POZNAN UNIVERSITY OF TECHNOLOGY ACADEMIC JOURNALS
No 89 Electrical Engineering 2017

DOI 10.21008/j.1897-0737.2017.89.0001

Tadeusz KACZOREK*

CHARACTERISTIC EQUATIONS OF THE STANDARD AND DESCRIPTOR LINEAR ELECTRICAL CIRCUITS

The problem of calculation of the characteristic equations of the standard and descriptor linear electrical circuits is addressed. It is shown that the characteristic equation of the linear electrical circuit is independent of the method used for its analysis. The well-known three methods: the state space method, the mesh method and the node method are analyzed.

KEYWORDS: computation, characteristic equation, standard, descriptor, linear, electrical circuit

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [3, 9]. Variety of models having positive behavior can be found in engineering, especially in electrical circuits [16], economics, social sciences, biology and medicine, etc. [3, 9]. The analysis of linear systems and electrical circuits has been addressed in [1, 2, 16-20].

The positive electrical circuits have been analyzed in [4-8, 10, 16]. The constructability and observability of standard and positive electrical circuits has been addressed in [5], the decoupling zeros in [6] and minimal-phase positive electrical circuits in [7]. A new class of normal positive linear electrical circuits has been introduced in [8]. Positive fractional linear electrical circuits have been investigated in [11], positive linear systems with different fractional orders in [12, 13] and positive unstable electrical circuits in [14]. Zeroing of state variables in descriptor electrical circuits has been addressed in [15].

In this paper the problem of calculation of the characteristic equations of the standard, positive and descriptor linear electrical circuits will be analyzed.

The paper is organized as follows. The problem of calculation of the characteristic equations by the use of three methods: the state space method, the

* Białystok University of Technology.

mesh method and the node method is investigated in section 2. An extension of the results of section 2 to descriptor linear electrical circuits is presented in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. CHARACTERISTIC EQUATIONS OF THE ELECTRICAL CIRCUITS

2.1. State space method

Consider the linear continuous-time electrical circuit described by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$. It is well-known [1, 2, 16-20] that any standard linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources can be described by the equation (2.1). Usually as the state variables $x_1(t)$, …, $x_n(t)$ (the components of the vector $x(t)$) the currents in the coils and voltages on the capacitors are chosen.

Definition 2.1. The electrical circuit (2.1) is called (internally) positive if $x(t)\in\Re_+^n$, for any initial condition $x(0)\in\Re_+^n$ and every $u(t)\in\Re_+^m$, $t\in[0,+\infty)$.

Theorem 2.1. The electrical circuit (2.1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m}. \tag{2.2}$$

The positive electrical circuit (2.1) for $u(t)=0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \ \text{ for all } x(0)\in\Re_+^n. \tag{2.3}$$

Theorem 2.2. The positive electrical circuit (2.1) is asymptotically stable if and only if

$$\operatorname{Re}\lambda_k < 0 \ \text{ for } k=1,\dots,n, \tag{2.4}$$

where λ_k is the eigenvalue of the matrix $A\in M_n$ and

$$\det[I_n\lambda - A] = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n). \tag{2.5}$$

2.2. Mesh method

Any linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources in transient states can be also analyzed by the use of the mesh method [2, 16].

Using the mesh method and the Laplace transform we can describe the electrical circuit in transient states by the equation

$$Z(s)X(s) = E(s), \tag{2.6a}$$

where $X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$ ($\mathcal{L}$ is the Laplace operator),

$$Z(s) = \begin{bmatrix} Z_{11}(s) & Z_{12}(s) & \cdots & Z_{1n}(s) \\ Z_{21}(s) & Z_{22}(s) & \cdots & Z_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{n1}(s) & Z_{n2}(s) & \cdots & Z_{nn}(s) \end{bmatrix}, \; E(s) = \begin{bmatrix} E_1(s) \\ E_2(s) \\ \vdots \\ E_n(s) \end{bmatrix}. \tag{2.6b}$$

For example for the electrical circuit with given resistances R_1, R_2, R_3, inductances L_1, L_2 and voltage sources e_1, e_2 shown in Figure 2.1 using the mesh method we obtain the following.

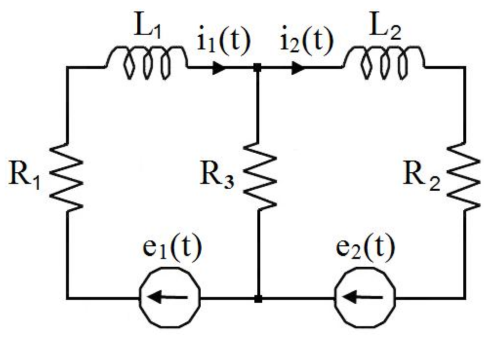

Fig. 2.1. Electrical circuit

Using the Kirchhoff's laws for the electrical circuit we obtain the equations

$$\begin{aligned} e_1 &= R_1 i_1 + L_1 \frac{di_1}{dt} + R_3(i_1 - i_2), \\ e_2 &= R_2 i_2 + L_2 \frac{di_2}{dt} + R_3(i_2 - i_1), \end{aligned} \tag{2.7}$$

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_1\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_1\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.8a}$$

where

$$A_1 = \begin{bmatrix} -\dfrac{R_1+R_3}{L_1} & \dfrac{R_3}{L_1} \\ \dfrac{R_3}{L_2} & -\dfrac{R_2+R_3}{L_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \dfrac{1}{L_1} & 0 \\ 0 & \dfrac{1}{L_2} \end{bmatrix}. \tag{2.8b}$$

The electrical circuit is positive since $A_1 \in M_2$ and $B_1 \in \Re^{2\times 2}$.

The characteristic equation of the electrical circuit has the form

$$\det[I_2 s - A_1] = \begin{vmatrix} s+\dfrac{R_1+R_3}{L_1} & -\dfrac{R_3}{L_1} \\ -\dfrac{R_3}{L_2} & s+\dfrac{R_2+R_3}{L_2} \end{vmatrix}$$
$$= s^2 + \left(\frac{R_1+R_3}{L_1} + \frac{R_2+R_3}{L_2}\right)s + \frac{R_1(R_2+R_3)+R_2R_3}{L_1L_2} = 0. \tag{2.9}$$

Using the mesh method and the Laplace transform we obtain

$$\begin{bmatrix} R_1+R_3+sL_1 & -R_3 \\ -R_3 & R_2+R_3+sL_2 \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} E_1(s) \\ E_2(s) \end{bmatrix}, \tag{2.10a}$$

where $I_k(s) = \mathcal{L}[i_k(t)]$, $E_k(s) = \mathcal{L}[e_k(t)]$, $k = 1,2$.

In this case we have

$$Z(s) = \begin{bmatrix} R_1+R_3+sL_1 & -R_3 \\ -R_3 & R_2+R_3+sL_2 \end{bmatrix}, \quad X(s) = \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}, \quad E(s) = \begin{bmatrix} E_1(s) \\ E_2(s) \end{bmatrix}. \tag{2.10b}$$

Note that

$$\det Z(s) = \begin{vmatrix} R_1+R_3+sL_1 & -R_3 \\ -R_3 & R_2+R_3+sL_2 \end{vmatrix}$$
$$= L_1L_2s^2 + [(R_1+R_3)L_2 + (R_2+R_3)L_1]s + R_1(R_2+R_3) + R_2R_3 \tag{2.11}$$

and after multiplication by $\dfrac{1}{L_1L_2}$ we obtain

$$\det Z(s) = L_1L_2 \det[I_2 s - A_1]. \tag{2.12}$$

From (2.12) we have the following conclusion.

Conclusion 2.1. The characteristic equation (2.9) of the electrical circuit can be also obtained by computation of the determinant of the matrix $Z(s)$ in the mesh method.

2.3. Node method

Any linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources in transient states can be also analyzed by the use of the node method. Using the node method and the Laplace transform we can describe the electrical circuit in transient states by the equation [2, 16]

$$Y(s)V(s) = I_z(s), \tag{2.13a}$$

where

$$Y(s) = \begin{bmatrix} Y_{11}(s) & Y_{12}(s) & \cdots & Y_{1q}(s) \\ Y_{21}(s) & Y_{22}(s) & \cdots & Y_{2q}(s) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{q1}(s) & Y_{q2}(s) & \cdots & Y_{qq}(s) \end{bmatrix}, \; V(s) = \begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_q(s) \end{bmatrix}, \; I_z(s) = \begin{bmatrix} I_{z1}(s) \\ I_{z2}(s) \\ \vdots \\ I_{zq}(s) \end{bmatrix}, \tag{2.13b}$$

q is the number of linearly independent nodes, $Y_{ij}(s)$ and $V_i(s)$, $i, j = 1,...,q$ are Laplace transforms of conductances and current sources of the electrical circuit, respectively.

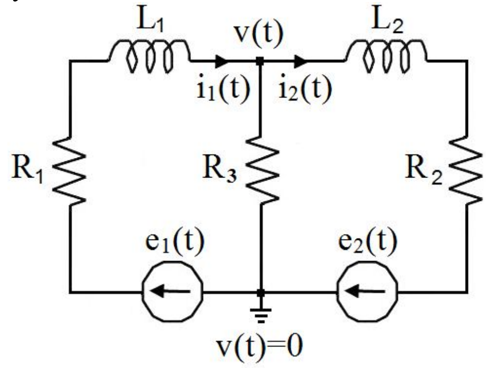

Fig. 2.2. Electrical circuit

For example for the electrical circuit shown in Figure 2.2 using the node method we obtain

$$Y(s)V(s) = I_z(s), \tag{2.14a}$$

where $V(s) = \mathcal{L}[v(t)]$, $E_k(s) = \mathcal{L}[e_k(t)]$, $k = 1,2$ and

$$
\begin{aligned}
Y(s) &= Y_{11}(s) = \frac{1}{R_1 + sL_1} + \frac{1}{R_2 + sL_2} + \frac{1}{R_3}, \\
I_z(s) &= -\frac{E_1(s)}{R_1 + sL_1} + \frac{E_2(s)}{R_2 + sL_2}.
\end{aligned} \tag{2.14b}
$$

Note that

$$
\begin{aligned}
\det Y(s) = Y(s) &= \frac{1}{R_1 + sL_1} + \frac{1}{R_2 + sL_2} + \frac{1}{R_3} \\
&= \frac{L_1 L_2 s^2 + [(R_1 + R_3)L_2 + (R_2 + R_3)L_1]s + R_1(R_2 + R_3) + R_2 R_3}{(R_1 + sL_1)(R_2 + sL_2)R_3}
\end{aligned} \tag{2.15}
$$

and after multiplication by $\dfrac{(R_1 + sL_1)(R_2 + sL_2)R_3}{L_1 L_2}$ we obtain

$$
\det Y(s) = \frac{L_1 L_2}{(R_1 + sL_1)(R_2 + sL_2)R_3} \det[I_2 s - A_1]. \tag{2.16}
$$

From (2.16) we have the following conclusion.

Conclusion 2.2. The characteristic equation (2.9) of the electrical circuit can be also obtained by computation of the determinant of the matrix $Y(s)$ in the node method.

In general case we shall prove the following theorem.

Theorem 2.3. The characteristic equation of any linear circuit composed of resistors, coils and capacitors is given in state equations method by

$$
\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0,
$$

in mesh method by $\det Z(s) = 0$ and in node method by $\det Y(s) = 0$.

Proof. Applying the Laplace transform to (2.1) with zero initial conditions we obtain

$$
X(s) = [I_n s - A]^{-1} BU(s) = \frac{[I_n s - A]_{ad}}{\det[I_n s - A]} BU(s), \tag{2.17}
$$

where $[I_n s - A]_{ad}$ is the adjoint matrix of $[I_n s - A]$.

From (2.6a) we have

$$
X(s) = Z^{-1}(s)E(s) = \frac{Z_{ad}(s)}{\det Z(s)} E(s). \tag{2.18}
$$

Comparing the denominators of (2.17) and (2.18) we obtain that $\det[I_n s - A] = 0$ is equivalent to $\det Z(s) = 0$.

From (2.13a) we have

$$
V(s) = Y^{-1}(s) I_z(s). \tag{2.19}
$$

Note that knowing $V(s)$ we can always find such matrix $P(s) \in \Re^{n \times q}(s)$ that

$$X(s) = P(s)V(s). \tag{2.20}$$

Substituting (2.19) into (2.20) we obtain

$$X(s) = P(s)Y^{-1}(s)I_z(s) = \frac{P(s)Y_{ad}(s)I_z(s)}{\det Y(s)}. \tag{2.21}$$

Comparing the denominators of (2.17) and (2.21) we obtain that $\det[I_n s - A] = 0$ is equivalent to $\det Y(s) = 0$. □

Remark 2.1. The characteristic polynomial and characteristic equation of any linear electrical circuit is independent of the voltage (current) sources. Therefore, computing the characteristic equation (polynomial) of the electrical circuit the voltage (current) sources can be assumed as zero.

Example 2.1. Consider the electrical circuit shown in Figure 2.3 with given resistances R_k, $k = 1,\ldots,5$, inductances L_1, L_2, capacitance C and source voltages e_1, e_2.

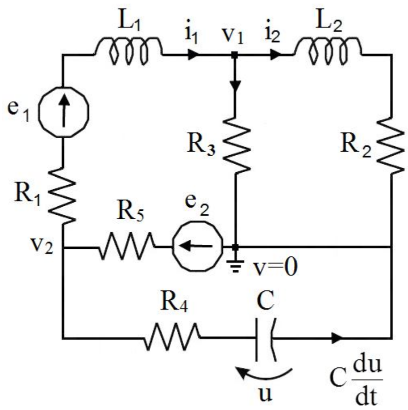

Fig. 2.3. Electrical circuit of Example 2.1

Using the Kirchhoff's laws for the electrical circuit we can write the equations

$$\begin{aligned} e_1 + e_2 &= R_{11}i_1 - R_3 i_2 + R_5 C\frac{du}{dt} + L_1\frac{di_1}{dt}, \\ 0 &= R_{22}i_2 - R_3 i_1 + L_2\frac{di_1}{dt}, \\ e_2 &= u + R_{33}C\frac{du}{dt} + R_5 i_1, \end{aligned} \tag{2.22}$$

where $R_{11} = R_1 + R_3 + R_5$, $R_{22} = R_2 + R_3$, $R_{33} = R_4 + R_5$.

The equations (2.22) can be written in the form

$$\begin{bmatrix} L_1 & 0 & R_5 C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33}C \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = \begin{bmatrix} -R_{11} & R_3 & 0 \\ R_3 & -R_{22} & 0 \\ -R_5 & 0 & -1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (2.23)$$

From (2.23) we have

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A_2 \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + B_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (2.24a)$$

where

$$A_2 = \begin{bmatrix} L_1 & 0 & R_5 C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33}C \end{bmatrix}^{-1}\begin{bmatrix} -R_{11} & R_3 & 0 \\ R_3 & -R_{22} & 0 \\ -R_5 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1}\left(\frac{R_5^2}{R_{33}} - R_{11}\right) & \frac{R_3}{L_1} & \frac{R_5}{L_1 R_{33}} \\ \frac{R_3}{L_2} & -\frac{R_{22}}{L_2} & 0 \\ -\frac{R_5}{R_{33}C} & 0 & -\frac{1}{R_{33}C} \end{bmatrix}, \quad (2.24b)$$

$$B_2 = \begin{bmatrix} L_1 & 0 & R_5 C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33}C \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & \frac{1}{L_1}\left(1 - \frac{R_5}{R_{33}}\right) \\ 0 & 0 \\ 0 & \frac{1}{R_{33}C} \end{bmatrix}.$$

Note that the electrical circuit is positive if and only if $R_5 = 0$.
The characteristic equation of the electrical circuit has the form

$$\det[I_3 s - A_2] = \begin{bmatrix} s + \frac{1}{L_1}\left(R_{11} - \frac{R_5^2}{R_{33}}\right) & -\frac{R_3}{L_1} & -\frac{R_5}{L_1 R_{33}} \\ -\frac{R_3}{L_2} & s + \frac{R_{22}}{L_2} & 0 \\ \frac{R_5}{R_{33}C} & 0 & s + \frac{1}{R_{33}C} \end{bmatrix} =$$

$$= \frac{L_1 L_2 R_{33} C s^3 + (L_1 L_2 - L_2 R_5^2 C + L_2 R_{11} R_{33} C + L_1 R_{22} R_{33} C) s^2}{L_1 L_2 R_{33} C} \quad (2.25)$$

$$+ \frac{(L_1 R_{22} + L_2 R_{11} - R_{22} R_5^2 C - R_{33} R_3^2 C + R_{11} R_{22} R_{33} C) s + R_{11} R_{22} - R_3^2}{L_1 L_2 R_{33} C}.$$

Applying to the electrical circuit in Figure 2.3 the mesh method we choose as the state variables the currents i_1, i_2, $C\frac{du}{dt}$ and we obtain

$$Z(s)=\begin{bmatrix} R_{11}+sL_1 & -R_3 & -R_5 \\ -R_3 & R_{22}+sL_3 & 0 \\ -R_5 & 0 & R_{33}+\frac{1}{sC} \end{bmatrix}, \tag{2.26}$$

where $R_{11}=R_1+R_3+R_5$, $R_{22}=R_2+R_3$, $R_{33}=R_4+R_5$ and

$$\det Z(s)=\frac{L_1L_2R_{33}Cs^3+(L_1L_2-L_2R_5^2C+L_2R_{11}R_{33}C+L_1R_{22}R_{33}C)s^2}{sC} \\ +\frac{(L_1R_{22}+L_2R_{11}-R_{22}R_5^2C-R_{33}R_3^2C+R_{11}R_{22}R_{33}C)s+R_{11}R_{22}-R_3^2}{sC}. \tag{2.27}$$

From comparison of (2.25) and (2.27) it follows that the characteristic equations are equivalent.

Applying to the electrical circuit in Figure 2.3 the node method we obtain

$$Y(s)=\begin{bmatrix} \frac{1}{R_1+sL_1}+\frac{1}{R_2+sL_2}+\frac{1}{R_3} & -\frac{1}{R_1+sL_1} \\ -\frac{1}{R_1+sL_1} & \frac{1}{R_1+sL_1}+\frac{1}{R_4+\frac{1}{sC}}+\frac{1}{R_5} \end{bmatrix} \tag{2.28}$$

and

$$\det Y(s)=\frac{L_1L_2R_{33}Cs^3+(L_1L_2-L_2R_5^2C+L_2R_{11}R_{33}C+L_1R_{22}R_{33}C)s^2}{R_3R_5(sR_4Cs+1)(R_1+sL_1)(R_2+sL_2)} \\ +\frac{(L_1R_{22}+L_2R_{11}-R_{22}R_5^2C-R_{33}R_3^2C+R_{11}R_{22}R_{33}C)s+R_{11}R_{22}-R_3^2}{R_3R_5(sR_4Cs+1)(R_1+sL_1)(R_2+sL_2)}, \tag{2.29}$$

where $R_{11}=R_1+R_3+R_5$, $R_{22}=R_2+R_3$, $R_{33}=R_4+R_5$.

From comparison of (2.25) and (2.29) if follows that the characteristic equations are equivalent.

3. DESCRIPTOR ELECTRICAL CIRCUITS

In this section the previous results will be extended to descriptor linear electrical circuits.

Consider the linear electrical circuit described by the equation

$$E\dot{x}=Ax+Bu, \tag{3.1}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that

$$\det E = 0 \text{ and } \det[Es - A] \neq 0$$

for some $s \in \mathbf{C}$ (the field of complex numbers). (3.2)

Definition 3.1. The linear electrical circuit described by (3.1) satisfying the assumption (3.2) is called a descriptor (singular) electrical circuit.

Theorem 3.1. Linear electrical circuit is descriptor if it contains at least one mesh consisting of only ideal capacitors and voltage sources or at least one node with branches with coils.

Proof. The proof is given in [16].

Consider the descriptor electrical circuit shown in Figure 3.1 with given resistances R_k, $k = 1,2,3$, inductances L_k, $k = 1,2,3$ and voltage sources e_1, e_2,

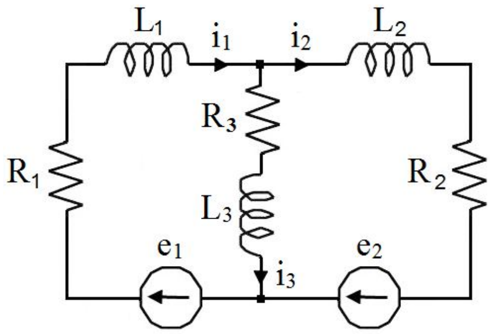

Fig. 3.1. Descriptor electrical circuit

Using the Kirchhoff's laws we obtain the equations

$$\begin{aligned} e_1 &= R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt}, \\ e_2 &= R_2 i_2 + L_2 \frac{di_2}{dt} - R_3 i_3 - L_3 \frac{di_3}{dt}, \\ i_1 &= i_2 + i_3, \end{aligned} \tag{3.3}$$

which can be written in the form

$$E\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.4a}$$

where

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & -L_3 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & R_3 \\ 1 & -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.4b}$$

The condition (3.2) is satisfied since $\det E = 0$ and

$$\det[Es - A] = \begin{vmatrix} R_1 + sL_1 & 0 & R_3 + sL_3 \\ 0 & R_2 + sL_2 & -R_3 - sL_3 \\ -1 & 1 & 1 \end{vmatrix} = [L_1(L_2 + L_3) + L_2 L_3]s^2$$
$$+ [L_1 R_3 + L_3 R_1 + L_2(R_1 + R_3) + (L_1 + L_3)R_2]s + R_1 R_3 + R_2(R_1 + R_2) \neq 0. \tag{3.5}$$

Therefore, the characteristic equation of the electrical circuit has the form

$$s^2 + \frac{L_1 R_3 + L_3 R_1 + L_2(R_1 + R_3) + (L_1 + L_3)R_2}{L_1(L_2 + L_3) + L_2 L_3} s + \frac{R_1 R_3 + R_2(R_1 + R_2)}{L_1(L_2 + L_3) + L_2 L_3]} = 0. \tag{3.6}$$

The descriptor electrical circuit shown in Figure 3.1 is positive if $i_k(t) \geq 0$, $k = 1,2,3$ for any initial conditions $i_k(0) \geq 0$, $k = 1,2,3$ and all $e_i(t) \geq 0$, $i = 1,2$ for $t \geq 0$.

Substituting $i_1(t) = i_2(t) + i_3(t)$ into (3.3) we obtain

$$\begin{bmatrix} L_1 & L_1 + L_3 \\ L_2 & -L_3 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1 & -(R_1 + R_3) \\ -R_2 & R_3 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.7}$$

and

$$\frac{d}{dt} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \overline{A} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} + \overline{B} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{3.8a}$$

where

$$\overline{A} = \begin{bmatrix} L_1 & L_1 + L_3 \\ L_2 & -L_3 \end{bmatrix}^{-1} \begin{bmatrix} -R_1 & -(R_1 + R_3) \\ -R_2 & R_3 \end{bmatrix} = \begin{bmatrix} \frac{R_1 L_3 + R_2(L_1 + L_3)}{L_1 L_3 + L_2(L_1 + L_3)} & \frac{(R_1 + R_3)L_2 - R_3 L_1}{L_1 L_3 + L_2(L_1 + L_3)} \\ \frac{R_1 L_2 - R_2 L_1}{L_1 L_3 + L_2(L_1 + L_3)} & \frac{(R_1 + R_3)L_2 - R_3 L_1}{L_1 L_3 + L_2(L_1 + L_3)} \end{bmatrix},$$
$$\overline{B} = \begin{bmatrix} L_1 & L_1 + L_3 \\ L_2 & -L_3 \end{bmatrix}^{-1} = \frac{1}{L_1 L_3 + L_2(L_1 + L_3)} \begin{bmatrix} L_3 & L_1 + L_3 \\ L_2 & -L_1 \end{bmatrix}. \tag{3.8b}$$

From (3.8b) it follows that $\overline{A} \in M_2$ if and only if $(R_1 + R_3)L_2 \geq R_3 L_1$ and $R_1 L_2 \geq R_2 L_1$, and $\overline{B} \in \Re_+^{2\times 2}$ if and only if $L_1 = 0$.

Therefore, the descriptor electrical circuit is not positive for all values of the resistances R_k, $k = 1,2,3$ and inductances L_k, $k = 1,2,3$.

Using the mesh method to the electrical circuit we obtain

$$Z(s) = \begin{bmatrix} R_1 + R_3 + s(L_1 + L_3) & -R_3 - sL_3 \\ -R_3 - sL_3 & R_2 + R_3 + s(L_2 + L_3) \end{bmatrix} \tag{3.9}$$

and

$$\begin{aligned} \det Z(s) &= [R_1 + R_3 + s(L_1 + L_3)][R_2 + R_3 + s(L_2 + L_3)] - (R_3 + sL_3)^2 \\ &= [L_1(L_2 + L_3) + L_2L_3]s^2 + [L_1R_3 + L_3R_1 + L_2(R_1 + R_3) + (L_1 + L_3)R_2]s \\ &+ R_1R_3 + R_2(R_1 + R_2). \end{aligned} \tag{3.10}$$

From comparison of (3.5) and (3.10) it follows that the characteristic equation obtained in the mesh method is identical with (3.6).

Using the node method to the electrical circuit we obtain

$$Y(s) = \frac{1}{R_1 + sL_1} + \frac{1}{R_2 + sL_2} + \frac{1}{R_3 + sL_3} \tag{3.11}$$

and

$$\begin{aligned} \det Y(s) &= \frac{(R_2 + sL_2)(R_3 + sL_3) + (R_1 + sL_1)(R_3 + sL_3) + (R_1 + sL_1)(R_2 + sL_2)}{(R_1 + sL_1)(R_2 + sL_2)(R_3 + sL_3)} \\ &= \frac{L_1(L_2 + L_3) + L_2L_3]s^2 + [L_1R_3 + L_3R_1 + L_2(R_1 + R_3) + (L_1 + L_3)R_2]s}{(R_1 + sL_1)(R_2 + sL_2)(R_3 + sL_3)} \\ &+ \frac{R_1R_3 + R_2(R_1 + R_2)}{(R_1 + sL_1)(R_2 + sL_2)(R_3 + sL_3)}. \end{aligned} \tag{3.12}$$

Therefore, the characteristic equation obtained in the node method is identical with (3.6). In general case we have the following theorem.

Theorem 3.1. The characteristic equations of the descriptor linear electrical circuit composed of resistors, coils and capacitors obtained by the state space method, mesh method and node method are equivalent.

Proof. The proof is similar to the proof of Theorem 2.3.

4. CONCLUDING REMARKS

The problem of calculation of the characteristic equations of the standard positive and descriptor linear electrical circuits has been addressed. It has been shown that the characteristic equation of the linear electrical circuit is independent of the method used for its analysis. Three well-known methods of the analysis of linear electrical circuits: state space method, mesh method and node method have been analyzed. The characteristic equation of the standard or descriptor linear electrical circuit can be computed using one of the three methods (Theorems 2.3 and 3.2). The considerations have been illustrated by simple examples of the linear electrical circuits and can be easily extended to fractional linear electrical circuits.

ACKNOWLEDGMENT

This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

REFERENCES

[1] Antsaklis P.J., Michel A.N.: Linear Systems. Birkhauser, Boston, 2006.
[2] Cholewicki T.: Theoretical Electrotechnics, WNT, Warszawa, 1967 (in Polish).
[3] Farina L., Rinaldi S.: Positive Linear Systems; Theory and Applications. J. Wiley, New York, 2000.
[4] Kaczorek T.: A class of positive and stable time-varying electrical circuits. Electrical Review, vol. 91, no. 5, 2015, 121-124.
[5] Kaczorek T.: Constructability and observability of standard and positive electrical circuits. Electrical Review, vol. 89, no. 7, 2013, 132-136.
[6] Kaczorek T.: Decoupling zeros of positive continuous-time linear systems and electrical circuits. Advances in Systems Science. Advances in Intelligent Systems and Computing, vol. 240, 2014, Springer, 1-15.
[7] Kaczorek T.: Minimal-phase positive electrical circuits. Electrical Review, vol. 92, no. 3, 2016, 182-189.
[8] Kaczorek T.: Normal positive electrical circuits. IET Circuits Theory and Applications, vol. 9, no. 5, 2015, 691-699.
[9] Kaczorek T.: Positive 1D and 2D Systems. Springer-Verlag, London, 2002.
[10] Kaczorek T.: Positive electrical circuits and their reachability. Archives of Electrical Engineering, vol. 60, no. 3, 2011, 283-301.
[11] Kaczorek T.: Positive fractional linear electrical circuits. Proceedings of SPIE, vol. 8903, Bellingham WA, USA, Art. No 3903-35.
[12] Kaczorek T.: Positive linear systems with different fractional orders. Bull. Pol. Acad. Sci. Techn., vol. 58, no. 3, 2010, 453-458.
[13] Kaczorek T.: Positive systems consisting of n subsystems with different fractional orders. IEEE Trans. Circuits and Systems – regular paper, vol. 58, no. 6, June 2011, 1203-1210.
[14] Kaczorek T.: Positive unstable electrical circuits. Electrical Review, vol. 88, no. 5a, 2012, 187-192.
[15] Kaczorek T.: Zeroing of state variables in descriptor electrical circuits by state-feedbacks. Electrical Review, vol. 89, no. 10, 2013, 200-203.
[16] Kaczorek T., Rogowski K.: Fractional Linear Systems and Electrical Circuits. Studies in Systems, Decision and Control, vol. 13, Springer, 2015.
[17] Kailath T.: Linear Systems. Prentice-Hall, Englewood Cliffs, New York, 1980.
[18] Rosenbrock H.: State-Space and Multivariable Theory. J. Wiley, New York, 1970.
[19] Wolovich W.A.: Linear Multivariable Systems, Springer-Verlag, New York, 1974.
[20] Żak S.H.: Systems and Control. Oxford University Press, New York, 2003.

(Received: 06. 02. 2017, revised: 20. 02. 2017)