the time interval 31.1.1995 – 31.1.2018): SPX (Standard and Poor's Index is designed to measure performance of the broad US economy through the aggregate market value of 500 stocks representing all major industries), DAX (The German Stock Index is a total return index of 30 selected German blue chip stocks traded on the Frankfurt Stock Exchange), UKX (The FTSE 100 Index is a capitalization-weighted index of the 100 most highly capitalized companies traded on the London Stock Exchange), NKY (The Nikkei-225 Stock Average is a price-weighted average of 225 top-rated Japanese companies listed in the First Section of the Tokyo Stock Exchange), HSI (The Hang Seng Index is a free-float capitalization-weighted index of a selection of companies from the Stock Exchange of Hong Kong), LEGATRUU (The Bloomberg Barclays Global Aggregate Bond Index is a flagship measure of global investment grade debt from twenty-four local currency markets), SPGSCITR (The S&P GSCI Total Return Index in USD is widely recognized as the leading measure of general commodity price movements and inflation in the world economy).

The paper is organized as follows. The second section is devoted to a brief overview of the theory of hierarchical Archimedean copulas, vine copulas, factor copulas and methodology of copula fitting to multi–dimensional time series. The third section contains application to real data modeling. Finally we discuss results and conclude.

2. Theory

Copula represents a multivariate distribution that captures the dependence structure between/among random variables leaving alone their marginal distributions. Due to Sklar [25]

$$F(x_1, ..., x_n) = C\left[F_1(x_1), ..., F_n(x_n)\right],$$

where F is joint cumulative distribution function of random vector $(X_1, ..., X_n)$, F_i is marginal cumulative distribution function of X_i, and $C : [0,1]^n \to [0,1]$ is a copula which is a n-increasing function with 1 as neutral element and 0 as annihilator, see e.g. monograph Nelsen (2006) [20]. Besides three fundamental copulas

$$M(x_1, ..., x_n) = \min(x_1, ..., x_n)$$
$$W(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$$
$$\Pi(x_1, ..., x_n) = \prod_{i=1}^{n} x_i$$

which model perfect positive dependence, perfect negative dependence (not applicable for $n > 2$) and independence, respectively, there exist numerous parametric classes, such as Archimedean, Extreme-Value and elliptical copulas. Within the last one there belong such important parametric families as *Gaussian* copulas

$$C_G(x_1, ..., x_n) = \Phi\left[\Phi_1^{-1}(x_1), ..., \Phi_n^{-1}(x_n)\right]$$

and *Student t*-copulas

$$C_t(x_1, ..., x_n) = t\left[t_1^{-1}(x_1), ..., t_n^{-1}(x_n)\right],$$

(where Φ and t are joint distribution functions of multivariate normal and Student *t* distributions, similarly Φ_i^{-1} and t_i^{-1}, $i = 1, ..., n$ are univariate quantile functions related to X_i), able to flexibly describe dependence in multidimensional random vector.

2.1. Hierarchical Archimedean Copulas

Archimedean copulas are easy to handle, however in more than two dimensions their dependence structure is too simplistic. Nevertheless, they are used as building blocks in other, more flexible classes of copulas. One such class, that is suitable for modeling multidimensional stochastic dependence, are hierarchical Archimedean copulas. Let us recall fundamentals. Archimedean copulas are defined (for any dimension n) by formula

$$C(x_1, \dots, x_n) = \phi(\phi^{-1}(x_1) + \ldots + \phi^{-1}(x_n))$$

where the so-called generator $\phi\colon [0,\infty) \searrow [0,1]$ satisfies boundary conditions $\phi(0) = 1$, $\phi(\infty) = 0$ (strict Archimedean copulas) and absolute monotonicity (for further details see [17]). Such a construction is analytically convenient and very flexible in bivariate setting, however it is too restrictive in higher dimensions since the whole dependence structure is rendered by a single univariate function, and – moreover – it is exchangeable.

Hierarchical Archimedean copulas (HAC) overcome this problem by nesting simple Archimedean copulas. Since the general multivariate structure is notationally too complex, we illustrate the principle in four dimensions. For example, fully nested HAC (Fig. 1, left) can be given by

$$\begin{aligned} C_{(s)}(x_1, \dots, x_4) &= C_3\Big(C_2\big(C_1(x_1,x_2), x_3\big), x_4\Big) = \\ &= \phi_3\left(\phi_3^{-1} \circ \phi_2\Big(\phi_2^{-1} \circ \phi_1(\phi_1^{-1}(x_1) + \phi_1^{-1}(x_2)) + \right. \\ &\qquad \left. \phi_2^{-1}(u_3)\Big) + \phi_1^{-1}(u_4)\right), \quad (1) \end{aligned}$$

where C_j, $j = 1, \dots, n-1$ are Archimedean copulas with their corresponding generators ϕ_j and $s = (((1,2),3),4)$ the nesting structure. An example of partially nested Archimedean copula (Fig. 1, right) is given by

$$C_{(s)}(x_1, \dots, x_4) = C_3\Big(C_1(x_1,x_2), C_2(x_3,x_4)\Big), \quad (2)$$

where $s = ((1,2),(3,4))$. Fully and partially nested Archimedean copulas form a class of hierarchical Archimedean copulas which can adopt arbitrarily complex structure s, generally $s = (\dots,(i_a,i_b),i_c,\dots)$, where $i. \in \{1,\dots,n\}$ is reordering of the indices of variables with $a,b,c \in \{1,\dots,n \mid a \neq b \neq c\}$, see, e.g., [9, 12, 22]. This makes it a very flexible yet parsimonious distribution model. The generators within a single HAC can come either from a single generator family or from different families. In the first case there is required complete monotonicity of composition $\phi_i^{-1} \circ \phi_j$, $(i \neq j)$, which imposes some constraints on their parameters, see sufficient conditions given by [16]. For majority of generators HAC requires decreasing parameters from top to bottom in its hierarchy. In the case of different generator families, the condition of complete monotonicity is not always fulfilled. The software implementation in R, the *HAC* package [22] which we use in our study, considers only single-parameter generators from the same family. Then the

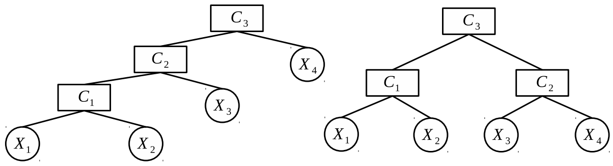

Fig. 1. Fully nested and partially nested Archimedean copulas structure.

whole distribution is specified with at most $n-1$ parameters which can be seen as an alternative to covariance driven models, as remarked in [22], nevertheless, besides the parameters also structure s needs to be estimated. As there are already $n!/k!$ possibilities of combining n variables to fully nested HAC with k-dimensional AC on its lowest level, the greedy approach to structure estimation would be unreasonable even in moderate dimensions, therefore HAC package offers computationally efficient recursive procedure suggested by [21]

2.2. Vine Copulas

Another class that can use bivariate Archimedean copulas as building blocks are Vine copulas. However, they are not restricted to that class and can combine copulas of arbitrary kind via a vine tree structure, which can be estimated (by default following the correlation strength ordering), visualized and interpreted, see [1, 4, 24].

Formally, an n–dimensional regular vine tree structure $\mathcal{S} = \{T_1, \dots, T_n\}$ is a sequence of $n-1$ linked trees with properties (see [3, 4]):

- Tree T_1 is a tree on nodes 1 to n;
- Tree T_j has $n+1-j$ nodes and $n-j$ edges;
- Edges in tree T_j become nodes in tree T_{j+1};
- Two nodes in tree T_{j+1} can be joined by an edge only if the corresponding edges in tree T_j share a node.

In the following, we outline the construction of three-dimensional probability density function f

$$\begin{aligned} f(x_1,x_2,x_3) &= f_1(x_1)\cdot f_{2|1}(x_1,x_2)\cdot f_{3|12}(x_1,x_2,x_3) = \\ &= f_1(x_1)\cdot c_{12}\left[F_1(x_1), F_2(x_2)\right]\cdot f_2(x_2)\cdot \\ &\cdot c_{31|2}\left[F_{x_3|x_2}(x_2,x_3), F_{x_1|x_2}(x_1,x_2)\right]\cdot \\ &\cdot c_{23}\left[F_2(x_2), F_3(x_3)\right]\cdot f_3(x_3) \quad (3) \end{aligned}$$

where f_i is a (marginal) probability density function of $X_i,\ i = 1,2,3$,

$$f_{i|j}(x_i,x_j) = \frac{f(x_i,x_j)}{f_j(x_j)}$$

is conditional density function of X_i given X_j. A copula density c_{ij} couples X_i and X_j while $c_{ij|k}$ couples bivariate conditional distributions of $X_i|X_k$ and $X_j|X_k, i,j,k \in \{1,2,3\}, i \neq j \neq k \neq i$. Finally,

$$F_{x_i|x_j} = \frac{\partial C_{ij}\left[F_i(x_i), F_j(x_j)\right]}{\partial F_j(x_j)}$$

is a conditional cumulative distribution function of X_i given X_j.

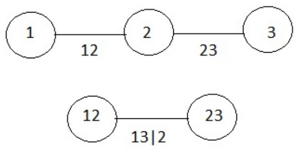

Fig. 2. Vine trees corresponding to (3)

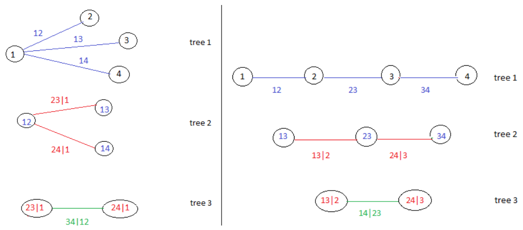

Fig. 3. C-Vine tree (left) and D-Vine tree (right)

The construction (3) represented by two vine trees shown in Fig. 2 is one of the three possible pair–copula decompositions. Since $n = 3$, its vine structure coincides both with

- canonical (C–) vines: each tree has a unique node connected to $n - j$ edges (use only star like tree - useful for ordering by importance); and
- drawable (D–) vines: no node is connected to more than 2 edges (use only path like trees - useful for temporal ordering of variables)

and these differ for $n \geq 4$ as illustrated by Fig. 3, see [5]. However in higher dimensions, C–vines and D–vines are just small subsets of a more general class - regular vines, see [4, 3].

Besides the well-known 2–dimensional product copula and elliptical copulas (Gaussian and Student), as construction blocks of vine copula we utilized also numerous 2–dimensional families of Archimedean and Extreme-value copulas, as well as their rotations, described below in the section Methods.

2.3. Factor Copulas

Yet another subclass within pair-copula construction approach is getting considerable attention: factor copulas. According to [13] factor copula models are conditional independence models where observed variables $(U_1, \ldots, U_n)$ are conditionally independent given one or more latent variables $(V_1, \ldots, V_p)$. These models extend the multivariate Gaussian model with factor correlation structure. They can be also viewed as p-truncated C-vine copulas rooted at the latent variables, one just needs to integrate out latent variables in the joint copula density to get density of observables. The most popular are 1-factor copulas defined as

$$C(u_1, \ldots, u_n) = \int_0^1 \prod_{i=1}^{n} C_{i|V_1}(u_i, v_1) dv_1 \qquad (4)$$

with the density

$$c(u_1, \ldots, u_n) = \int_0^1 \prod_{i=1}^{n} c_{iV_1}(u_i, v_1) dv_1$$

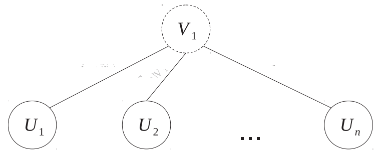

Fig. 4. 1-factor copula graphical model of dependence corresponding to construction (4)

thus the dependence among U_is is induced by the so-called linking copulas C_{iV_1}, $i = 1, \ldots, n$, and there are no constraints among the bivariate copulas. The dependence structure of n-dimensional 1-facor copula is graphically illustrated in Fig. 4.

It is interesting to note, that the Archimedean copulas generated by universal generators (those based on Laplace transform, see e.g. [2] for examples) are special case of 1-factor copulas, with exceptionally simple form. A main advantage of factor copula models comparing to Archimedean and Gaussian copulas is that it allows for asymmetric dependence structure (both reflection asymmetry and non-exchangeability) among observables. Later we will see that they are flexible enough to compete with more complex class of vine copulas while keeping relative parsimony and interpretability. The main drawback nowadays, however, is the lack of software implementation. Commercial programs are rather conservative in bringing new statistical methods and from the open source tools, only in R, the most popular environment for statistical calculations and visualizations [23], we found single package related to factor copula: *FDGcopulas*. In this package a Durante class of bivariate copulas defined by

$$C(u, v) = \min(u, v) f(\max(u, v)),$$

are used as linking copulas, where the generator $f\colon [0, 1] \to [0, 1]$ is differentiable and increasing function such that f (1) = 1 and $t \to f(t)/t$ is decreasing. There may be chosen four different parametric families, such as Cuadras-Augé $f(t) = t^{1-\theta}$, $\theta \in [0, 1]$, Fréchet $f(t) = (1 - \theta)t + \theta$, $\theta \in [0, 1]$, Durante-exponential $f(t) = \exp\left(\frac{t^\theta - 1}{\theta}\right)$, $\theta > 0$, and Durante-sinus $f(t) = \frac{\sin(\theta t}{\sin(\theta)}$ with parameter $\theta \in (0, \pi/2]$, please refer to [15] for finer details. The downside of this class is a singular component present in the model, which is not natural for most economic, hydrologic or other frequently analyzed phenomenons. However, as the authors argue, it is not of that much importance in higher dimensions, where just certain features of distribution is preferred (such as critical levels or tail behavior) instead of its overall shape. On the other hand, the class of 1-factor copulas with Durante generators reduce the computational burden of general 1-factor copulas while giving a good fit to observed data, as we will see in the results.

2.4. Methods

Within the considered classes of 2–dimensional copulas as well as n-dimensional elliptical copulas, the optimal models were selected using the Maximum likelihood estimation (MLE) method. Recall that for given *m* observations $\{X_{j,i}\}_{i=1,\ldots,m}$ of *j*-th random variable $X_j,\ \ j = 1,...,n$, the parameters θ of all copulas under consideration were estimated by maximizing the likelihood function

$$L(\theta) = \sum_{i=1}^{m} \log\left[c_\theta(U_{1,i},\ldots,U_{n,i})\right], \qquad (5)$$

where c_θ denotes density of a parametric copula family C_θ, and

$$U_{j,i} = \frac{1}{m+1}\sum_{k=1}^{m} \mathbf{1}(X_{j,k} \le X_{j,i}), \quad i = 1,...,m,$$

are so-called pseudo-observations. The higher dimensional structures of HAC, vine and factor copulas were estimated as described in [22], [6] and [15], respectively.

Goodness-of-fit was performed by a test proposed by Genest et al. [8] and based on empirical copula process using Cramer-von Misses test statistic

$$S_{CM} = \sum_{i=1}^{m} \left[C_\theta(U_{1,i},\ldots,U_{n,i}) - C_m(U_{1,i},\ldots,U_{n,i})\right]^2 \qquad (6)$$

with empirical copula

$$C_m(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^{m}\prod_{j=1}^{n} \mathbf{1}(X_{j,i} \le x_j)$$

and indicator function $\mathbf{1}(A) = 1$ whenever *A* is true, otherwise $\mathbf{1}(A) = 0$.

All calculations were done in R [23] with the specific help of packages *copula* [10], *HAC* [22], *VineCopula* [19], and *FDGcopulas* [14]. Because their goodness-of-fit methods, including the Cramer-von Misses metric, are not directly comparable and computing the values of vine copula cumulative distribution function involves integration over 7-dimensional space, we rather approximated C_θ from (6) by empirical copula of the random samples generated from the corresponding copulas each counting 100 000 realizations.

Besides the usual parametric families of Archimedean class such as Gumbel, Clayton, Frank, Joe, copulas BB1, BB6, BB7, BB8 and Tawn copulas (see e.g. [11, 18, 20, 26]) in bivariate case, to build vine copulas we used also their rotations C_α by angle α defined

$$C_{90}(x_1,x_2) = x_2 - C(1-x_1,x_2),$$
$$C_{180}(x_1,x_2) = x_1 + x_2 - 1 + C(1-x_1,1-x_2),$$
$$C_{270}(x_1,x_2) = x_1 - C(x_1,1-x_2),$$

that are implemented in the package *VineCopula*.

As a preliminary analysis of dependence between random variables, we employ classical measures of dependence such as Pearson's and Kendall's correlation coefficients, moreover to inspect the conditional (in)dependence (which is further investigated parametrically with vines) the partial correlation matrix comes handy. Given a Pearson's correlation matrix Σ, the partial correlation between variables X_i, X_j conditional on all the other pairs in vector $(X_1,\ldots,X_n)$ can be computed

$$\rho_{ij|-ij} = \frac{-p_{ij}}{\sqrt{p_{ii}p_{jj}}}$$

where p_{ij} $(i,j = 1,\ldots,n)$ are elements of the matrix $P = \Sigma^{-1}$. Recall that partial correlation is a measure of the strength and direction of a linear relationship between two continuous random variables that takes into account (removes) the influence of some other continuous random variables.

Partial correlations are important, e.g., a) when building (Gaussian) graphical models, where insignificant connections are removed to obtain more parsimonious model, as well as b) to better understand the structure of estimated vine copula.

The R source script used for calculations can be obtained from the corresponding author upon request or on his web page.

3. Results

All indexes are computed in terms of returns

$$return_i = \frac{index_i - index_{i-1}}{index_{i-1}}, \quad i = 2,3,...,n.$$

Before further analysis, we filtered all considered time series of returns by ARIMA-GARCH filters ([7]). For all investigated series of returns, the best filters were identified (by the system Mathematica, Version 11) in the class GARCH(1,1).

The obtained residuals have pairwise Kendall correlation coefficients τ in the interval $(-0.08, 0.42)$, maximal value was achieved for the couples SPX–DAX and SPX–UKX, see Fig. 5.

Fig. 6 reveals partial correlations, showing that the relations of (filtered) returns of SPX–UKX, SPX–DAX, SPX–HSI and DAX–UKX attain the largest values, which is in accordance with corresponding strongest dependencies between couples in the first tree of the optimal global vine copula in Table 1.

Subsequently, the residuals were transformed by their respective empirical distribution function into so-called pseudo-observations uniformly distributed over unit interval (see diagonal of Fig. 5). Results served as inputs to calculations of 7–dimensional copula models.

We extended our analysis by examining evolution of the Kendall's correlations. We have chosen frequency of calculations of Kendall's correlation coefficients over the intervals of 72 months overlapping by 36 months with the neighboring intervals, the last time period spans only 60 months. For each of the couples of considered indexes, we calculated a sequence of 7 local Kendall correlation coefficients on individual local time intervals. We can see (in Fig. 7) that just 6 out of all 21 couples of indexes have Kendall correlation coefficients in most of the periods significantly positive.

In the following subsections, first, an overall dependence structure is modeled by means of Elliptical,

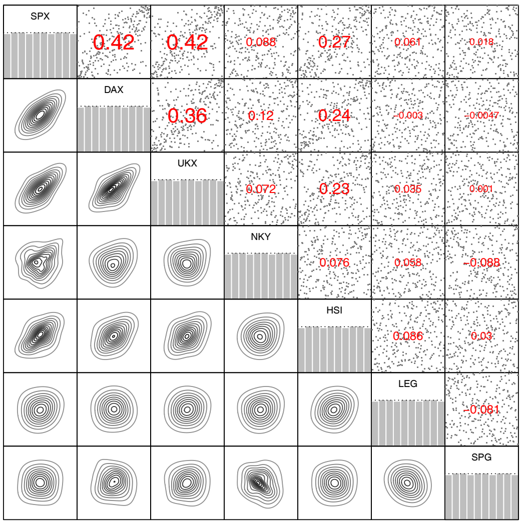

Fig. 5. Pairwise scatter plots with Kendall's tau (upper triangle), bivariate density contour plots with standard normal margins (lower) and marginal density (diagonal) of pseudo-observations.

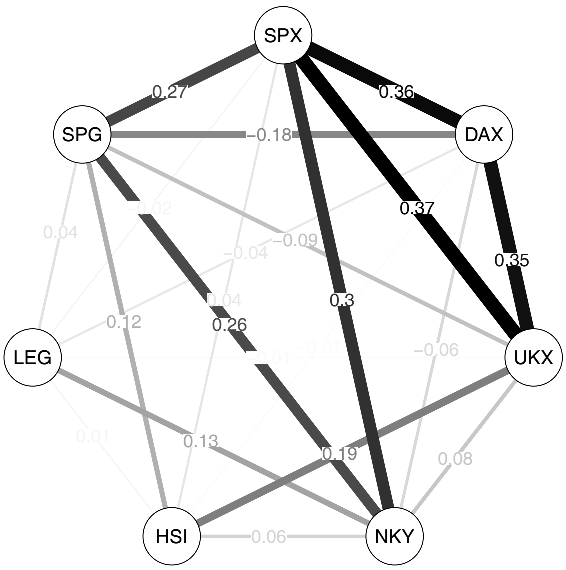

Fig. 6. Partial correlation coefficients for the residuals (conditioned on the remaining elements of the considered group of residuals).

nested Archimedean, vine and 1-factor copulas. Then, with the same classes of models we examine dependence in subsequent periods.

3.1. Global Models of Dependence

The best 7–dimensional vine copula (based on forward selection of trees and AIC criterion for pair-copulas, see [19]) is summarized in Table 1, Fig. 8 and . We observe that at the lowest tree there are modeled stronger links between SPX with the triple UKX, DAX and HSI. It illustrates a very strong international position of the US economy. (It is also interesting to realize that there exist historically strong ties of HSI to the increasingly influential Chinese economy.) At the second tree, we can clearly observe a modest dependence between UKX and DAX, conditioned on SPX. All other elements of the second tree are clearly weaker. Interestingly, at the very last tree, a slight dependence (negative) between LEGATRUU with SPGSCITR, conditioned on all considered stock indexes can be observed.

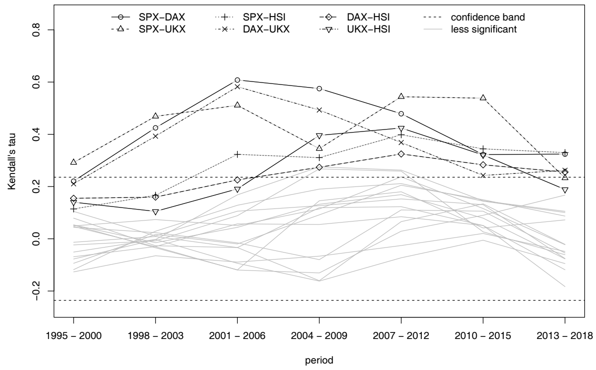

Fig. 7. Evolution of Kendall's τ for all couples of the (filtered) returns

Tab. 1. The summary of the best 7–dimensional vine copula (I1=SPX, I2=DAX, I3=UKX, I4=NKY, I5=HSI, I6=LEGATRUU, I7=SPGSCITR)

tree	edge	family	par_1	par_2	τ
1	I4 - I7	t	-0.13	6.04	-0.08
	I2 - I4	SC	0.32	0.00	0.14
	I1 - I2	t	0.61	2.80	**0.42**
	I1 - I3	t	0.62	2.75	**0.43**
	I5 - I1	t	0.43	3.19	**0.28**
	I6 - I5	G	1.09	0.00	0.08
2	I2 - I7; I4	t	-0.01	5.83	-0.01
	I1 - I4; I2	J	1.11	-	0.06
	I3 - I2; I1	t	0.26	3.87	**0.17**
	I5 - I3; I1	t	0.12	3.76	0.08
	I6 - I1; I5	Tawn	17.67	0.01	0.01
3	I1 - I7; I2 - I4	SJ	1.08	0.00	0.05
	I3 - I4; I1 - I2	t	0.07	7.19	0.05
	I5 - I2; I3 - I1	t	0.09	5.18	0.05
	I6 - I3; I5 - I1	I	-	-	0.00
4	I3 - I7; I1 - I2 - I4	I	-	-	0.00
	I5 - I4; I3 - I1 - I2	Tawn90	-9.33	0.00	0.00
	I6 - I2; I5 - I3 - I1	t	-0.10	6.14	-0.06
5	I5 - I7; I3 - I1 - I2 - I4	I	-	-	0.00
	I6 - I4; I5 - I3 - I1 - I2	G	1.06	0.00	0.06
6	**I6 - I7; I5 - I3 - I1 - I2 - I4**	F	-0.81	0.00	**-0.09**

type: R-vine logLik: 257.52 AIC: -455.03 BIC: -346.42

The best HAC is shown in Fig. 9.

According to the GoF test statistics, see Tab. 2, the best models for the investigated data are in the class of 1–factor copula with Durante generators followed by Student t–copula and HAC with bivariate Gumbel Archimedean copula. For comparison, the distance of product copula to empirical copula is equal to 0.167.

There is no interesting structure to be illustrated about both 1-factor copulas, Fig. 10 shows scatter-plot of 300 simulated observations, one may observe a singular component present in the FDG copula dependence model. Although not clearly visible, bivariate

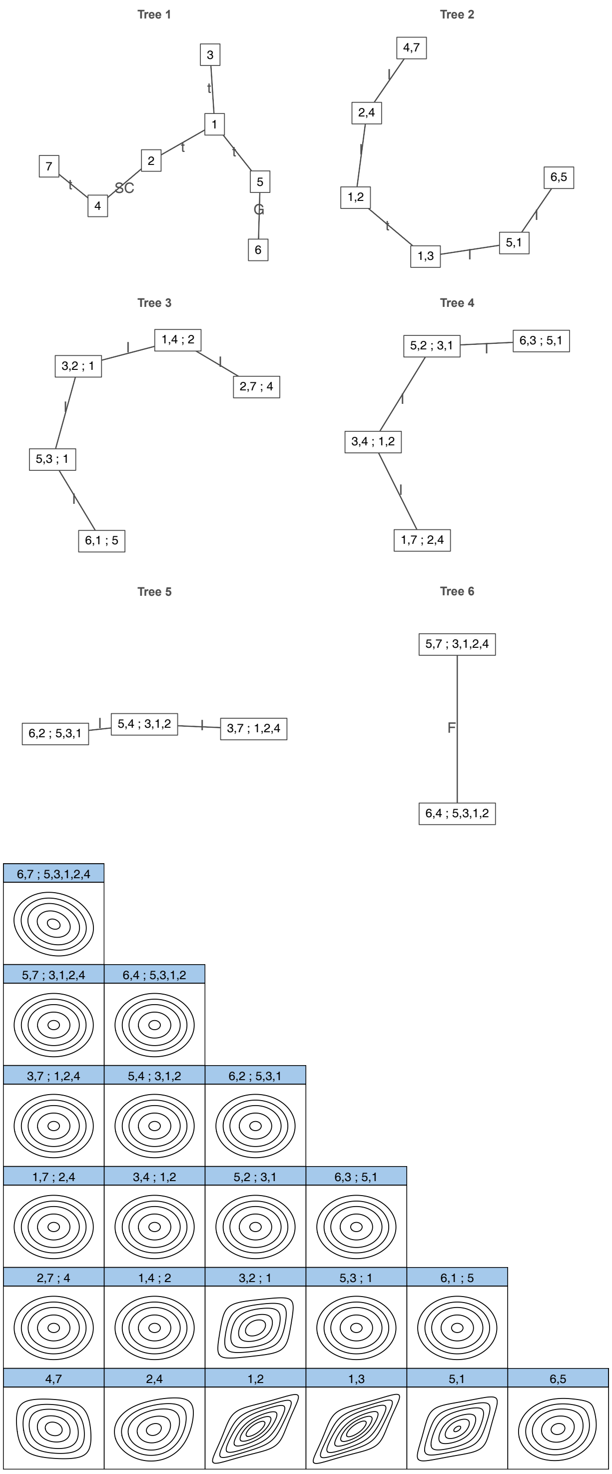

Fig. 8. The best global Vine copula: trees with pair copula family indicated on edges (up) and density contour plots (down), see Tab. 1 for a legend

margins of 1-factor copula with generator of Frechet family are radially symmetric (equal tail dependence), those Cuadras-Auge family generator based have zero lower tail dependence.

3.2. Modeling Evolution of Dependence by Means of Local Models

We continued by searching models for the 7 time intervals described above (for which sequence of Kendall's correlation coefficient was calculated). A best vine copula was identified (tree structure) for each interval but estimated (in the same structure) also for all the other intervals. This way we got the selection of 7 different, locally best fitting vine copula structures and their corresponding sequences of estimated vine copulas. Through almost whole considered time period, the best vine copula was that one from period 2007-2012 (V5). Similarly we estimated a sequence of 7 eliptic, HAC and factor copulas. Among the elliptic co-

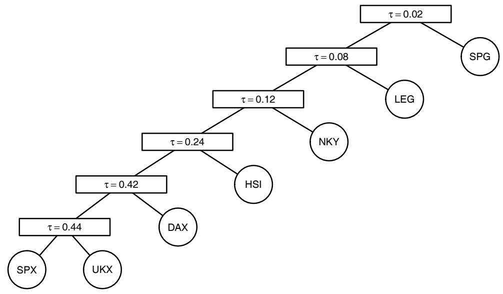

Fig. 9. The best global Gumbel HAC copula

Tab. 2. GoF test statistics for all four global multi–dimensional models

class	family			
elliptical	Gaussian 0.022	Student 0.017		
HAC	Gumbel 0.019	Clayton 0.052	Frank 0.026	Joe 0.023
Vine	(Fig. 8) 0.027			
Factor FDG	Fréchet 0.016	Cuadras-Auge 0.014		

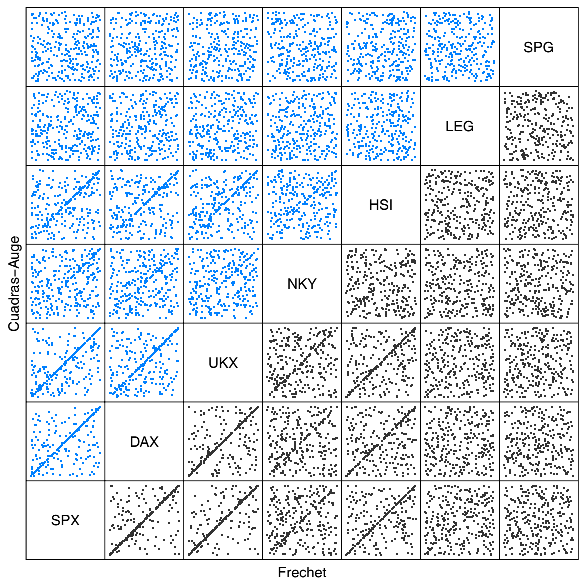

Fig. 10. Simulated observations from bivariate margins of the two best 1-factor copulas with Durante generators

pulas, t-copula was mostly better (except for the last 2 subintervals). We have selected HAC from the classes Gumbel, Clayton, Frank, Joe and Ali–Mikhail–Haq. Throughout all considered time intervals, the best model among them was Gumbel HAC (H3, H4) with the same hierarchical structure as the global HAC but with DAX and SPX swaped. Factor copulas performs similarly except for the first period, when Cuadras-Auge family fitted better. The corresponding GoF test statistic (for the best copulas in each class) is displayed in Fig. 11 and it shows slightly superior performance of hierarchical Archimedean copula over elliptical, factor and vine copulas throughout the whole analyzed period (except for the first 3 subintervals).

Here come two interesting observations. First, the breath-taking performance of HAC considering its parsimony: for illustration take now only bivariate copulas used in vines, factor and HAC copulas, then number of parameters needed for construction of n-dimensional normal copula are $n(n-1)/2$, vine copulas $n + (n-1) + \ldots + 1$, 1-factor copulas n and HAC copulas only $n-1$. It is true that when (conditional) independence takes place in the random vector, vine copula gets significantly reduced, however in our particular case as for global copula 20 parameters are involved comparing to 6 of HAC, and as for the optimal evolving copula the vine structure contains 8 parameters.

Second, unlike elliptical copulas, the best HAC, vine and factor copulas reveal some asymmetry with respect to tail behavior, and while vines are better for directly displaying conditional relationships, hierarchical Archimedean copulas shows clusters of random variables in somewhat clearer way.

4. Conclusion and Future Work

Modeling dependencies between international financial market indexes is interesting and important for investors, risk managers and policy makers. Application of more dimensional copulas is bringing a new insight and experience for modeling activities.

Interestingly, while global optimal model (for the whole period of 277 months) belong to the Factor FDG copulas class, the optimal local models can be found (with very minor differences in the values of GoF test statistic) in the class of HAC. The dominance of this model is most striking over the interval of the financial market crisis, where the quality of the best Student class model was providing a substantially poorer fit.

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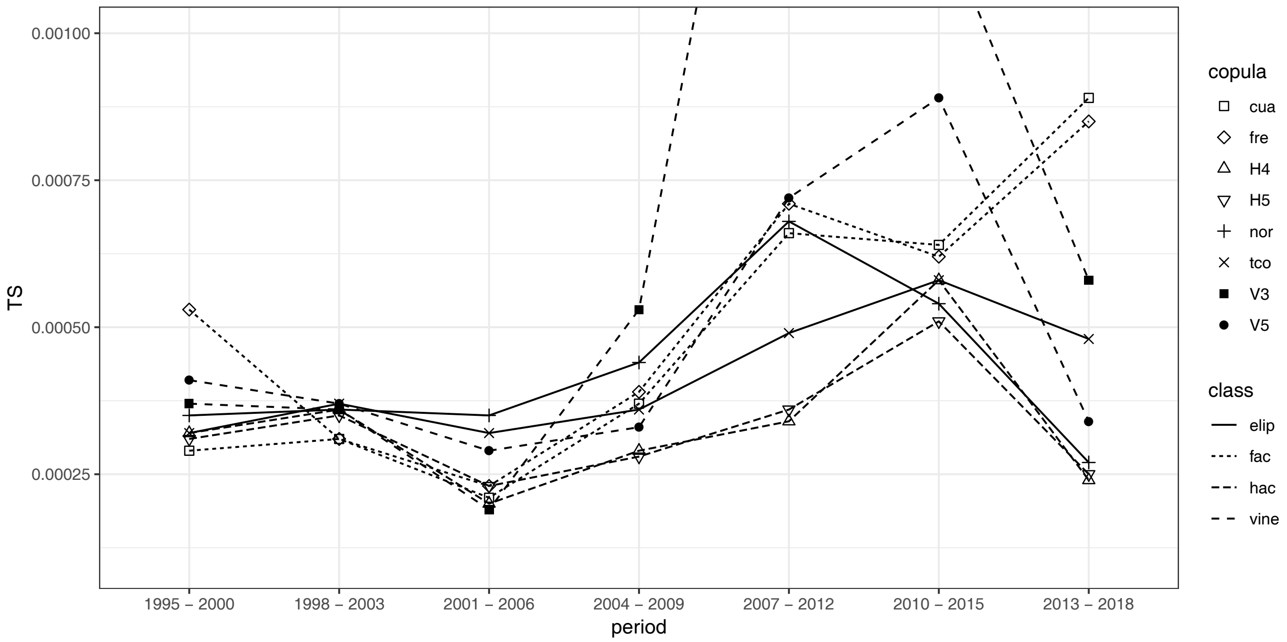

Fig. 11. Evolution of GoF test statistic for the best copulas in each class

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