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AN ANALYTICAL STUDY ON THE ELASTIC-PLASTIC BEHAVIOR OF METAL MATRIX COMPOSITES UNDER TENSILE LOADING

Sadaf Khosoussi
Aerospace Engineering Department, Sharif University of Technology, Tehran, Iran

Mehdi Mondali
Department of Mechanical and Aerospace Engineering, Islamic Azad University, Science and Research Branch, Tehran, Iran

Ali Abedian
Aerospace Engineering Department, Sharif University of Technology, Tehran, Iran; e-mail: abedian@sharif.ir

An analytical approach is proposed for studying the elastic-plastic behavior of short fiber reinforced metal matrix composites under tensile loading. In the proposed research, a micromechanical approach is employed considering an axi-symmetric unit cell including one fiber and the surrounding matrix. First, the governing equations and the boundary conditions are derived and the elastic solution is obtained based on some shear lag type methods. Since under normal loading conditions and according to the fiber material characteristics the metal matrix undergoes plastic deformation, while the fiber remains within the elastic region, a plastic deformation is obtained for the matrix under each small tensile loading step. Then, the elastic-plastic stress transfer behavior of the composite is studied considering this plastic deformation. The results are finally compared with the numerical results obtained from the FE analysis of the considered micromechanical model.

Keywords: metal matrix composites, plasticity in matrix, shear lag

1. Introduction

The use of metal matrix composite materials (MMC) in aerospace structures and engine parts has become more frequent recently. Stress transfer from the matrix to the fiber is known as the most important mechanism governing the deformation and the fracture response of such MMCs. The stress transfer characteristic of fiber reinforced composite materials under various mechanical and thermal loadings has been studied frequently.

Since the analytical study of the behavior of MMCs and the assumptions involved are complex, the majority of the investigations is limited to the numerical finite element or experimental methods. Recent efforts for analytical study of such materials have led to development of various micromechanical models including the fiber and the surrounding matrix, which considering several simplifying assumptions, have tried to interpret the behavior of MMCs. Due to the complexity of the model and various assumptions involved, most of the performed studies have been limited to the elastic behavior analysis of the fiber and the matrix. Though, since the matrix is metallic and thus the plasticity will occur at low strains, unpredicted failure may happen in the parts made from MMCs.

The solution to the concerned problem consists of two parts, first the elastic analysis of the MMCs and then the plastic solution for the metal matrix, which is based on the stress/strain fields obtained from the former elastic analysis. In general, numerous methods have been developed for elastic analysis of such MMCs. One of the main approaches to such a problem is the Shear Lag Method (Cox, 1952; Kelly, 1966; Piggott, 1980; Fukuda and Chou, 1981; Nardone and Prewo, 1986; Clyne, 1989; Karbhari and Wilkins, 1991; Starink and Syngellakis, 1999; Gao and Li, 2005), which due to the good description of the load transfer mechanism from the fiber to the matrix is of major importance and application. But, it should be noted that the various simplifying assumptions involved are considered as the main disadvantage of this model. Since due to such assumptions, the model is not that much accurate, it has been modified by others authors, see Hsueh (1988-1999), leading to increased efficiency, enabling the calculation of the stresses at the end of the fiber and the shear stress at the fiber/matrix interface. In this method, called the Imaginary Fiber Technique, the problem is initially solved for a continuous long fiber, and by means of applying the boundary conditions, the consistency condition, and considering a fiber with the same matrix material at the end section, the solution to the short fiber problem is obtained accordingly. The next effort for modeling the concerned elastic problem was performed by Jiang *et al.* (1998, 2004). In the initial steps, he also applied some of the previous simplifying assumptions involved, but eliminating some of these assumptions in the final stages, he achieved a better compatibility with the existing FE results.

After this brief review of the existing solutions for the elastic analysis of the current problem, some of the efforts for solving the plastic problem will be discussed shortly. The Shear Lag Method first introduced by Cox (1952) was vastly used due to its mathematical simplicity and its good prediction of this mechanism. After the elastic analysis of the existing problem, Jiang *et al.* (2004) studied the plastic behavior of short fiber reinforced MMCs, applying the Shear Lag Method. In this study, the effect of the matrix plastic deformation on the load transfer mechanism has been defined via introduction of a plastic strain. The results show that the efficiency of the load transfer in the elastic region decreases due to local plastic deformations in the matrix. Though, in this study, a very simple approach has been used for plastic analysis of the matrix. Since the general shear lag model applied is not capable of predicting the stress distribution in the matrix, the problem has been simplified by means of assuming an average axial plastic deformation in the matrix, defined by some linear distribution assumptions made later. Furthermore, the plastic strain term considered has been limited to the axial strain only, and the effect of the shear stress has been totally neglected by applying an approximate relation between the axial stress and the average axial plastic deformation in the matrix.

In the present study, an analytical approach is proposed for studying the elastic-plastic behavior of short fiber reinforced metal matrix composites under small tensile loading steps. In the proposed research, employing a micromechanical approach, an axi-symmetric unit cell including one fiber and the surrounding matrix is considered. Using a shear-lag based formulation, first, the governing equations and the boundary conditions are derived and the elastic solution is obtained for both the fiber and matrix. Since under normal loading conditions and according to the fiber material characteristics, the metal matrix undergoes plastic deformation first, plastic deformation is considered in the matrix. The governing relations are then obtained, considering both axial and shear stress terms, e.g. the equivalent stress and the plastic strain components. Finally, an approximate estimate for plastic strain distribution in the matrix and the effects of this plastic deformation on the stress transfer mechanism of the composite are resulted. Some numerical results obtained by FE analysis of the model are shown as well.

2. Mathematical formulation

In order to study the elastic-plastic behavior of MMCs under simple tensile loading, a cylindrical axi-symmetric unit cell consisting of a fiber and the surrounding matrix has been considered as shown in Figs. 1a and 1b. The cell is subjected to a uniform tensile stress σ_0.

According to FEM results and the governing equations, which can be admitted by common sense also, the plastic deformation in the matrix will start at somewhere in the adjacent region to the fiber, region I shown in Fig. 1b, due to the low matrix yield stress and high stress

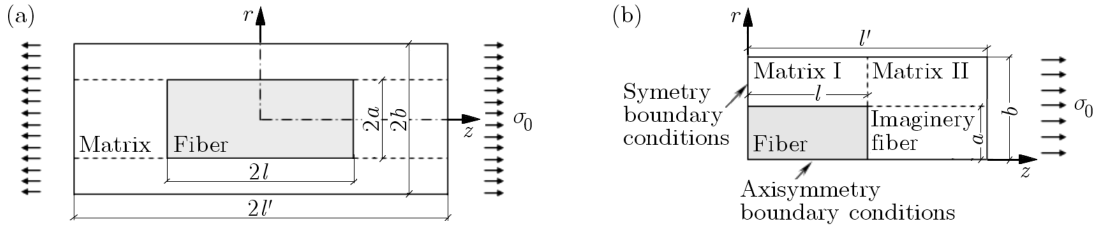

Fig. 1. (a) Micromechanical unit cell. (b) Axi-symmetric model

concentrations in that region. Considering the occurrence of plastic deformation in the matrix and focusing on region I of matrix from now on, the stress-strain relations for the matrix in this region can be re-written as (Mendelson, 1986)

$$\begin{aligned}
\varepsilon_r^m &= \frac{1}{E^m}[\sigma_r^m - \nu^m(\sigma_z^m + \sigma_\theta^m)] + \varepsilon_r^p \qquad & \varepsilon_\theta^m &= \frac{1}{E^m}[\sigma_\theta^m - \nu^m(\sigma_r^m + \sigma_z^m)] + \varepsilon_\theta^p \\
\varepsilon_z^m &= \frac{1}{E^m}[\sigma_z^m - \nu^m(\sigma_\theta^m + \sigma_r^m)] + \varepsilon_z^p \qquad & \varepsilon_{rz}^m &= \frac{1+\nu^m}{E^m}\tau_{rz}^m + \varepsilon_{rz}^p
\end{aligned} \tag{2.1}$$

where E^m and ν^m are Young's modulus and Poisson's ratio of the matrix, respectively, and ε^p terms define the plastic strain components. It is clear though that as the loading is increased, the plastic region expands within the matrix, while the points for which the yield criterion is not satisfied yet, still remain in the elastic region, with ε^p terms being zero in Eqs. (2.1). To summarize, at each loading step, the yield criterion must be checked for the matrix in order to determine the plastic region as will be shown later.

Considering the axi-symmetry of the model, and thus neglecting the derivatives with respect to θ, the general equilibrium equations for both the fiber and matrix can be written as

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0 \qquad \frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2.2}$$

Assuming $\partial\sigma_z^m/\partial z = g(z)$ and using Eq. $(2.1)_1$, it can be shown that (Gao, 2005)

$$\tau_{rz}^m(z) = \frac{a}{b^2 - a^2}\Big(\frac{b^2}{r} - r\Big)\tau_i(z) \tag{2.3}$$

where $\tau_i(z)$ is the shear stress at the fiber/matrix interface. According to the last of the strain--displacement relations

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \varepsilon_{rz} = \frac{1}{2}\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\Big) \tag{2.4}$$

with u being the radial and w the axial displacement, and neglecting the term $\partial u^m/\partial z$ for matrix, according to the assumption that $|\partial u^m/\partial z| \ll |\partial w^m/\partial r|$, which due to the tensile loading condition and the symmetry of the model is a reasonable assumption, the radial displacement of the matrix w^m is resulted as follows

$$w^m(r,z) = w_a^m + \frac{w_b^m - w_a^m - 2\int\limits_a^b \varepsilon_{rz}^p(r,z)\,dr}{b^2\ln\Big(\frac{b}{a}\Big) - \frac{b^2-a^2}{2}}\Big[b^2\ln\Big(\frac{r}{a}\Big) - \frac{r^2-a^2}{2}\Big] + 2\int\limits_a^r \varepsilon_{rz}^p(r,z)\,dr \tag{2.5}$$

where w_a^m is the matrix axial displacement at the interface $r = a$, and w_b^m is the axial displacement at $r = b$. From the other hand, and assuming that the radial and tangential stresses in the matrix are much smaller compared to the axial stress term, $|\sigma_r^m + \sigma_\theta^m| \ll \sigma_z^m$, and thus neglecting the term $(\sigma_r^m + \sigma_\theta^m)$ in Eq. $(2.1)_3$, differentiating Eq. (2.5) with respect to z and finally considering the strain-displacement relations, Eqs. (2.4), it can be shown that

$$\begin{aligned} \sigma_z^m(r,z) &= \sigma_a^m + E^m\varepsilon_z^p(a,z) + 2E^m\frac{\partial}{\partial z}\int\limits_a^r \varepsilon_{rz}^p(r,z)\;dr \\ &+ \frac{\sigma_b^m + E^m\varepsilon_z^p(b,z) - \sigma_a^m - E^m\varepsilon_z^p(a,z) - 2E^m\frac{\partial}{\partial z}\int\limits_a^b \varepsilon_{rz}^p(r,z)\;dr}{b^2\ln\Big(\frac{b}{a}\Big) - \frac{b^2-a^2}{2}} \\ &\cdot \Big[b^2\ln\Big(\frac{r}{a}\Big) - \frac{r^2-a^2}{2}\Big] - E^m\varepsilon_z^p(r,z) \end{aligned} \tag{2.6}$$

where σ_a^m and σ_b^m are the matrix axial stresses at $r=a$ and $r=b$, respectively. Writing the equilibrium equation of a cross section of the cell, it can be concluded that

$$\frac{a^2}{2}\overline{\sigma}_z^f(z) + \int\limits_a^b r\sigma_z^m(r,z)\;dr = \frac{b^2}{2}\sigma_0 \tag{2.7}$$

in which $\overline{\sigma}_z^f(z)$ is the average axial stress in fiber. Substituting $\sigma_z^m(r,z)$ from Eq. (2.6) into (2.7) and re-writing Eq. (2.6) with the resulted σ_b^m, axial stress in the matrix can be found as

$$\begin{aligned} \sigma_z^m(r,z) &= \sigma_a^m + E^m\varepsilon_z^p(a,z) - E^m\varepsilon_z^p(r,z) + 2E^m\frac{\partial}{\partial z}\int\limits_a^r \varepsilon_{rz}^p(r,z)\;dr \\ &+ \frac{b^2\ln\Big(\frac{r}{a}\Big) - \frac{1}{2}(r^2-a^2)}{b^4\ln\Big(\frac{b}{a}\Big) - \frac{1}{4}(b^2-a^2)(3b^2-a^2)}\Big\{ - a^2\overline{\sigma}_z^f + b^2\sigma_0 - (b^2-a^2)[\sigma_a^m + E^m\varepsilon_z^p(a,z)] \\ &- 4E^m\frac{\partial}{\partial z}\int\limits_a^b r\int\limits_a^r \varepsilon_{rz}^p(r,z)\;dr\;dr + 2E^m\int\limits_a^b \varepsilon_z^p(r,z)r\;dr\Big\} \end{aligned} \tag{2.8}$$

On the other hand, from the first of equilibrium equations $(2.1)_1$ for fiber using the average axial stress introduced in Eq. (2.7), the following relation known as shear lag equation can be derived

$$\frac{d\overline{\sigma}_z^f(z)}{dz} = -\frac{2}{a}\tau_i(z) \tag{2.9}$$

Combining Eq. (2.9), Eq. (2.3), stress-strain relations, Eqs. (2.1) and strain-displacement equations Eqs. (2.4) and finally substituting for $\sigma_z^m(r,z)$ from Eq. (2.8), the following relation will be derived for the fiber average axial stress $\overline{\sigma}_z^f(z)$

$$\begin{aligned} \frac{d^2\overline{\sigma}_z^f(z)}{dz^2} &= -\frac{b^2-a^2}{a^2(1+\nu^m)\Big[b^4\ln\Big(\frac{b}{a}\Big) - \frac{1}{4}(b^2-a^2)(3b^2-a^2)\Big]} \\ &\cdot\Big\{ - a^2\overline{\sigma}_z^f + b^2\sigma_0 - (b^2-a^2)[\sigma_a^m + E^m\varepsilon_z^p(a,z)] \\ &- 4E^m\frac{\partial}{\partial z}\int\limits_a^b r\int\limits_a^r \varepsilon_{rz}^p(r,z)\;dr\;dr + 2E^m\int\limits_a^b r\varepsilon_z^p(r,z)\;dr\Big\} \end{aligned} \tag{2.10}$$

As can be seen, the only unknown term in Eq. (2.10) is the matrix axial stress at the interface σ_a^m, which can be derived as follows. The assumption of perfect bond between the fiber and matrix at the interface implies the boundary condition of equality of axial strains along the interface. Referring to Eq. $(2.1)_3$ and remembering the previously made assumption according to which the summation of the radial and tangential stresses were neglected compared to the axial term of stress, the equality of axial strains at the boundary will lead to the following

$$\sigma_a^m = E^m\Big(\frac{\overline{\sigma}_z^f(z)}{E^f} - \varepsilon_z^p(a,z)\Big) \tag{2.11}$$

where E^f is Young's modulus of the fiber. It should be noted that the stress-strain relation used for the fiber is in the elastic state, which is obvious according to the basic assumption of the problem, previously stated in the introduction.

Therefore, substituting Eq. (2.11) into Eq. (2.10) and solving the resulted ordinary differential equation, one can easily derive the relation for fiber average axial stress, including the effect of plastic deformation in the matrix

$$\begin{aligned}\overline{\sigma}_z^f(z) &= \exp(-\sqrt{AB}z)\Big\{C_1 - \frac{1}{2}\sqrt{\frac{A}{B}}\int \exp(\sqrt{AB}z)[C+F(z)]\;dz\Big\}\\ &\quad + \exp(\sqrt{AB}z)\Big\{C_2 + \frac{1}{2}\sqrt{\frac{A}{B}}\int \exp(-\sqrt{AB}z)[C+F(z)]\;dz\Big\}\end{aligned} \tag{2.12}$$

in which

$$\begin{aligned}A &= \frac{b^2-a^2}{a^2(1+\nu^m)\Big[b^4\ln\Big(\frac{b}{a}\Big) - \frac{1}{4}(b^2-a^2)(3b^2-a^2)\Big]}\\ B &= a^2 + (b^2-a^2)\frac{E^m}{E^f} \qquad\qquad C = -b^2\sigma_0\\ F(z) &= 4E^m\frac{\partial}{\partial z}\int\limits_a^b r\int\limits_a^r \varepsilon_{rz}^p(r,z)\;dr\;dr - 2E^m\int\limits_a^b r\varepsilon_z^p(r,z)\;dr\end{aligned} \tag{2.13}$$

Using Eq. (2.9), fiber shear stress can be derived accordingly

$$\begin{aligned}\tau_i(z) &= -\frac{a}{2}\Big\{-\sqrt{AB}\exp(-\sqrt{AB}z)\Big[C_1 - \frac{1}{2}\sqrt{\frac{A}{B}}\int \exp(\sqrt{AB}z)[C+F(z)]\;dz\\ &\quad + \sqrt{AB}\exp(\sqrt{AB}z)\Big[C_2 + \frac{1}{2}\sqrt{\frac{A}{B}}\int \exp(-\sqrt{AB}z)[C+F(z)]\;dz\Big\}\end{aligned} \tag{2.14}$$

As cleared at the beginning of the formulation, all the above discussions up to this point are valid for region I of the matrix shown in Fig. 1b and the fiber within $0 \leqslant z \leqslant l$ range. Considering the $l \leqslant z \leqslant l'$ area, the same approach as the one proposed by Hsueh (1995), Hsueh and Becher (1996), known as the Imaginary Fiber Technique has been applied. In other words, the fiber is considered to be continuous along the whole cell length, with the $l \leqslant z \leqslant l'$ region being known as the imaginary fiber. The same treatment as the one discussed above is applied to this fiber, except that at the end of the process, the material properties of this imaginary section will be replaced by the matrix properties. Considering the local plastic deformation in region I of the matrix as discussed, the same argument with Jiang *et al.* (2004) has been used as follows. As stated in that study, when the matrix local plastic deformation occurs in region I, it will also occur in the region near the fiber end face with the same magnitude. Therefore, the stress transfer in the fiber end region will not be affected by the plastic deformation in the fiber region. Following such an argument, the governing differential equation for the imaginary fiber axial stress can be written as

$$\frac{d^2\overline{\sigma}'^f_z(z)}{dz^2} = -\frac{b^2-a^2}{a^2(1+\nu^m)\Big[b^4\ln\Big(\frac{b}{a}\Big)-\frac{1}{4}(b^2-a^2)(3b^2-a^2)\Big]}(-b^2\overline{\sigma}'^f_z+b^2\sigma_0) \tag{2.15}$$

Therefore, the axial and shear stress for imaginary fiber can be derived as follows

$$\begin{aligned} \overline{\sigma}'^f_z(z) &= C_3\exp(-\sqrt{AB'}z)+C_4\exp(\sqrt{AB'}z)-\frac{C}{B'} \\ \tau'_i(z) &= -\frac{a\sqrt{AB'}}{2}\Big[-C_3\exp(-\sqrt{AB'}z)+C_4\exp(\sqrt{AB'}z)\Big] \end{aligned} \tag{2.16}$$

where constants A and C are previously defined in Eq. $(2.13)_1$ and $(2.13)_3$, while B' is

$$B' = b^2 \tag{2.17}$$

As can be seen in Eqs. (2.12), (2.14) and (2.16), four unknown constants C_1, C_2, C_3 and C_4 are still to be calculated. For this reason, the following boundary conditions are applied to the axial and shear stresses previously derived

$$\overline{\sigma}'^f_z = \begin{cases} \sigma_0 & \text{at} \quad z=l' \\ \overline{\sigma}^f_z & \text{at} \quad z=l \end{cases} \qquad \tau'_i = \begin{cases} \tau_i & \text{at} \quad z=l \\ 0 & \text{at} \quad z=0 \end{cases} \tag{2.18}$$

In order to calculate the plastic deformation and its effect on stress transfer in the composite, first of all, the yield criterion should be determined in order to indentify the region in which the plastic deformation occurs under each loading step. Considering the von-Mises yield criterion (Mendelson, 1986) as follows

$$\sigma_e = \sigma_y \tag{2.19}$$

the yielding will occur as soon as the equivalent stress σ_e in a certain point reaches the yield stress σ_y in which the equivalent stress is defined as

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\tau_{rz}^2} \tag{2.20}$$

According to the previously stated assumption, $|\sigma_r^m+\sigma_\theta^m| \ll \sigma_z^m$, the yield criterion for the current problem can be restated as

$$\sqrt{\sigma_z^{m2}+3\tau_{rz}^{m2}} = \sigma_y \tag{2.21}$$

As described before and will be shown in the numerical results, it is evident that the yielding in the matrix will start at the interface somewhere at the end of the fiber. Thus, substituting for σ_z^m and τ_{rz}^m at $(r,z)=(a,l)$ into Eq. (2.21), the critical stress σ_0 for which the yielding will start for the first time in the composite, will be calculated. Thereafter, for stress values above this critical threshold, the region in the matrix in which the plastic deformation occurs will be expanded. In order to obtain an estimate for the behavior of the model considering plastic deformations in the matrix, the relations obtained up to this point will be simplified as follows. The plastic strain increments are related to the stresses through the yield criterion and the associated flow rule. According to the von-Mises yield criterion, Eq. (2.19) previously applied and the Prandtl-Reuss relations considered here, the z and rz strain increments can be obtained as follows (Mendelson, 1986)

$$\varDelta\varepsilon_z^p = \frac{\varDelta\varepsilon_p^e}{2\sigma_e}(2\sigma_z-\sigma_r-\sigma_\theta) \qquad \varDelta\varepsilon_{rz}^p = \frac{3}{2}\frac{\varDelta\varepsilon_p^e}{\sigma_e}\tau_{rz} \tag{2.22}$$

in which

$$\Delta\varepsilon_p^e = \frac{\sqrt{2}}{3}\sqrt{(\Delta\varepsilon_r^p - \Delta\varepsilon_\theta^p)^2 + (\Delta\varepsilon_\theta^p - \Delta\varepsilon_z^p)^2 + (\Delta\varepsilon_z^p - \Delta\varepsilon_r^p)^2 + 6\Delta{\varepsilon_{rz}^p}^2} \tag{2.23}$$

Assuming a proportional loading, i.e. considering a loading level for which the yielded area is relatively small compared to the fiber diameter, it can be assumed that the stress state in the yielded region in the matrix is the same as the elastic state, e.g.

$$\frac{\sigma_z^{mE}(r,z)}{\tau_{rz}^{mE}(r,z)} = \frac{\sigma_z^{mP}(r,z)}{\tau_{rz}^{mP}(r,z)} \tag{2.24}$$

where E and P superscripts refer to elastic and plastic stress fields, respectively. With such an argument and rewriting Eqs. (2.22) for the proportional loading case and thus omitting the increment notation Δ according to the deformation or total theory of plasticity and remembering that the radial and axial stresses were neglected compared to the axial term, the following relation can be derived between z and rz plastic strains

$$\frac{\varepsilon_z^p}{\varepsilon_{rz}^p} = \frac{2}{3}\frac{\sigma_z(r,z)}{\tau_{rz}(r,z)} \tag{2.25}$$

Thereafter, as previously stated, considering small loading steps, and thus a small plastic region, a linear distribution for plastic deformations can be assumed over the plastic region for a certain loading step

$$\begin{aligned}
\varepsilon_z^p(r,z) &= \frac{\varepsilon_{z1}^p(a-r)}{da} - \frac{\varepsilon_{z1}^p(l-z)}{dz} + \varepsilon_{z1}^p \\
\varepsilon_{rz}^p(r,z) &= \frac{\varepsilon_{rz1}^p(a-r)}{da} - \frac{\varepsilon_{rz1}^p(l-z)}{dz} + \varepsilon_{rz1}^p
\end{aligned} \tag{2.26}$$

in which ε_{z1}^p and ε_{rz1}^p are the maximum values of the plastic deformation at $(r,z) = (a,l)$, and da and dz are radial and axial boundaries of the plastic region, respectively. The analysis applied will be then as follows. For a loading step σ_{01} greater than the initial critical stress, the yield boundaries, e.g. da and dz in Eqs. (2.26), in the matrix are determined. Using Eq. (2.12), (2.14) and (2.16) and rewriting the relations, substituting for one of the plastic strains in terms of the other according to Eq. (2.25) at $(r,z) = (a,l)$, the axial stress in the fiber and the interface shear stress will be calculated as functions of the considered plastic strain. Thereafter, substituting the results into Eq. (2.8) and (2.3), the axial and shear stress in the matrix will be determined accordingly. Calculating the equivalent stress for the matrix, the equivalent plastic strain will be determined from the uni-axial stress-strain curve as follows

$$\sigma_e = \sigma_y + \frac{EE^p}{E - E^p}\varepsilon_e^p \tag{2.27}$$

in which E^p is the plastic modulus in the uni-axial stress-strain curve and ε_e^p is the equivalent plastic strain, previously defined incrementally in Eq. (2.23). It should be noted that this relation is valid for a perfectly plastic material. Also, since the radial and the tangential plastic strains are considered to be

$$\varepsilon_r^p = \varepsilon_\theta^p = -\frac{1}{2}\varepsilon_z^p \tag{2.28}$$

the equivalent strain term, Eq. (2.23), can be simplified to

$$\varepsilon_p^e = \frac{\sqrt{2}}{3}\sqrt{\frac{9}{2}\varepsilon_z^{p2} + 6{\varepsilon_{rz}^p}^2} \tag{2.29}$$

Finally, solving Eq. (2.27) at $(r,z) = (a,l)$ by a MAPLE code developed, the plastic strain terms, and therefore the stresses, can be derived.

3. Results and discussion

In order to understand the effects of the plastic deformation in the matrix on stress transfer behavior of the composite, the effective stresses contributing in this mechanism, $\overline{\sigma}_z^f$ and τ_i, will be studied. The calculations are done for Al6061/SiC20% composite, with the following specifications

Table 1. Geometrical specifications of Al6061/SiC20% composite

s	k	f'	f
5	1	0.342	0.2

Table 2. Mechanical properties of Al6061/SiC20% composite

Material	ρ [g/cm^3]	E [GPa]	E^p [GPa]	υ
Al6061	2.7	68.3	5.667	0.345
SiC	3.2	470	–	0.17

Figure 2a shows the equivalent stress distribution in the matrix for an arbitrary elastic loading. It is clear that this stress is maximum at the interface, at the fiber end point $(r,z)=(a,l)$, as claimed before. As shown in this figure, for an applied loading of $\sigma_0 = 274$ MPa, the equivalent stress at the critical point of $(r,z)=(a,l)$ reaches the yield value $\sigma_y = 276$ MPa for the first time. Thereafter, comparing this equivalent stress with the yield stress according to the von-Mises criterion, the expansion of plastic region in the matrix for different loading steps σ_0 is shown in Fig. 2b. As can be seen, the boundaries of this plastic region da and dz in Eqs. (2.26) expand with the increase of the loading applied. As described before, the relation between the two plastic strain components can be estimated from Eq. (2.25). The calculated value at $(r,z)=(a,l)$ has been rounded to 1 for simplicity, i.e. $\varepsilon_{z1}^p = \varepsilon_{rz1}^p$ and the z and rz terms of the plastic strain have been considered to be equal $\varepsilon_z^p(r,z) = \varepsilon_{rz}^p(r,z)$. The variation of the plastic strain components have been previously defined by linear functions in Eqs. (2.26). With this assumption, the fiber average axial stress, the interfacial shear stress, and the z/rz terms of the matrix plastic strain have been calculated for each loading step as shown. The plastic strain functions are illustrated in Figs. 3a and 3b at $z=l$ and $r=a$, respectively, for different loading steps.

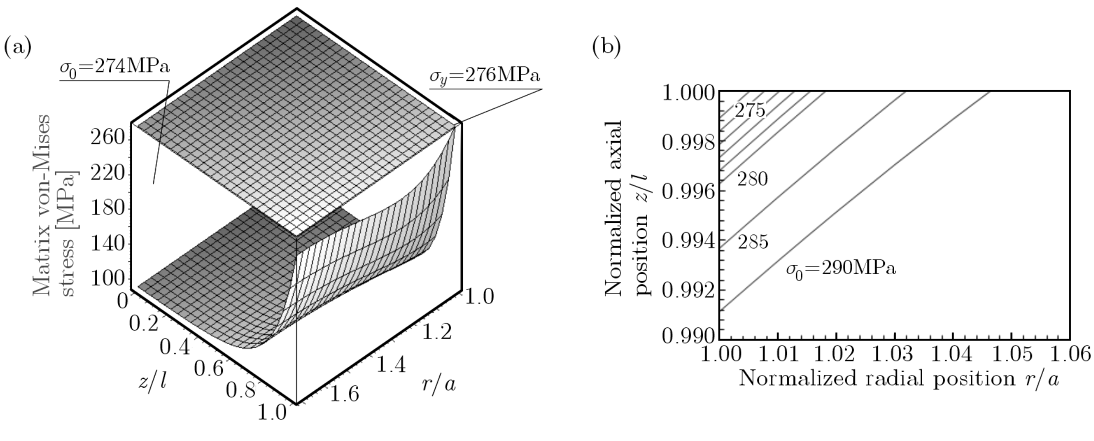

Fig. 2. (a) Matrix equivalent stress for $\sigma_0 = 274$ MPa, (b) expansion of the plastic region in the matrix model

General behavior of the shear stress at $r=a$ and fiber average axial stress are shown in Figs. 4a and 4b, respectively. A closer view of these stress curves are shown in Figs. 5a and 5b

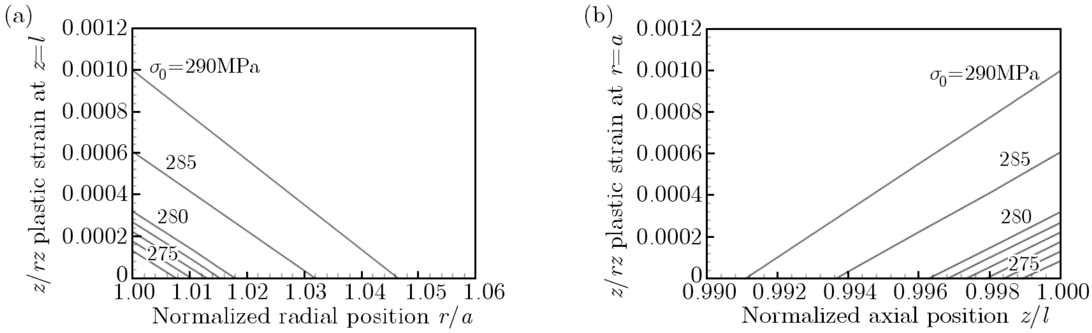

Fig. 3. $\varepsilon_z^p(r,l)/\varepsilon_{rz}^p(r,l)$, plastic strains at: (a) $z=l$, (b) $r=a$

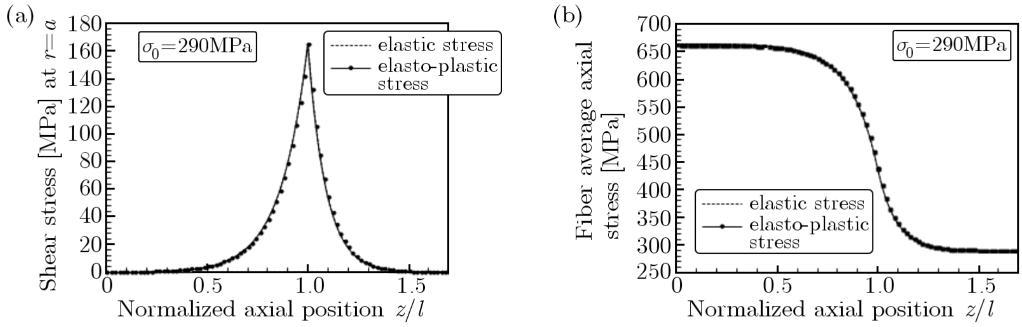

Fig. 4. Shear stress at $r=a$ (a) and fiber average axial stress (b) for $\sigma_0 = 290\,\mathrm{MPa}$

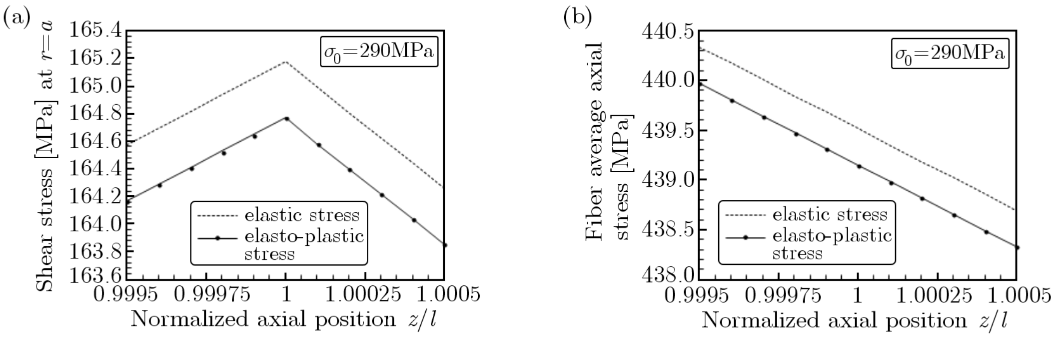

Fig. 5. Shear stress at $r=a$ (a) and fiber average axial stress (b) for $\sigma_0 = 290\,\mathrm{MPa}$

for a small region about the fiber end $z=l$. The stresses have been calculated for $\sigma_0 = 290\,\mathrm{MPa}$ both for the elastic and elastic-plastic conditions. As can be seen, the results comply with the initial expectations that the occurrence of plasticity in the matrix will reduce the efficiency of the load transfer mechanism in the composite. Since the shear stress at the interface and the axial stress in the fiber are of major importance in the process of the load transfer from the matrix to the fiber, this decrease in their values will affect the initially expected load bearing characteristics of the material. Thereafter, Figs. 6a and 6b show the variation of the shear stress at the interface and the fiber average axial stress for different loading steps. It can be noted that increasing the loading, the stresses in presence of the plastic strains will increase more slowly compared to the elastic condition. This is another verification of the adverse effect of the plasticity in the matrix over the mechanical characteristics of the MMCs. Finally, some numerical FEM results using ANSYS 10 can be found in Fig. 7 to Fig. 9 for a general comparison. Taking a glance over these results, first of all it can be noticed that the assumption of the equality of the plastic strain terms, $\varepsilon_z^p(r,z) = \varepsilon_{rz}^p(r,z)$, is acceptable within the loading region applied. Furthermore, as can be seen, the linear distribution considered for plastic strain distribution over the plastic region is also a reasonable assumption for the considered loading steps. Thereafter, the values derived numerically for the sample loading of $\sigma_0 = 290\,\mathrm{MPa}$ show a relatively good compatibility with the theoretical results obtained.

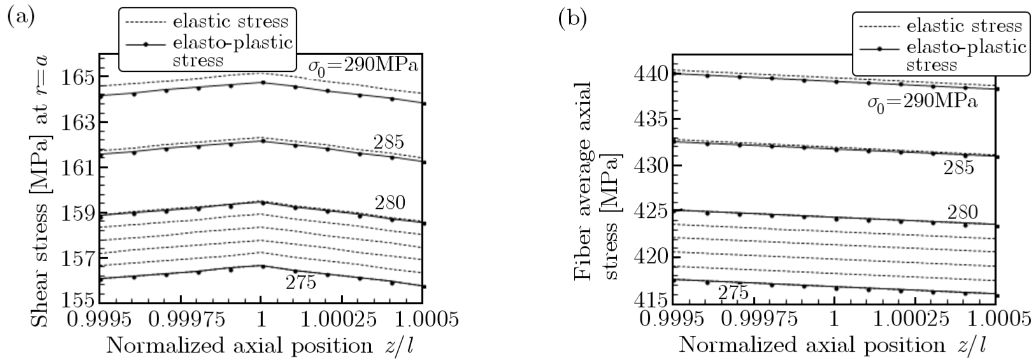

Fig. 6. Shear stress at $r = a$ (a) and fiber average axial stress (b), different loadings σ_0

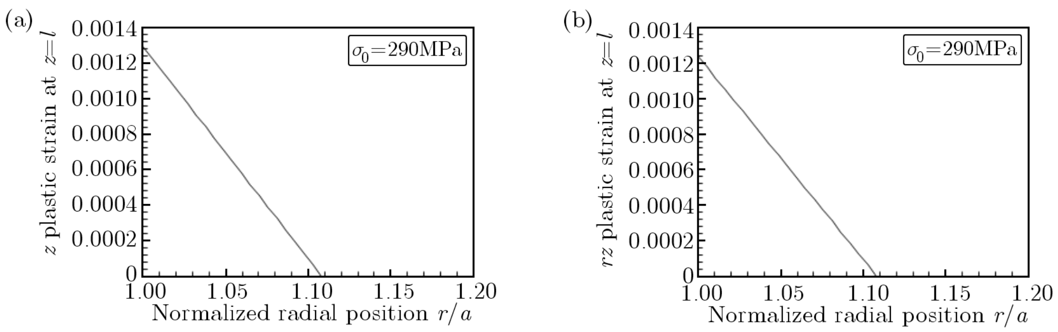

Fig. 7. Numerical results – $\varepsilon_z^p(r,l)$ at: (a) $z = l$ and (b) $z = l$

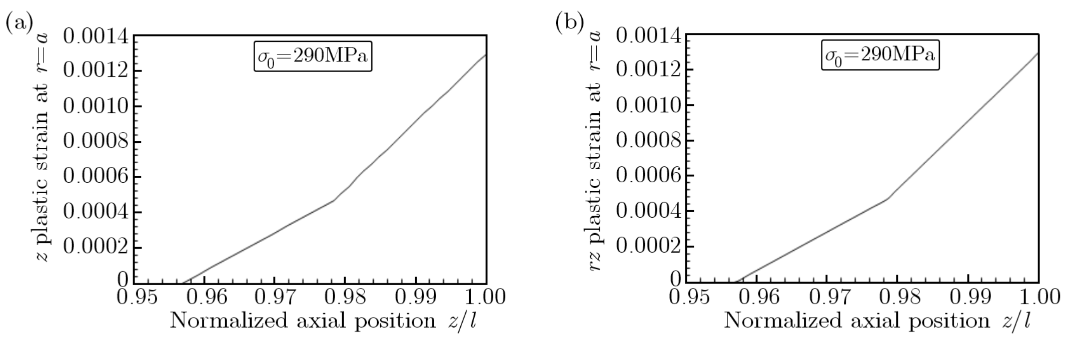

Fig. 8. Numerical results – $\varepsilon_z^p(a,z)$ at: (a) $r = a$ and (b) $r = a$

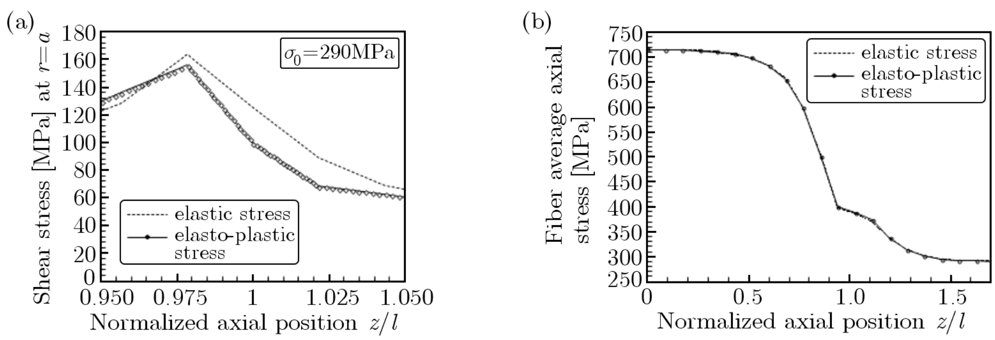

Fig. 9. Numerical results – shear stress at $r = a$ (a) and fiber average axial stress (b) for $\sigma_0 = 290\,\mathrm{MPa}$

4. Conclusion and remarks

In the present study, an analytical shear lag based model was proposed to study the effects of plastic deformations in the matrix on overall stress transfer behavior of a fiber reinforced metal matrix composite. For this reason, a cylindrical unit cell consisting of a fiber and the surrounding metal matrix was considered under tensile loading. Writing the shear lag based relations for this model, the stress terms for both the matrix and fiber were calculated considering the occurrence of plasticity in the matrix due to the ductility of the metal matrix compared to the brittle characteristics of the hard fiber. Unlike a relatively similar study performed by Jiang *et al.* (2004a) based on the shear lag approach, the present work has taken into account the stress distribution in the matrix and has derived the governing relations for that region. Also, the effect of the equivalent stress, i.e. both shear and axial stresses, have been considered rather than the mere axial stress applied in that work. Moreover, both of the plastic strain terms, i.e. axial and shear strain components have been taken into account also via the equivalent plastic strain term in the plasticity relations. Furthermore, the performed study is able to predict the plastic region boundaries in the matrix for a given loading and is capable of proposing an estimate for the growth of the yielded region with an increase in the loading.

To summarize, having obtained the shear lag based relations for the problem, the plastic strain distribution in the matrix has been derived for a given loading provided that the yield area is small compared to the fiber diameter. Thereafter, the effects of these plastic strains on the interface shear and the fiber average axial stress have been obtained. It was also verified that the occurrence of local plasticity in the matrix has an adverse effect over the stress transfer efficiency of the composite via reduction in these two critical load transfer mechanisms compared to the stress distribution in the absence of such deformations. Moreover, it was shown that as the local plastic deformation occurs in the matrix, the increase in stresses happens more slowly compared to the elastic case, which is another evidence for the efficiency loss in the presence of the matrix plasticity.

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