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THE CROSSING NUMBERS OF JOIN PRODUCTS OF PATHS WITH THREE GRAPHS OF ORDER FIVE

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Abstract. The main aim of this paper is to give the crossing number of the join product $G^* + P_n$ for the disconnected graph G^* of order five consisting of the complete graph K_4 and one isolated vertex, where P_n is the path on n vertices. The proofs are done with the help of a lot of well-known exact values for the crossing numbers of the join products of subgraphs of the graph G^* with the paths. Finally, by adding new edges to the graph G^*, we are able to obtain the crossing numbers of the join products of two other graphs with the path P_n.

Keywords: graph, crossing number, join product, cyclic permutation, path.

Mathematics Subject Classification: 05C10, 05C38.

1. INTRODUCTION

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane (for the definition of a *drawing* see Klešč [10]). It is easy to see that a drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G ($D(G)$). We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold [10]:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

The investigation on the crossing number of graphs is a classical and very difficult problem. Garey and Johnson [7] proved that this problem is NP-complete. Note that the exact values of the crossing numbers are known for only a few families of graphs, see Clancy *et al.* [4]. The purpose of this article is to extend the known results concerning this topic. Some parts of proofs will be based on Kleitman's result [9] on the crossing numbers for some complete bipartite graphs. He showed that

$$\mathrm{cr}(K_{m,n}) = \Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor \Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor, \quad \text{for} \quad m \leq 6.$$

The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let D_n and P_n be the *discrete graph* and the *path* on n vertices, respectively. The crossings numbers of the join products of the paths with all graphs of order at most four have been well-known for a long time by Klešč [11], and Klešč and Schrötter [18], and therefore it is understandable that our immediate goal is to establish the exact values for the crossing numbers of $G + P_n$ also for all graphs G of order five. The crossing numbers of $G + P_n$ are already known for a lot of graphs G of order five and six [3, 5, 6, 10, 14, 17, 22, 24, 27, 28]. In all these cases, the graph G is connected and contains usually at least one cycle. Note that the crossing numbers of the join product $G + P_n$ are known only for some disconnected graphs G on five or six vertices [2, 19, 23], and so the purpose of this article is to extend the known results concerning this topic to new disconnected graphs. The minimal number of crossings in the Cartesian product and in the strong product of paths have been studied by Klešč *et al.* in [15] and [16].

In this paper, we will use definitions and notation of the crossing numbers of graphs presented by Klešč [11]. We will also use special designation of some graphs that are represented by lower indexes in the order originally designated by [11, 13] (except in the case of the disconnected graph G^*), and in which an upper index represents the number of vertices of the examined graph. Let G^* be the disconnected graph of order five consisting of one isolated vertex and the complete graph K_4. The crossing number of $G^* + D_n$ was determined for any $n \geq 1$ by Staš [26] using the properties of cyclic permutations. The main aim of the paper is to establish the crossing numbers of the join products of G^* with paths P_n. Due to the special drawing of $G^* + P_2$ in Figure 2 with only 3 crossings, the result of the main Theorem 2.4 can be estimated for the paths P_n on at least 3 vertices. The paper concludes by giving the crossing numbers of $G^5_{16} + P_n$ and $G^5_{18} + P_n$ in Corollaries 3.2 and 3.3, and Theorem 4.3, respectively. Note that the result in Corollary 3.3 has already been claimed by Li [20]. Since this paper does not seem to be available in English, we have not been able to verify this result but we can certainly say that the author's result is incorrect for $G^5_{16} + P_2$ according to Corollary 3.2. The result in Theorem 4.3 has also been claimed by Li [21], but again not in English. Clancy *et al.* [4] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals do not have a sufficiently rigorous peer-review process.

In our paper, certain parts of proofs can be also simplified with the help of software COGA generating all cyclic permutations of five elements. Its description can be found in Berežný and Buša [1]. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

2. CYCLIC PERMUTATIONS AND POSSIBLE DRAWINGS OF G^*

In the rest of the paper, let $V(G^*) = \{v_1, v_2, \dots, v_5\}$, and let v_5 be the vertex notation of the isolated vertex of G^* in all considered good subdrawings of the graph G^*. We consider the join product of G^* with the discrete graph D_n. It is not difficult to see that the graph $G^* + D_n$ consists of just one copy of the graph G^* and n vertices $t_1, t_2, \dots, t_n$, where each vertex $t_i,\ i = 1, 2, \dots, n$, is adjacent to every vertex of G^*. Let T^i, $1 \leq i \leq n$, denote the subgraph which is uniquely induced by the five edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{5,n}$ and

$$G^* + D_n = G^* \cup K_{5,n} = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

The graph $G^* + P_n$ contains $G^* + D_n$ as a subgraph. For the subgraphs of the graph $G^* + P_n$ which are also subgraphs of the graph $G^* + nK_1$ we use the same notation as above. Let P_n^* denote the path induced on n vertices of $G^* + P_n$ not belonging to the subgraph G^*. Hence, P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and of the edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \dots, n-1$. One can easily see that

$$G^* + P_n = G^* \cup K_{5,n} \cup P_n^* = G^* \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2.2}$$

Let D be a good drawing of the graph $G^* + D_n$. The *rotation* of a vertex t_i in the drawing D ($\mathrm{rot}_D(t_i)$) is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see Hernández-Vélez *et al.* [8] or Woodall [29]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We emphasize that a rotation is a cyclic permutation, that is, (12345), (23451), (34512), (45123), and (51234) denote the same rotation. Thus, $5!/5 = 24$ different $\mathrm{rot}_D(t_i)$ can appear in a drawing of the graph $G^* + D_n$. In the given drawing D, we separate all subgraphs T^i of the graph $G^* + D_n$ into three mutually disjoint subsets depending on how many times the considered T^i crosses the edges of G^* in D. For $i = 1, \dots, n$, $T^i \in R_D$ if $\mathrm{cr}_D(G^*, T^i) = 0$ and $T^i \in S_D$ if $\mathrm{cr}_D(G^*, T^i) = 1$. Every other subgraph T^i crosses the edges of G^* at least twice in D. Clearly, the idea of dividing the subgraphs T^i into three mentioned subsets is also retained in all drawings of the graphs $G^* + P_n$. Due to arguments in the proof of Theorem 2.4, if we would like to obtain an optimal drawing D of $G^* + P_n$, at least one of the sets R_D and S_D must be nonempty.

For $T^i \in R_D \cup S_D$, let F^i denote the subgraph $G^* \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $G^* + D_n$ and let $D(F^i)$ be its subdrawing induced by D. In [26], three possible non isomorphic drawings of G^* were described. They are presented in Figure 1 with the corresponding vertex notation.

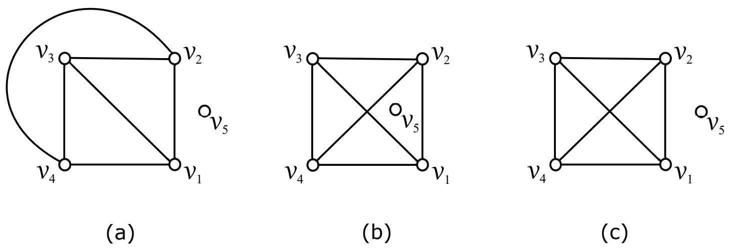

Fig. 1. Three possible non isomorphic drawings of the graph G^*: (a) the planar drawing of G^*; (b) the drawing of G^* with $\mathrm{cr}_D(G^*) = 1$ and the isolated vertex v_5 located in the triangular region of subdrawing $G^* \setminus v_5$; (c) the drawing of G^* with $\mathrm{cr}_D(G^*) = 1$ and the isolated vertex v_5 located in the quadrangular region of subdrawing $G^* \setminus v_5$

In the proof of Theorem 2.4, several parts will be based on the following theorem presented in [26].

Theorem 2.1 ([26, Theorem 3.1]). $\mathrm{cr}(G^* + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Lemma 2.2. $\mathrm{cr}(G^* + D_2) = \mathrm{cr}(G^* + P_2) = 3$.

Proof. Figure 2 shows the drawing of $G^* + P_2$ with three crossings, that is, $\mathrm{cr}(G^* + P_2) \leq 3$. The graph $G^* + D_2$ is a subgraph of $G^* + P_2$, and therefore, $\mathrm{cr}(G^* + P_2) \geq \mathrm{cr}(G^* + D_2) = 3$ by Theorem 2.1. This completes the proof of Lemma 2.2. □

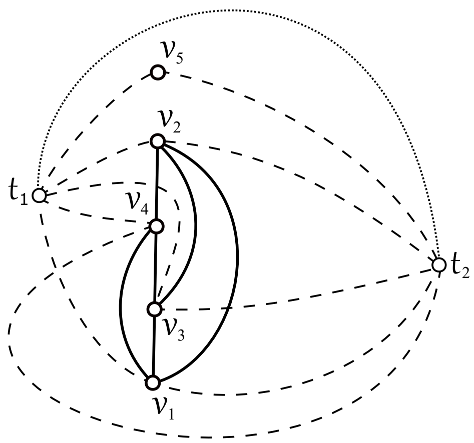

Fig. 2. The drawing of $G^* + P_2$ with 3 crossings

Lemma 2.3. $\mathrm{cr}(G^* + P_3) = 9$.

Proof. Figure 3 offers the subdrawing of G^*+P_3 with 9 crossings, and so $\mathrm{cr}(G^*+P_3) \leq 9$. The graph G^*+P_3 contains a subgraph that is a subdivision of the graph K_4+C_3, and it was proved by Klešč [12] that $\mathrm{cr}(K_4+C_3) = 9$. As $\mathrm{cr}(G^* + P_3) \geq \mathrm{cr}(K_4 + C_3) = 9$, the proof of Lemma 2.3 is done. □

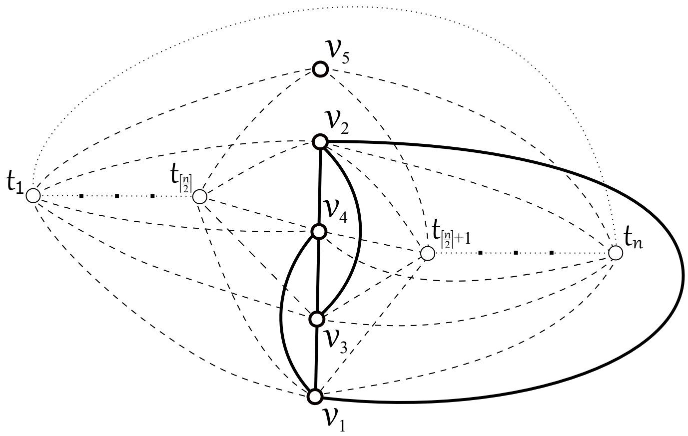

Fig. 3. The good drawing of $G^* + P_n$, $n \geq 3$, with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings

Two vertices t_i and t_j of the graph G^*+D_n are *antipodal* in a drawing of G^*+D_n if the subgraphs T^i and T^j do not cross. A drawing is *antipode-free* if it has no antipodal vertices. The same idea of two noncrossing subgraphs is also retained in all drawings of the graphs $G^* + P_n$. Now we are able to prove the main result of this paper.

Theorem 2.4. $\mathrm{cr}(G^* + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 3$.

Proof. By Lemma 2.3, the result is true for $n = 3$. In Figure 3, the edges of $K_{5,n}$ cross each other

$$4\binom{\lceil \frac{n}{2} \rceil}{2} + 4\binom{\lfloor \frac{n}{2} \rfloor}{2} = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

times, each subgraph T^i, $i = 1, \ldots, \lceil \frac{n}{2} \rceil$ on the left side crosses the edges of G^* exactly once and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n$ on the right side crosses the edges of G^* exactly twice. The path P_n^* crosses G^* once, and so $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings appear among the edges of the graph $G^* + P_n$ in this drawing. Thus, $\mathrm{cr}(G^* + P_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 4$, there is a drawing D with

$$\mathrm{cr}_D(G^* + P_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{2.3}$$

and that

$$\mathrm{cr}(G^* + P_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ for any integer } 3 \leq m < n. \tag{2.4}$$

As the graph $G^* + D_n$ is a subgraph of the graph $G^* + P_n$, by Theorem 2.1, the edges of $G^* + P_n$ are crossed exactly $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ times, and therefore, no edge of the path P_n^* is crossed in D. This also enforces that all vertices t_i of the path P_n^* must be placed in the same region of the considered good subdrawing of G^*. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (2.3) together with the well-known fact $\mathrm{cr}(K_{5,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ imply that in D:

$$\mathrm{cr}_D(G^*) + \sum_{T^i \in R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) + \sum_{T^i \notin R_D \cup S_D} \mathrm{cr}_D(G^*, T^i) \leq n + \left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$\mathrm{cr}_D(G^*) + 0r + 1s + 2(n - r - s) \leq n + \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.5}$$

This readily forces that $2r + s \geq \lceil \frac{n}{2} \rceil + \mathrm{cr}_D(G^*)$, that is, $r + s \geq 1$, and so there is at least one subgraph T^i whose edges cross the edges of G^* at most once. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the considered drawing D and we will show that in all cases a contradiction with the assumption (2.3) is obtained.

Case 1. We suppose the drawing with the vertex notation of G^* in such a way as shown in Figure 1(a). The graph G^* contains the cycle $v_1v_2v_4v_1$ as a subgraph by which the vertices v_3 and v_5 are separated in $D(G^*)$, that is, each T^i crosses the edges of $v_1v_2v_4v_1$ at least once. Because no region is incident to all vertices in $D(G^*)$, there is no possibility to obtain a subdrawing of $G^* \cup T^i$ for a $T^i \in R_D$. As $r = 0$, there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i by which the edges of G^* are crossed just once. Since all vertices t_i of the path P_n^* are placed in the same region of the considered good subdrawing of G^*, all such vertices t_i of P_n^* must be placed in the outer region of subdrawing G^* with the vertices v_1, v_2, v_4, and v_5 on its boundary.

Let us denote by H the subgraph of G^* with the vertex set $V(G^*)$, and the edge set $E(G^*) \setminus \{v_1v_2, v_2v_4, v_4v_1\}$. Since the exact value for the crossing number of the graph $H + P_n$ is given by Klešč and Staš [19], i.e., $\mathrm{cr}(H + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, the edges of $H + P_n$ are crossed at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ times in D. The graph G^* contains the cycle $v_1v_2v_4v_1$ as a subgraph by which the vertices v_3 and v_5 are separated in $D(G^*)$, that is, each subgraph T^i crosses the edges of this cycle at least once. However, by the assumption (2.3), any such T^i must cross the edges of the cycle $v_1v_2v_4v_1$ exactly once. As $s = n$, if we apply the similar discussion of a fixing as in Case 1 in [26], we obtain at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings for all subcases in D. Let $\mathcal{M}_D$ be the set of all configurations for the drawing D belonging to $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6\}$, where a subdrawing of any subgraph $G^* \cup T^i$ has the configuration either $\mathcal{A}_p$ or $\mathcal{B}_p$ represented by some cyclic permutation with either $\mathrm{rot}_D(t_i) = \mathcal{A}_p$ or $\mathrm{rot}_D(t_i) = \mathcal{B}_p$ for some $p \in \{1, \ldots, 6\}$, respectively. The lower bounds for the number of crossings of two configurations $\mathrm{cr}(\mathcal{X}_p, \mathcal{Y}_q)$ are presented in Table 1. (They were also established in Table 1 of [26], where $\mathcal{X}, \mathcal{Y} \in \{\mathcal{A}, \mathcal{B}\}$ and $p, q \in \{1, \ldots, 6\}$.)

Table 1
The necessary number of crossings between T^i and T^j for the configurations $\mathcal{X}_p$, $\mathcal{Y}_q$

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$
$\mathcal{A}_1$	4	1	2	3	2	3	4	3	2	3	2	3
$\mathcal{A}_2$	1	4	3	2	3	2	3	4	3	2	3	2
$\mathcal{A}_3$	2	3	4	1	2	3	2	3	4	3	2	3
$\mathcal{A}_4$	3	2	1	4	3	2	3	2	3	4	3	2
$\mathcal{A}_5$	2	3	2	3	4	1	2	3	2	3	4	3
$\mathcal{A}_6$	3	2	3	2	1	4	3	2	3	2	3	4
$\mathcal{B}_1$	4	3	2	3	2	3	4	3	4	3	4	3
$\mathcal{B}_2$	3	4	3	2	3	2	3	4	3	4	3	4
$\mathcal{B}_3$	2	3	4	3	2	3	4	3	4	3	4	3
$\mathcal{B}_4$	3	2	3	4	3	2	3	4	3	4	3	4
$\mathcal{B}_5$	2	3	2	3	4	3	4	3	4	3	4	3
$\mathcal{B}_6$	3	2	3	2	3	4	3	4	3	4	3	4

We discuss over all possible subsets of $\mathcal{M}_D$ in the following subcases:

a) $\{\mathcal{A}_o, \mathcal{A}_{o+1}\} \subseteq \mathcal{M}_D$ for some $o \in \{1,3,5\}$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in S_D$ such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^k) \geq 5$ is fulfilling for any $T^k \in S_D$ with $k \neq n-1, n$ by summing the values in the first two rows for each column of Table 1. As $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \geq 3$, by fixing the graph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^{n-1} \cup T^n) \\
&\quad + \mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \\
&\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(n-2) + 3 \\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1.
\end{aligned}$$

This contradicts the assumption of D. Due to the symmetry, the same arguments are applied for the cases $\{\mathcal{A}_3, \mathcal{A}_4\}$ and $\{\mathcal{A}_5, \mathcal{A}_6\}$.

b) $\{\mathcal{A}_o, \mathcal{A}_{o+1}\} \not\subseteq \mathcal{M}_D$ for $o = 1,3,5$. First, suppose $\{\mathcal{A}_p, \mathcal{A}_{p+2}, \mathcal{A}_{p+4}\} \subseteq \mathcal{M}_D$ for some $p \in \{1,2\}$ or there are three mutually different $o, p, q \in \{1,\dots,6\}$ with $o \equiv p \equiv q \pmod 2$ such that $\{\mathcal{A}_o, \mathcal{A}_p, \mathcal{B}_q\} \subseteq \mathcal{M}_D$. In the rest of the paper, let us assume three different subgraphs T^{n-2}, T^{n-1}, $T^n \in S_D$ such that F^{n-2}, F^{n-1} and F^n have configurations $\mathcal{A}_1$, $\mathcal{A}_3$ and $\mathcal{B}_5$, respectively. Then, $\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n, T^k) \geq 8$ holds for any $T^k \in S_D$ with $k \neq n-2, n-1, n$ by summing of three corresponding values of Table 1. As $\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n) \geq 6$, by fixing the graph $G^* \cup T^{n-2} \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}\mathrm{cr}_D(G^*+P_n) &\geq \mathrm{cr}_D(K_{5,n-3})+\mathrm{cr}_D(K_{5,n-3},G^*\cup T^{n-2}\cup T^{n-1}\cup T^n)\\ &\quad+\mathrm{cr}_D(G^*\cup T^{n-2}\cup T^{n-1}\cup T^n)\\ &\geq 4\Big\lfloor\frac{n-3}{2}\Big\rfloor\Big\lfloor\frac{n-4}{2}\Big\rfloor+9(n-3)+6+3\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor+n+\Big\lfloor\frac{n}{2}\Big\rfloor+1.\end{aligned}$$

This also contradicts the assumption of D, and therefore, suppose that $\{\mathcal{A}_p,\mathcal{A}_{p+2},\mathcal{A}_{p+4}\}\not\subseteq\mathcal{M}_D$ for any $p=1,2$, and also $\{\mathcal{A}_o,\mathcal{A}_p,\mathcal{B}_q\}\not\subseteq\mathcal{M}_D$ with $o\equiv p\equiv q\,(\mathrm{mod}\,2)$ for any three mutually different $o,p,q=1,\dots,6$. Now, for $T^i\in S_D$, we will discuss the possibility of obtaining a subdrawing of $G^*\cup T^i\cup T^j$ in D with $\mathrm{cr}_D(T^i,T^j)=2$ for some $T^j\in S_D$.

Let us consider that there are two different subgraphs T^i, $T^j\in S_D$ with $\mathrm{cr}_D(T^i,T^j)=2$ such that F^i and F^j have configurations $\mathcal{X}_p$ and $\mathcal{Y}_q$, respectively, where $\mathcal{X},\mathcal{Y}\in\{\mathcal{A},\mathcal{B}\}$ and $p,q\in\{1,\dots,6\}$. Then, $\mathrm{cr}_D(T^i\cup T^j,T^k)\geq 6$ holds for any $T^k\in S_D$, $k\neq i,j$ by summing of two corresponding values of Table 1. Thus, by fixing the graph $G^*\cup T^{n-1}\cup T^n$, we have

$$\begin{aligned}\mathrm{cr}_D(G^*+P_n) &\geq \mathrm{cr}_D(K_{5,n-2})+\mathrm{cr}_D(K_{5,n-2},G^*\cup T^{n-1}\cup T^n)\\ &\quad+\mathrm{cr}_D(G^*\cup T^{n-1}\cup T^n)\\ &\geq 4\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor+7(n-2)+2+2\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor+n+\Big\lfloor\frac{n}{2}\Big\rfloor+1.\end{aligned}$$

Finally, assume that there are no two different subgraphs $T^i,T^j\in S_D$ with $\mathrm{cr}_D(T^i,T^j)\leq 2$. Hence, for each $T^i\in S_D$, $\mathrm{cr}_D(G^*\cup T^i,T^j)\geq 1+3=4$ is fulfilling for any $T^j\in S_D$ with $j\neq i$. Consequently, by fixing the graph $G^*\cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(G^*+P_n) &\geq \mathrm{cr}_D(K_{5,n-1})+\mathrm{cr}_D(K_{5,n-1},G^*\cup T^i)+\mathrm{cr}_D(G^*\cup T^i)\\ &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor+4(n-1)+1\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor+n+\Big\lfloor\frac{n}{2}\Big\rfloor+1.\end{aligned}$$

Case 2. We consider the drawing of G^* given in Figure 1(b). In this case, the same idea of the separating cycle $v_1v_2v_4v_1$ can be also used. Again, let H the subgraph of G^* defined by the same form as in Case 1. By [19], there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$ crossings on the edges of $H+P_n$, and the edges of the cycle $v_1v_2v_4v_1$ are crossed at least n times by the edges of subgraphs T^i and once by the edges of the subgraph H.

Case 3. We consider the drawing of G^* given in Figure 1(c). Since the set S_D is empty, there are at least $\lceil\frac{\lceil\frac{n}{2}\rceil+1}{2}\rceil$ subgraphs T^i whose edges do not cross the edges of G^* provided by $2r \geq \lceil \frac{n}{2} \rceil + 1$. So, there are 4 different possible rotations systems with no crossing depending on in which region of $D(F^i \setminus v_5)$ the edge $t_i v_5$ is placed, see also [26]. These four possibilities under our consideration are denoted by $\mathcal{E}_p$, for $p = 1, 2, 3, 4$, and they are represented by the cyclic permutations (14325), (14532), (14352), and (15432), respectively. They have been already introduced in [26]. Let $\mathcal{N}_D$ be the set of all configurations for the drawing D belonging to $\mathcal{N} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$. The lower bounds for the number of crossings of two configurations $\mathrm{cr}(\mathcal{E}_p, \mathcal{E}_q)$ are presented in Table 2 (they were also established in Table 2 of [26]).

Table 2

The necessary number of crossings between T^i and T^j for the configurations $\mathcal{E}_p$, $\mathcal{E}_q$

$-$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_1$	4	2	3	3
$\mathcal{E}_2$	2	4	3	3
$\mathcal{E}_3$	3	3	4	2
$\mathcal{E}_4$	3	3	2	4

Let us show that the considered drawing D must be antipode-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^k, T^l) = 0$. If at least one of T^k and T^l, say T^k, does not cross G^*, it is not difficult to verify that $\mathrm{cr}_D(G^*, T^k \cup T^l) \geq 4$ holds by four possible subdrawings of F^k with the configuration $\mathcal{E}_p$ for some $p \in \{1, \dots, 4\}$, for more also see [26]. By [9], we already know that $\mathrm{cr}(K_{5,3}) = 4$, which yields that any T^m, $m \neq k, l$, crosses the edges of the subgraph $T^k \cup T^l$ at least four times. So, the number of crossings of $G^* + P_n$ in D is given by

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &= \mathrm{cr}_D\left(G^* + P_{n-2}\right) + \mathrm{cr}_D(T^k \cup T^l) + \mathrm{cr}_D(K_{5,n-2}, T^k \cup T^l) \\
&\quad + \mathrm{cr}_D(G^*, T^k \cup T^l) \\
&\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + n - 2 + \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 4(n-2) + 4 \\
&= 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1.
\end{aligned}$$

This contradiction with the assumption (2.3) confirms that D is antipode-free. Now, we consider the following subcases:

a) $\{\mathcal{E}_p, \mathcal{E}_{p+1}\} \subseteq \mathcal{N}_D$ for some $p \in \{1, 3\}$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Also, by summing the values in the first two rows for each column of Table 2, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^k) \geq 6$ holds for any $T^k \in R_D$ with $k \neq n-1, n$.

Thus, by fixing the graph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^{n-1} \cup T^n) \\
&\quad + \mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \\
&\geq 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 6(r-2) + 6(n-r) + 2 + 1 \\
&= 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 6n - 9 \\
&\geq 4\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + n + \Big\lfloor \frac{n}{2} \Big\rfloor + 1.
\end{aligned}$$

This subcase confirms a contradiction with the assumption in D, and therefore, suppose that $\{\mathcal{E}_1, \mathcal{E}_2\} \not\subseteq \mathcal{N}_D$ and $\{\mathcal{E}_3, \mathcal{E}_4\} \not\subseteq \mathcal{N}_D$ in all following cases. Moreover, for each $T^i \in R_D$, let us denote $L_D(T^i) = \{T^k \notin R_D : \mathrm{cr}_D(G^* \cup T^i, T^k) = 3\}$, and $l_i = |L_D(T^i)|$. If there is a subgraph $T^i \in R_D$ such that $2l_i \geq \lfloor \frac{n+2}{2} \rfloor$, then by fixing the subgraph $G^* \cup T^i \cup T^k$ with some $T^k \in L_D(T^i)$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^i \cup T^k) + \mathrm{cr}_D(G^* \cup T^i \cup T^k) \\
&\geq 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 5(r-1) + 7(l_i - 1) + 5(n - r - l_i) + 3 + 1 \\
&= 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 5n + 2l_i - 8 \\
&\geq 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 5n + \Big\lfloor \frac{n+2}{2} \Big\rfloor - 8 \\
&\geq 4\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + n + \Big\lfloor \frac{n}{2} \Big\rfloor + 1.
\end{aligned}$$

This contradicts the assumption (2.3) in D, and therefore, suppose that $2l_i < \lfloor \frac{n+2}{2} \rfloor$ for each $T^i \in R_D$, which yields that $l_i \leq \Big\lfloor \frac{\lfloor \frac{n+2}{2} \rfloor - 1}{2} \Big\rfloor$.

b) $\mathcal{N}_D = \{\mathcal{E}_p, \mathcal{E}_q\}$ for two different $p, q = 1, 2, 3, 4$ with respect to the restriction $3 < p + q < 7$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{E}_1$ and $\mathcal{E}_3$, respectively. Thus, by fixing the graph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G^* \cup T^{n-1} \cup T^n) \\
&\quad + \mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \\
&\geq 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 7(r-2) + 5(l_i + l_j) + 6(n - r - l_i - l_j) + 4 \\
&= 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 6n + r - l_i - l_j - 10 \\
&\geq 4\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 6n + \Big\lceil \frac{\lceil \frac{n}{2} \rceil + 1}{2} \Big\rceil - 2\Big\lfloor \frac{\lfloor \frac{n+2}{2} \rfloor - 1}{2} \Big\rfloor - 10 \\
&\geq 4\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + n + \Big\lfloor \frac{n}{2} \Big\rfloor + 1.
\end{aligned}$$

Due to the symmetry, the proof proceeds in the similar way also for the remaining cases of two different configurations $\{\mathcal{E}_1, \mathcal{E}_4\}$, $\{\mathcal{E}_2, \mathcal{E}_3\}$, and $\{\mathcal{E}_2, \mathcal{E}_4\}$.

c) $\mathcal{N}_D = \{\mathcal{E}_p\}$ for only one $p \in \{1, 2, 3, 4\}$. Without lost of generality, we can assume that $T^n \in R_D$ with the configuration $\mathcal{E}_1$ of the subgraph F^n. By fixing the graph $G^* \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G^* \cup T^n) + \mathrm{cr}_D(G^* \cup T^n)\\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(r-1) + 3l_i + 4(n-r-l_i) + 1\\
&= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n - l_i - 3\\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n - \left\lfloor \frac{\lfloor \frac{n+2}{2} \rfloor - 1}{2} \right\rfloor - 3\\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1.
\end{aligned}$$

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $G^* + P_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings. This completes the proof of Theorem 2.4. □

3. THE CROSSING NUMBER OF $G^5_{16} + P_n$

Theorem 3.1 ([26, Corollary 4.1]). $\mathrm{cr}(G^5_{16} + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

In Figure 4, let G^5_{16} be the graph obtained from G^* by adding the edge v_2v_5 in the subdrawing in Figure 1(a).

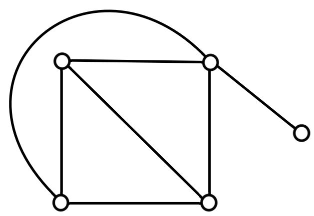

Fig. 4. The graph G^5_{16} by adding new edge to the graph G^*

Since we are able to add this edge to the graph G^* without additional crossings in Figure 2 and Figure 3, the drawings of the graph $G^5_{16} + P_2$ with 3 crossings and the graph $G^5_{16} + P_n$, $n \geq 3$, with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings are obtained, respectively. On the other hand, $G^* + P_n$ is a subgraph of $G^5_{16} + P_n$, and therefore, $\mathrm{cr}(G^5_{16} + P_n) \geq \mathrm{cr}(G^* + P_n)$. Thus, the next results are obvious.

Corollary 3.2. $\mathrm{cr}(G^5_{16} + P_2) = 3$.

Corollary 3.3. $\mathrm{cr}(G^5_{16} + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 3$.

4. THE CROSSING NUMBER OF $G^5_{18} + P_n$

Let G^5_{18} be the graph obtained by removing two edges incident with the same vertex from the complete graph K_5. Since the graph G^5_{18} contains the complete graph K_4 as a subgraph, all possible good drawings of G^5_{18} can be obtained from the drawings of the graph G^* by adding two new edges incident with the same vertex. In the rest of the paper, suppose that let v_5 be the vertex notation of this vertex of degree 2 in all considered good subdrawing of the graph G^5_{18}.

Theorem 4.1 ([26], Corollary 4.1)**.** $\mathrm{cr}(G^5_{18} + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Lemma 4.2. $\mathrm{cr}(G^5_{18} + P_2) = 5$ *and* $\mathrm{cr}(G^5_{18} + P_3) = 10$.

Proof. Notice that the graphs $G^5_{18}+P_2$ and $G^5_{18}+P_3$ are isomorphic to the join product of the cycle C_3 with the graphs G^4_9 and G^5_{17}, respectively. It was proven in [12] and [25] that $\mathrm{cr}(G^4_9 + C_3) = 5$ and $\mathrm{cr}(G^5_{17} + C_3) = 10$, respectively, and so $\mathrm{cr}(G^5_{18} + P_2) = 5$ and $\mathrm{cr}(G^5_{18} + P_3) = 10$. □

Theorem 4.3. $\mathrm{cr}(G^5_{18} + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 2$.

Proof. Lemma 4.2 confirms this result for $n = 2$ and $n = 3$. Figure 5 offers the drawing of the graph $G^5_{18} + P_n$ with exactly $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings.

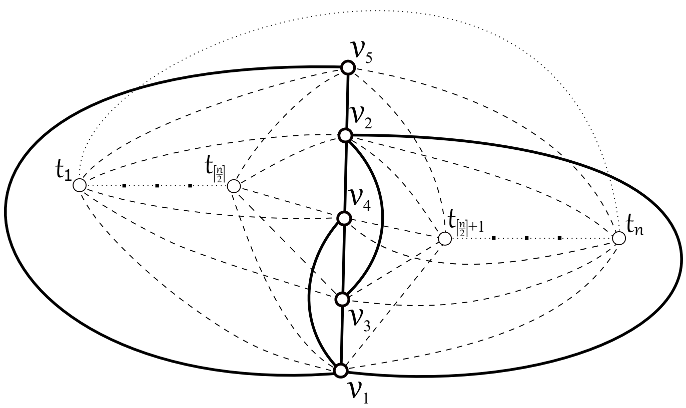

Fig. 5. The good drawing of $G^5_{18} + P_n$, $n \geq 2$, with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$ crossings

Thus, $\mathrm{cr}(G^5_{18} + P_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 2$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 4$, there is a drawing D with

$$\mathrm{cr}_D(G^5_{18} + P_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2, \tag{4.1}$$

and that

$$\mathrm{cr}(G^5_{18} + P_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ for any integer } 3 \leq m < n. \tag{4.2}$$

As the graphs $G^* + D_n$ and $G^* + P_n$ are some subgraphs of the graph $G^5_{18} + P_n$, by Theorems 2.1 and 2.4, the edges of $G^5_{18}+P_n$ are crossed exactly $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor+1$ times, and therefore, neither of the two edges of G^5_{18} incident with the vertex v_5 is crossed and at most one edge of the path P_n^* can be crossed in D. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (4.1) together with the well-known fact $\mathrm{cr}(K_{5,n}) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ imply that in D:

$$\mathrm{cr}_D(G^5_{18}) + \sum_{T^i\in R_D\cup S_D} \mathrm{cr}_D(G^5_{18}, T^i) + \sum_{T^i\notin R_D\cup S_D} \mathrm{cr}_D(G^5_{18}, T^i) \le n + \left\lfloor\frac{n}{2}\right\rfloor + 1,$$

i.e.,

$$\mathrm{cr}_D(G^5_{18}) + 0r + 1s + 2(n-r-s) \le n + \left\lfloor\frac{n}{2}\right\rfloor + 1. \tag{4.3}$$

This forces that $2r + s \ge \lceil\frac{n}{2}\rceil - 1 + \mathrm{cr}_D(G^5_{18})$, that is, $r + s \ge 1$, and so there is at least one subgraph T^i by which the edges of G^5_{18} are crossed at most once. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the considered drawing D and we will show that in all cases a contradiction with the assumption (4.1) is obtained.

Case 1. We suppose the subdrawing with the vertex notation of G^* in such a way as shown in Figure 1(a). The vertex v_5 cannot be adjacent with the vertex v_3, and therefore, two possible subcases may occur:

1) The vertex v_5 is not adjacent with the vertex v_1, that is, v_2v_5 and v_4v_5 are two edges of the graph G^5_{18}. Since there is no possibility to obtain a subdrawing of $G^5_{18}\cup T^i$ for a $T^i \in R_D$, there are at least $\lceil\frac{n}{2}\rceil - 1$ subgraphs T^i by which the edges of G^5_{18} are crossed exactly once. For a subgraph $T^i \in S_D$, the vertex t_i must be placed in the region with four vertices v_1, v_2, v_4, and v_5 of the graph G^5_{18} on its boundary. There are two possibilities to obtain the subdrawing of $F^i = G^5_{18} \cup T^i$ depending on which edge of the graph G^5_{18} is crossed by t_iv_3, but the same discussion will be used for both possible subdrawings. If all vertices of the path P_n^* are placed in this same region of subdrawing of G^5_{18}, then it is not difficult to verify in five possible regions of $D(G^5_{18} \cup T^i)$ that $\mathrm{cr}_D(G^5_{18} \cup T^i, T^j) \ge 4$ holds for each subgraph T^j, $j \ne i$. By fixing the subgraph $G^5_{18} \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(G^5_{18} + P_n) &\ge \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G^5_{18} \cup T^i) + \mathrm{cr}_D(G^5_{18} \cup T^i)\\ &\ge 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(n-1) + 1\\ &\ge 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor + 2.\end{aligned}$$

This contradicts the assumption of D. If all vertices of the path P_n^* are not placed in this considered region, they must be placed in the region with three vertices either v_1, v_2, v_3 or v_1, v_3, v_4 of the graph G^5_{18} on its boundary, because there is at most one crossing on the edges of P_n^* and neither of the edges v_2v_5, v_4v_5 is crossed in D. Without lost generality, based on their symmetry, let the edge v_1v_2 be crossed by some edge of the path P_n^*, that is, there is a vertex t_j placed in the region v_1, v_2, v_3 of G^5_{18} on its boundary. Let us denote by H the subgraph of G^5_{18} with the vertex set $V(G^5_{18})$, and the edge set $E(G^5_{18}) \setminus \{v_1v_2, v_1v_3, v_1v_4, v_2v_3\}$. Since the exact value for the crossing number of the graph $H + D_n$ is given in [23], i.e., $\mathrm{cr}(H + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, the edges of $H + D_n$ are crossed at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ times in D. As the subgraph T^j crosses edges of the cycle $v_1v_2v_3v_1$ twice, we obtain

$$\begin{aligned}\mathrm{cr}_D(G^5_{18} + P_n) &= \mathrm{cr}_D(H + D_n) + \mathrm{cr}_D(H + D_n, G^5_{18} - H) + \mathrm{cr}_D(G^5_{18}, P_n) \\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + (n+1) + 1 \\ &= 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2,\end{aligned}$$

where $G^5_{18} - H$ is the graph difference of graphs G^5_{18} and H. Hence, the discussed drawing contradicts the assumption of D again.

2) The vertex v_5 is adjacent with the vertex v_1, that is, v_1v_5 and v_kv_5 are the edges of the graph G^5_{18} for only one $k \in \{2, 4\}$. The proof proceeds in the same way as in the previous subcase.

Case 2. We consider the subdrawing of G^* given in Figure 1(b). In this case, both edges of the graph G^5_{18} incident with the vertex v_5 are uniquely designated as v_1v_5 and v_2v_5. Because no region is incident to all vertices in $D(G^5_{18})$, there is no possibility to obtain a subdrawing of $G^5_{18} \cup T^i$ for a $T^i \in R_D$. As $r = 0$, there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i whose edges cross the edges of G^5_{18} just once. For a $T^i \in S_D$, the vertex t_i is placed in the region with four vertices v_1, v_2, v_3, and v_4 of the graph G^5_{18} on its boundary. This enforces that the edge v_1v_2 of G^5_{18} must be crossed by the edge t_iv_5 and the subgraph $F^i = G^5_{18} \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (14325)$. Then, $\mathrm{cr}_D(G^5_{18} \cup T^i, T^j) \geq 1 + 4 = 5$ holds for any $T^j \in S_D$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, it is not difficult to verify in ten possible regions of $D(G^5_{18} \cup T^i)$ that $\mathrm{cr}_D(G^5_{18} \cup T^i, T^k) \geq 3$ is true for any subgraph $T^k \notin S_D$. Thus, by fixing the subgraph $G^5_{18} \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(G^5_{18} + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G^5_{18} \cup T^i) + \mathrm{cr}_D(G^5_{18} \cup T^i) \\ &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(s-1) + 3(n-s) + 1 + 1 \\ &= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 2s - 3 \\ &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 2\left\lceil \frac{n}{2} \right\rceil - 3 \\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 2.\end{aligned}$$

Case 3. We consider the subdrawing of G^* given in Figure 1(c). In this case, the vertex v_5 cannot be adjacent with both vertices v_k and v_{k+2}, for some $k \in \{1, 2\}$, otherwise, there is no possibility of obtaining a subgraph T^i from the nonempty set $R_D \cup S_D$ (the edges v_kv_5 and $v_{k+2}v_5$ cannot be crossed in D). Without lost of generality, let v_1v_5 and v_2v_5 be two edges of the graph G^5_{18}. Since the set S_D is empty, the vertex t_i of a subgraph $T^i \in R_D$ must be placed in the region with five vertices v_1, v_2, v_3, v_4, and v_5 of the graph G^5_{18} on its boundary. This enforces that the subgraph $F^i = G^5_{18} \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (14325)$, and we can also easy to verify in eight possible regions of $D(G^5_{18} \cup T^i)$ that $\mathrm{cr}_D(G^5_{18} \cup T^i, T^j) \geq 4$ holds for any subgraph T^j, $j \neq i$. Thus, by fixing the subgraph $G^5_{18} \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(G^5_{18} + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G^5_{18} \cup T^i) + \mathrm{cr}_D(G^5_{18} \cup T^i)\\ &\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(n-1) + 1\\ &\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor + 2.\end{aligned}$$

We have shown, in all cases, that there is no good drawing D of the graph $G^5_{18} + P_n$ with fewer than $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n + \lfloor\frac{n}{2}\rfloor + 2$ crossings. The proof of Theorem 4.3 is done. □

5. CONCLUSION

Determining the crossing number of the join products $G + D_n$ and $G + P_n$ is an essential step in establishing the so far unknown value of the number of crossings of the graph $G + C_n$, where C_n is the cycle on n vertices. Using the results in Theorems 2.4 and 4.3, Corollary 3.3, and the optimal drawings of $G^* + P_n$ and $G^5_{18} + P_n$ in Figures 3 and 5, we are able to postulate that the crossings numbers of $G^* + C_n$, $G^5_{16} + C_n$, and $G^5_{18} + C_n$ are at most two more than $G^* + P_n$, $G^5_{16} + P_n$, and $G^5_{18} + P_n$, respectively.

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