Probability Measures and Logical Connectives on Quantum Logics

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Abstract:
The present paper is devoted to modelling of a probability measure of logical connectives on a quantum logic via a G-map, which is a special map on it. We follow the work in which the probability of logical conjunction (AND), disjunction (OR), symmetric difference (XOR) and their negations for non-compatible propositions are studied. Now we study all remaining cases of G-maps on quantum logic, namely a probability measure of projections, of implications, and of their negations. We show that unlike classical (Boolean) logic, probability measures of projections on a quantum logic are not necessarilly pure projections. We indicate how it is possible to define a probability measure of implication using a G-map in the quantum logic, and then we study some properties of this measure which are different from a measure of implication in a Boolean algebra. Finally, we compare the properties of a G-map with the properties of a probability measure related to logical connectives on a Boolean algebra.

Keywords: *logical connectives, orthomodular lattice, quantum logic, probability measure, state*

1. Introduction

The problem of modelling of probability measures for logical connectives of non-compatible propositions started by publishing the paper Birkhoff, G., von Neumann, J. [2]. Quantum logic allows to model situations with non-compatible events (events that are not simultaneously measurable). Methods of quantum logic appear in data processing, economic models, and in other domains of application e.g. [2, 28, 9, 19, 27].

Calculus for non-compatible observables has been described in [16], while modelling of logical connectives in terms of their algebraic properties and algebraic structures can be found in [7, 8, 21].

The present paper follows up the work [13], where the authors studied logical connectives: conjuction, disjunction, and symmetric difference together with their negations, from the perspective of a probability measure. An overview of various insights into this issue is provided in [25].

The paper is organized as follows. Section 2 reminds some basic notions and their properties. A special function that associates a probability measure to some logical connectives on a quantum logic is defined and studied in Section 3 – Section 5. In the last Section 6 properties of a G-map are compared with properties of a probability measure related to logical connectives on a Boolean algebra.

2. Basic Definitions and Properties

In the first part of this section, we recall fundamental notions: orthomodular lattice, compatibility, orthogonality, state, and their basic properties. For more details, see [6, 24]. In the second subsection, we recall some situations with two-dimensional states allowing to model a probability measure of logical connectives in the case of non-compatible events [16], [15]- [11], [26].

2.1. Quantum logic

Definition 2.1 *An orthomodular lattice (OML) is a lattice L with 0_L and 1_L as the smallest and the greatest element, respectively, endowed with a unary operation $a \mapsto a'$ that satisfies:*
(i) $a'' := (a')' = a$;
(ii) $a \leq b$ implies $b' \leq a'$;
(iii) $a \vee a' = 1_L$;
(iv) $a \leq b$ implies $b = a \vee (a' \wedge b)$ (the orthomodular law).

Definition 2.2 *Elements a, b of an orthomodular lattice L are called*
– orthogonal if $a \leq b'$; (notation $a \perp b$);
– compatible if

$$a = (a \wedge b) \vee (a \wedge b');$$

(notation $a \leftrightarrow b$).

Definition 2.3 *A state on an OML L is a function $m : L \to [0, 1]$ such that*
(i) $m(1_L) = 1$;
(ii) $a \perp b$ implies

$$m(a \vee b) = m(a) + m(b).$$

Note that the notions *state* and *probability measure* are closely tied, and it is clear that $m(0_L) = 0$.

There exist three kinds of OMLs: without any state, with exactly one state and with infinite number of states (see e.g. [20]). The first and the second type of OLMs as a basic structure are not suitable to build a generalized probability theory. The last type of OMLs, which has infinite number of states is considered in the present paper.

Definition 2.4 *An OML L with infinite number of states is called a quantum logic (QL).*

When studying states on a quantum logic, one can meet some problems, that do not exist on a Boolean algebra. It means, that some of basic properties of probability measures are not necessarily satisfied for non-compatible random events. Here are some of them: Bell-type inequalities (e.g. [9,10,23,26]), Jauch-Piron state, (e.g. [4, 22]), problems of pseudometric (see [13]).

2.2. Probability Measures of Logical Connectives on QLs

In [14], the notion of *a map for simultaneous measurements (an s-map)* on a QL has been introduced. This function is a measure of conjunction even for non-compatible propositions, see [25].

A map $p : L \times L \to [0,1]$ is called a *map for simultaneous measurements* (abbr. *s-map*) if the following conditions hold:

(s1) $p(1_L, 1_L) = 1$;

(s2) if $a \perp b$ then $p(a,b) = 0$;

(s3) if $a \perp b$ then for any $c \in L$:

$$p(a \vee b, c) = p(a,c) + p(b,c),$$

$$p(c, a \vee b) = p(c,a) + p(c,b).$$

The following properties of s-map have been proved: Let $p : L \times L \to [0,1]$ be an s-map and $a,b,c \in L$. Then

1) if $a \leftrightarrow b$ then $p(a,b) = p(a \wedge b, a \wedge b) = p(b,a)$;
2) if $a \le b$ then $p(a,b) = p(a,a)$;
3) if $a \le b$ then
$$p(a,c) \le p(b,c)$$
$$p(c,a) \le p(c,b)$$
for any $c \in L$;
4) $p(a,b) \le \min\{p(a,a), p(b,b)\}$;
5) the map $m_p : L \to [0,1]$ defined as $m_p(a) = p(a,a)$ is a state on L, induced by p.

The property 1. shows that s-maps can be seen as providing probabilities of 'virtual' conjunctions of propositions, even non-compatible ones, for in the case of compatible propositions the value $p(a,b)$ coincides with the value that a state m_p generated by p takes on the meet $a \wedge b$, which in this case really represents conjunction of a and b [25].

On the other hand, the identity $p(a,b) = p(b,a)$ may not be true in general. So an s-map can be used for describing of stochastic causality [16–18]. Moreover, for any $a \in L$: $m_p(a) = p(a,a) = p(1_L, a) = p(a, 1_L)$.

Logical connectives disjunction (*j-map*) and symetric difference (*d-map*) are studied on a QL [13,5].

Let L be a QL. A map $q : L \times L \to [0,1]$ is called a *join map* (*j-map*) if the following conditions hold:

(j1) $q(0_L, 0_L) = 0, \quad q(1_L, 1_L) = 1$;

(j2) if $a \perp b$ then $q(a,b) = q(a,a) + q(b,b)$;

(j3) if $a \perp b$ then for any $c \in L$:

$$\begin{aligned} q(a \vee b, c) &= q(a,c) + q(b,c) - q(c,c) \\ q(c, a \vee b) &= q(c,a) + q(c,b) - q(c,c). \end{aligned}$$

If p is an s-map on a QL, m_p is a state induced by p and $q_p : L \times L \to [0,1]$ such that for any $a,b \in L$

$$q_p(a,b) = m_p(a) + m_p(b) - p(a,b),$$

then q_p is a j-map. It is easy to see that if $a \leftrightarrow b$, then

$$q_p(a,b) = m_p(a) + m_p(b) - m_p(a \wedge b) = m_p(a \vee b)$$

which explains its name.

Let L be a QL. A map $d : L \times L \to [0,1]$ is called a *difference map* (*d-map*), if the following conditions hold:

(d1)

$$\begin{aligned} d(1_L, 1_L) = d(0_L, 0_L) &= 0 \\ d(1_L, 0_L) = d(0_L, 1_L) &= 1. \end{aligned}$$

(d2) if $a \perp b$ then $d(a,b) = d(a, 0_L) + d(0_L, b)$;

(d3) if $a \perp b$ then for any $c \in L$:

$$\begin{aligned} d(a \vee b, c) &= d(a,c) + d(b,c) - d(0_L, c) \\ d(c, a \vee b) &= d(c,a) + d(c,b) - d(c, 0_L). \end{aligned}$$

If $a \leftrightarrow b$, then

$$d(a,b) = m_d(a \triangle b) = m_d(a \wedge b') + m_d(a' \wedge b),$$

where m_d is a state induced by d.

3. Special Bivariables Maps on QLs

3.1. Measures and Boolean Functions

Let $\mathcal{B}$ be a Boolean algebra and $f : \mathcal{B}^n \to \mathcal{B}$ be a Boolean function. It means, that f is such n-ary operation on $\mathcal{B}$, which is composed of binary operations $\vee$, $\wedge$, a unary operation complement $'$, and brackets $()$.

For the sake of simplification, the expressions of the type

$$(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n)$$

will be written as $(\overline{y}_1, a_i, \overline{y}_2)$

Proposition 3.1 *Let $\mathcal{B}$ be a Boolean algebra, $f : \mathcal{B}^n \to \mathcal{B}$ a Boolean function and $m : \mathcal{B} \to [0,1]$ be a probability measure on $\mathcal{B}$. Then the composition of functions $m \circ f : \mathcal{B}^n \to [0,1]$,*

$$(m \circ f)(x_1, \cdots, x_n) = m(f(x_1, \cdots, x_n))$$

satisfies following properties:

(G1) Let $x_1, \cdots, x_n \in \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}^n$. Then

$$m(f(x_1, \cdots, x_n)) \in \{0, 1\}.$$

(G2) Let $a_i, b_j \in \mathcal{B}, a_i \perp b_j$. Then

$$\begin{aligned} m(f(\overline{y}_1, a_i, \overline{y}_2, b_j, \overline{y}_3)) &= m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2, b_j, \overline{y}_3)) \\ &\quad + m(f(\overline{y}_1, a_i, \overline{y}_2, 0_{\mathcal{B}}, \overline{y}_3)) \\ &\quad - m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2, 0_{\mathcal{B}}, \overline{y}_3)). \end{aligned}$$

(G3) Let $a_i, b_i \in \mathcal{B}, a_i \perp b_i$. Then

$$\begin{aligned} m(f(\overline{y}_1, a_i \vee b_i, \overline{y}_2)) &= m(f(\overline{y}_1, a_i, \overline{y}_2)) \\ &+m(f(\overline{y}_1, b_i, \overline{y}_2)) \\ &-m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2)). \end{aligned}$$

Proof.

(G1) Let $f : \mathcal{B}^n \to \mathcal{B}$ be a Boolean function. Let $x_1, \cdots, x_n \in \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}^n$. Then

$$f(x_1, \cdots, x_n) \in \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}$$

and then

$$m(f(x_1, \cdots, x_n)) \in \{0, 1\}.$$

(G2) Let $f : \mathcal{B}^n \to \mathcal{B}$ be a Boolean function. Then for any $a, b \in \{x_1, \ldots, x_n\}$

$$f(\overline{y}_1, a, \overline{y}_2, b, \overline{y}_3) = f(x_1, \cdots, x_n) \wedge U, \quad (1)$$

where $U = (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge b))$. This can be rewritten as

$$\begin{aligned} f(\overline{y}_1, a, \overline{y}_2, b, \overline{y}_3) &= (a \wedge b' \wedge Q_1) \vee (a' \wedge b \wedge Q_2) \vee \\ &\vee (a' \wedge b' \wedge Q_3) \vee (a \wedge b \wedge Q_4), \end{aligned}$$

where $Q_i, i = 1, 2, 3, 4$, are boolean expressions that do not contain any of the elements a, a', b, b'. Assume that $a \perp b$. Then

$$f(\overline{y}_1, a, \overline{y}_2, b, \overline{y}_3) = (a \wedge Q_1) \vee (b \wedge Q_2) \vee (a' \wedge b' \wedge Q_3).$$

If we put $m(f(\overline{y}_1, a, \overline{y}_2, b, \overline{y}_3)) = \mu$, then

$$\mu = m(a \wedge Q_1) + m(b \wedge Q_2) + m(a' \wedge b' \wedge Q_3). \quad (2)$$

Since m is a probability measure, it follows that

$$\begin{aligned} \mu &= m(a \wedge Q_1) + m(b \wedge Q_2) + m(Q_3) \\ &-m((a \vee b) \wedge Q_3) \\ &= m(a \wedge Q_1) + m(b \wedge Q_2) + m(Q_3) \\ &-m(a \wedge Q_3) - m(b \wedge Q_3) \\ &= m(a \wedge Q_1) + m(a' \wedge Q_3) + m(b \wedge Q_2) \\ &+m(b' \wedge Q_3) - m(Q_3). \end{aligned}$$

On the other side, from (2) we obtain

$$\begin{aligned} m(f(\overline{y}_1, a, \overline{y}_2, 0_{\mathcal{B}}, \overline{y}_3)) &= m(a \wedge Q_1) + m(a' \wedge Q_3), \\ m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2, b, \overline{y}_3)) &= m(b \wedge Q_2) + m(b' \wedge Q_3), \\ m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2, 0_{\mathcal{B}}, \overline{y}_3)) &= m(Q_3). \end{aligned}$$

Thus (G2) is satisfied.

(G3) Similarly, any Boolean function $f : B^n \to B$ can be written as

$$f(x_1, \ldots, x_n) = (x_i \wedge Q) \vee (x_i' \wedge P),$$

where the Boolean expressions Q, P do not contain x_i, x_i'. Thus

$$m(f(x_1, \ldots, x_n)) = m(x_i \wedge Q) + m(x_i' \wedge P). \quad (3)$$

Consider $a, b \in \mathcal{B}, a \perp b$, and put $x_i = a \vee b$. Then

$$\begin{aligned} & m(f(\overline{y}_1, a \vee b, \overline{y}_2)) \\ &= m((a \vee b) \wedge Q) + m((a \vee b)' \wedge P) \\ &= m(a \wedge Q) + m(b \wedge Q) + m(P) \\ &\quad -m(a \wedge P) - m(b \wedge P) \\ &= m(a \wedge Q) + m(a' \wedge P) + m(b \wedge Q) \\ &\quad +m(b' \wedge P) - m(P). \end{aligned}$$

On the other side, from (3) we obtain

$$\begin{aligned} m(f(\overline{y}_1, a, \overline{y}_2)) &= m(a \wedge Q) + m(a' \wedge P) \\ m(f(\overline{y}_1, b, \overline{y}_2)) &= m(b \wedge Q) + m(b' \wedge P) \\ m(f(\overline{y}_1, 0_{\mathcal{B}}, \overline{y}_2)) &= m(P). \end{aligned}$$

Thus (G3) is satisfied. (Q.E.D.)

It follows from the previous proposition that each probability measure of any boolean function has the properties (G1) – (G3). Then it should be interesing to study a function $G : \mathcal{B}^n \to [0, 1]$ which is endowed with properties (G1) – (G3). It is easy to see, that for $n = 1$ a function G is a classical measure ($G(1_{\mathcal{B}}) = 1$ and $G(0_{\mathcal{B}}) = 0$) or a negative measure ($G(1_{\mathcal{B}}) = 0$ and $G(0_{\mathcal{B}}) = 1$) on $\mathcal{B}$.

This article is devoted to functions G on a QL for $n = 2$.

3.2. Bivariable G-Maps on QLs

A special bivariable map G satisfying

$$G(0_L, 1_L) = G(1_L, 0_L)$$

has been introduced in [13]. The following definition brings an extended version of this G-map.

Definition 3.2 *Let L be a QL. A map*

$$G : L \times L \to [0, 1]$$

is called a G-map if the following holds:

(G1) if $a, b \in \{0_L, 1_L\}$ then $G(a, b) \in \{0, 1\}$;

(G2) if $a \perp b$ then

$$G(a, b) = G(a, 0_L) + G(0_L, b) - G(0_L, 0_L);$$

(G3) if $a \perp b$ then for any $c \in L$:

$$\begin{aligned} G(a \vee b, c) &= G(a, c) + G(b, c) - G(0_L, c) \\ G(c, a \vee b) &= G(c, a) + G(c, a) - G(c, 0_L). \end{aligned}$$

A G-map enables modelling of probability of logical connectives even for non-compatible propositions.

Lemma 3.3 *Let $G : L \times L \to [0, 1]$ be a G-map, where L is a QL. Then for $a \leftrightarrow b$ it holds*

$$\begin{aligned} G(a, b) &= G(a \wedge b, a \wedge b) + G(a \wedge b', 0_L) \\ &+G(0_L, a' \wedge b) - 2G(0_L, 0_L). \end{aligned}$$

Proof. See in [12].

Proposition 3.4 *Let $G : L \times L \to [0,1]$ be a G-map, where L is a QL. Then the map $G' = 1 - G$ is a G-map.*

Proof. See in [12].

There are sixteen families Γ_i, ($i = 1, ..., 16$) of maps G according to values in vertices

$$(1_L, 1_L),\ (1_L, 0_L),\ (0_L, 1_L),\ (0_L, 0_L).$$

Eight of them with $G(1_L, 0_L) = G(0_L, 1_L)$ are studied in [13]. More details can be found in Table 5, section 6.

Family Γ_2 is the set of all s-maps (measures of conjuntion), Γ_3 the set of all j-maps (measures of disjunction), and Γ_4 is that of all d-maps (measures of symmetric difference) on a QL (see [13] for more details).

In the present paper, the remaining cases Γ_i ($i = 9, ..., 16$) with

$$G(1_L, 0_L) \neq G(0_L, 1_L)$$

are focused on.

4. Probability Measures of Projections on QLs

This part is devoted to $\Gamma_9 - \Gamma_{12}$ with values in the vertices shown in the Table 1. As $G \in \Gamma_{11}$ iff $1 - G \in \Gamma_9$, and $G \in \Gamma_{12}$ iff $1 - G \in \Gamma_{10}$ (Proposition 3.4 and Table 1), and moreover, Γ_9 and Γ_{10} are analogical cases (Γ_{11} and Γ_{12} as well), only Γ_9 is studied in detail.

Lemma 4.1 *Let L be a QL and $G \in \Gamma_9$. Then for any $a, b \in L$ it holds*
1) $G(1_L, a) = 1$, $G(0_L, a) = 0$;
2) $G(a, 0_L) = G(a, a) = G(a, 1_L)$;
3) $G(a, 0_L) = \frac{1}{2}(G(a, b) + G(a, b'))$;
4)

$$G(a, 0_L) = \frac{1}{n} \sum_{i=1}^{n} G(a, b_i),$$

where $b_1, \cdots, b_n$ is an orthogonal partition of unity 1_L.

Proof. See in [12].

Proposition 4.2 *Let L be a QL, and $G \in \Gamma_9$. Then for any $a, b \in L$ it holds*
1) If $a \leftrightarrow b$ then $G(a, b) = G(a, 0_L)$.
2) For any choice of b, the map $m_b : L \to [0,1]$:

$$m_b(a) = G(a, b)$$

is a state on L.

Proof. See in [12].

From Proposition 4.2 it follows that any $G \in \Gamma_9$ is a probability measure of the projection onto the first coordinate. Analogical properties are fullfiled for any $G \in \Gamma_{10}$, which is a probability measure of the projection onto the second coordinate.

If L is a Boolean algebra, then for any $G \in \Gamma_9$ it holds $G(a, b) = G(a, 0_L)$ for all $a, b \in L$. Analogously for any $G \in \Gamma_{10}$ it holds $G(a, b) = G(0_L, b)$ for all $a, b \in L$.

If L is a QL but not a Boolean algebra, then the identity does not hold in general, as illustrates the following example.

Example 4.3 *Consider $L = \{0_L, 1_L, a, a', b, b'\}$, a horizontal sum of Boolean algebras*

$$\mathcal{B}_a = \{0_L, 1_L, a, a'\},$$

$$\mathcal{B}_b = \{0_L, 1_L, b, b'\}.$$

Consider $r_1, r_2, u_1, u_2 \in [0,1]$. Every $G \in \Gamma_9$ can be fully defined by Table 2, where

$$\alpha = \frac{1}{2}(r_1 + r_2),$$

$$\beta = \frac{1}{2}(u_1 + u_2)$$

according to Lemma 4.1. If $r_1 \neq r_2$ then

$$G(a, b) \neq G(a, 0_L).$$

From Table 2, one can extract all states on L, related to the choice of r_1, r_2, u_1, u_2. Each column in the Table 2 represents a state on L. As example, m_b and m_0 are in Table 3.

Definition 4.4 *Let $G \in \Gamma_9$. The map G is called a measure of pure projection (a pure projection) if*

$$G(a, b) = G(a, 0_L)$$

for any $a, b \in L$.

On a Boolean algebra, the projection onto the first coordinate may be expressed by a Boolean function

$$f(a, b) = (a \wedge b) \vee (a \wedge b') = (a \wedge b) \vee (b' \wedge a) = a,$$

what motivates us to define on a QL L four G-maps with the use of $p \in \Gamma_2$:

$$\begin{aligned} G_1(a, b) &= p(a, b) + p(a, b'), \\ G_2(a, b) &= p(b, a) + p(b', a), \\ G_3(a, b) &= p(a, b) + p(b', a), \\ G_4(a, b) &= p(b, a) + p(a, b'). \end{aligned}$$

Maps G_i are measures of projection onto the first coordinate, i.e. $G_i \in \Gamma_9$ what we prove below. If p is a commutative s-map, all G_i coincide,

$$G_i(a, b) = p(a, a)$$

what is a pure projection. If p is a non-commutative s-map, then

$$G_1(a, b) = G_2(a, b) = p(a, a)$$

is a pure projection, while G_3 and G_4 are not pure projections since:

$$G_3(a, b) = p(a, b) + p(a, a) - p(b, a),$$

Tab. 1. Γ_9 - Γ_{16} values in vertices

	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}	Γ_{13}	Γ_{14}	Γ_{15}	Γ_{16}
$G(0_L, 0_L)$	0	0	1	1	1	1	0	0
$G(0_L, 1_L)$	0	1	1	0	1	0	0	1
$G(1_L, 0_L)$	1	0	0	1	0	1	1	0
$G(1_L, 1_L)$	1	1	0	0	1	1	0	0

Tab. 2. G-maps from Γ_9 on a hor izontal sum of Boolean algebras

	a	a'	b	b'	0_L	1_L
a	α	α	r_1	r_2	α	α
a'	$1-\alpha$	$1-\alpha$	$1-r_1$	$1-r_2$	$1-\alpha$	$1-\alpha$
b	u_1	u_2	β	β	β	β
b'	$1-u_1$	$1-u_2$	$1-\beta$	$1-\beta$	$1-\beta$	$1-\beta$
0_L	0	0	0	0	0	0
1_L	1	1	1	1	1	1

Tab. 3. States on L

	a	a'	b	b'	0_L	1_L
m_b	r_1	$1-r_1$	β	$1-\beta$	0	1
m_0	α	$1-\alpha$	β	$1-\beta$	0	1

and

$$G_3(a, 0_L) = p(a, a),$$

and if $p(a, b) \neq p(b, a)$ then $G_3(a, b) \neq G_3(a, 0_L)$. Now we prove that G_3 is a projection (case G_4 is analogical).

(1) $G_3(a, b) \in [0, 1]$

$$0 \leq G_3(a, b) = p(a, b) + p(b', a) \leq p(b, b) + p(b', b') = 1.$$

(2) Values in vertices:

$$G_3(0_L, 0_L) = G_3(0_L, 1_L) = 0,$$
$$G_3(1_L, 0_L) = G_3(1_L, 1_L) = 1.$$

(3) If $a \perp b$, i.e. $a \leq b'$ then

$$G_3(a, b) = p(a, b) + p(b', a) = 0 + p(a, a).$$

From the other side

$$G_3(a, 0_L) + G_3(0_L, b) - G_3(0_L, 0_L)$$
$$= p(a, 0_L) + p(1_L, a) + p(0_L, b) + p(b', 0_L) - 0$$
$$= p(a, a).$$

(4) If $a \perp b$ and $c \in L$ then

$$G_3(a \vee b, c) = p(a \vee b, c) + p(c', a \vee b) = p(a, c) + p(b, c) + p(c', a) + p(c', b).$$

From the other side

$$G_3(a, c) + G_3(b, c) - G_3(0_L, c)$$
$$= p(a, c) + p(c', a) + p(b, c) + p(c', b) + p(0_L, c) + p(c', 0_L).$$

The second identity:

$$G_3(c, a \vee b)$$
$$= p(c, a \vee b) + p((a \vee b)', c)$$
$$= p(c, a) + p(c, b) + p(1_L, c) - p(a \vee b, c)$$
$$= p(c, a) + p(c, b) + p(1_L, c) - p(a, c) - p(b, c)$$
$$= p(c, a) + p(a', c) + p(c, b) + p(b', c) - p(1_L, c)$$
$$= G_3(c, a) + G_3(c, b) - G_3(c, 0_L).$$

Proposition 4.5 *For every s-map p there exists a G–map $G_p \in \Gamma_9$ such that*

$$G_p(a, b) = G_p(a, 0_L).$$

Proof. Let

$$G_p(a, b) = p(a, b) + p(a, b') = p(a, a),$$

where p is an arbitrary s-map. Then $G_p \in \Gamma_9$ and

$$G_p(a, b) = G_p(a, 0_L)$$

for any $b \in L$. (Q.E.D.)

The results for $\Gamma_9 - \Gamma_{12}$ are summarized in Table 4.

Tab. 4. Results for $\Gamma_9 - \Gamma_{12}$

	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}
probability of	a	b	a'	b'

5. Probability Measures of Implications on QLs

Values in vertices for families $\Gamma_{13} - \Gamma_{16}$ are in the Table 1. Similarly to the relations between Γ_9 - Γ_{12}, for families $\Gamma_{13} - \Gamma_{16}$ hold

$$G \in \Gamma_{13} \text{ iff } 1 - G \in \Gamma_{15},$$

$$G \in \Gamma_{14} \text{ iff } 1 - G \in \Gamma_{16}.$$

Γ_{15} and Γ_{16} are analogical cases. For these reasons only one of the famillies, Γ_{15}, will be focused on.

Lemma 5.1 *Let L be a QL and $G \in \Gamma_{15}$. Then for any $a, b \in L$ it holds*
1) $G(a, a) = G(a, 1_L) = G(0_L, a) = 0$;
2) $G(1_L, a) = 1 - G(a, 0_L) = G(a', 0_L)$;
3) If $a \leftrightarrow b$ then $G(a, b) = G(a \wedge b', 0_L)$.
4) If $a \le b$ then $G(a, b) = 0$.

Proof.
1) Let $G \in \Gamma_{15}$ and $a \in L$, then

$$\begin{aligned} 0 &= G(1_L, 1_L) \\ &= G(a, 1_L) + G(a', 1_L) - G(0_L, 1_L) \\ &= G(a, 1_L) + G(a', 1_L). \end{aligned}$$

Taking into account that $G(a, b) \in [0, 1]$, one concludes that $G(a, 1_L) = 0$ for any $a \in L$. Further

$$\begin{aligned} 0 &= G(a, 1_L) = G(a, a) + G(a, a') - G(a, 0_L) \\ &= G(a, a) + G(a, 0_L) + G(0_L, a') - G(0_L, 0_L,) \\ &\quad -G(a, 0_L) \\ &= G(a, a) + G(0_L, a'). \end{aligned}$$

Thus $G(a, a) = G(0_L, a) = 0$.

2) Let $G \in \Gamma_{15}$ and $a \in L$, then with the use of what preceeds,

$$\begin{aligned} G(1_L, a) &= G(a, a) + G(a', a) - G(0_L, a) \\ &= G(a', 0_L) + G(0_L, a) - G(0_L, 0_L) \\ &= G(a', 0_L). \end{aligned}$$

From the other side,

$$1 = G(1_L, 0_L) = G(a, 0_L) + G(a', 0_L).$$

Consequently,

$$G(1_L, a) = 1 - G(a, 0_L) = G(a', 0_L).$$

3) If $a \leftrightarrow b$ then $G(a, b) = G(a \wedge b', 0_L)$ follows directly from Lemma 3.3.

4) $a \le b$ is a particular case of $a \leftrightarrow b$, where $a \wedge b' = 0_L$. This leads immediatelly to

$$G(a, b) = G(a \wedge b', 0_L) = G(0_L, 0_L) = 0.$$

(Q.E.D.)

Lemma 5.2 *Let L be a QL and $G \in \Gamma_{15}$. Then the map $m_G : L \to [0, 1]$ defined as $m_G(a) = G(a, 0_L)$ is a state on L.*

Proof.
1) $m_G(1_L) = G(1_L, 0_L) = 1$

2) If $a \perp b$, then

$$\begin{aligned} m_G(a \vee b) &= G(a \vee b, 0_L) \\ &= G(a, 0_L) + G(b, 0_L) - G(0_L, 0_L) \\ &= m_G(a) + m_G(b). \end{aligned}$$

(Q.E.D.)

Proposition 5.3 *Let L be a QL. The famillies Γ_2 and Γ_{15} are isomorfic.*

Proof. Since Γ_2 is the set of all s-maps on L, it suffices to prove:
i) If $G \in \Gamma_{15}$ and $p_G(a, b) = G(a, b')$, then p_G is an s-map on L.
ii) If p is an s-map on L and $G_p(a, b) = p(a, b')$, then $G_p \in \Gamma_{15}$.

i) Let $G \in \Gamma_{15}$ and $p_G(a, b) = G(a, b')$. The properties (s1) – (s3) of s-map are verified bellow.
(s1) $p_G(1_L, 1_L) = G(1_L, 0_L) = 1$
(s2) If $a \perp b$, then $p_G(a, b) = G(a, b') = 0$. It implies from Lemma 5.1 as $a \le b'$.
(s3) If $a \perp b$ and $c \in L$, then

$$\begin{aligned} p_G(a \vee b, c) &= G(a \vee b, c') \\ &= G(a, c') + G(b, c') - G(0_L, c') \\ &= p_G(a, c) + p_G(b, c). \end{aligned}$$

The second identity:

$$p_G(c, a \vee b) = G(c, (a \vee b)') = G(c, a' \wedge b')$$
$$p_G(c, a) + p_G(c, b) = G(c, a') + G(c, b').$$

It suffices to show that $G(c, a') + G(c, b') = G(c, a' \wedge b')$. From the orthomodular law it follows that $a' = b \vee (b' \wedge a')$ and $b' = a \vee (a' \wedge b')$.

$$\begin{aligned} & G(c, a') + G(c, b') \\ =\ & G(c, b) + G(c, a' \wedge b') - G(c, 0_L) \\ & +G(c, a' \wedge b') + G(c, a) - G(c, 0_L) \\ =\ & (G(c, b) + G(c, a) - G(c, 0_L)) \\ & +G(c, a' \wedge b') - G(c, 0_L) \\ & +G(c, a' \wedge b') \\ =\ & G(c, a \vee b) + G(c, (a \vee b)') \\ & -G(c, 0_L) + G(c, a' \wedge b') \\ =\ & G(c, 1_L) + G(c, a' \wedge b') \\ =\ & G(c, a' \wedge b'). \end{aligned}$$

Consequently

$$p_G(c, a \vee b) = p_G(c, a) + p_G(c, b).$$

ii) Let p be an s-map and $G_p(a, b) = p(a, b')$. We want to prove $G \in \Gamma_{15}$.
- It is clear that the values of G_p in vertices match the maps of Γ_{15}.
- Let $a \perp b$. Then $G_p(a, b) = p(a, b') = p(a, a)$ as $a \le b'$. On the other hand

$$\begin{aligned} & G_p(a, 0_L) + G_p(0_L, b) - G_p(0_L, 0_L) \\ =\ & p(a, 1_L) + p(0_L, b') - p(0_L, 1_L) \\ =\ & p(a, a) = G_p(a, b). \end{aligned}$$

- Let $a, b, c \in L$ and $a \perp b$. Then

$$\begin{aligned} G_p(a \vee b, c) &= p(a \vee b, c') \\ &= p(a, c') + p(b, c') \\ &= G_p(a, c) + G_p(b, c) - G_p(0_L, c). \end{aligned}$$

The second identity:

$$\begin{aligned} & G_p(c, a \vee b) = p(c, a' \wedge b') \\ & G_p(c, a) + G_p(c, b) - G_p(c, 0_L) \\ = \; & p(c, a') + p(c, b') - p(c, 1_L). \end{aligned}$$

It suffices to show that

$$p(c, a' \wedge b') = p(c, a') + p(c, b') - p(c, 1_L).$$

Since

$$\begin{aligned} & p(c, a \vee b) = p(c, a) + p(c, b) \\ & p(c, a \vee b) = p(c, 1_L) - p(c, a' \wedge b') \\ & p(c, 1_L) - p(c, a' \wedge b') \\ & = p(c, 1_L) - p(c, a') + p(c, 1_L) - p(c, b') \end{aligned}$$

thus

$$p(c, a' \wedge b') = p(c, a') + p(c, b') - p(c, 1_L).$$

(Q.E.D.)

In a classical Boolean logic it holds (principle of a proof by contraposition)

$$a \Rightarrow b \quad \Leftrightarrow \quad b' \Rightarrow a'.$$

On a Boolean algebra is any measure of both the left and the right hand side the same. Quantum logics and some measures of implication $G \in \Gamma_{13}$ (induced by a non-commutative s-map) enable to model a situation where these measures are not equal. First look at basic properties of the class of implications, Γ_{13}.

Lemma 5.4 *Let L be a QL and $G \in \Gamma_{13}$. Then for any $a, b \in L$ it holds*
1) $G(a, a) = G(a, 1_L) = G(0_L, a) = 1$;
2) $G(1_L, a) = 1 - G(a, 0_L) = G(a', 0_L)$;
3) If $a \leftrightarrow b$ then $G(a, b) = G(a' \vee b, 0_L)$;
4) If $a \leq b$ then $G(a, b) = 1$.

Proposition 5.5 *Let L be a QL and $G \in \Gamma_{13}$. Then the map $m_G : L \to [0,1]$ defined as $m_G(a) = G(1_L, a)$ is a state on L.*

Proposition 5.6 *Let L be a QL. The families Γ_2 and Γ_{13} are isomorfic.*

Proof. The statement follows immediately from:
i) $p \in \Gamma_2$ iff $G_p \in \Gamma_{15}$, where $G_p(a, b) = p(a, b')$.
ii) $G \in \Gamma_{15}$ iff $1 - G \in \Gamma_{13}$.
From the above it is clear that $p \in \Gamma_2$ iff $G_p \in \Gamma_{13}$, where

$$G_p(a, b) = 1 - p(a, b')$$

The measure of implication G_p is called a measure induced by s-map p. (Q.E.D.)

Let us return to the tautology

$$a \Rightarrow b \text{ iff } b' \Rightarrow a'.$$

We would expect an equal measure of propositions

$$a \Rightarrow b \;\; \& \;\; b' \Rightarrow a',$$

or equivalently: for any $G \in \Gamma_{13}$ it holds $G(a, b) = G(b', a')$. As already noted, this is true on a Boolean algebra, but not necessarily on a quantum logic. Indeed, if a measure of implication G_p is induced by a non-commutative s-map p, and the events a, b are not compatible, one can obtain

$$G(a, b) = 1 - p(a, b')$$

different of

$$G(b', a') = 1 - p(b', a).$$

Note that, if a measure of implication is induced by a commutative s-map p, we have a classical situation.

6. Conclusion

An overview of all classes is in Table 5 and in Table 6. It is clear from these tables that on a Boolean algebra, a value of a G-map is a probability measure of a Boolean expression, according to the known table for the propositional logic. This leads to the interpretation of values of a function G on a quantum logic.

6.1. Relations between Classes Γ_1 - Γ_{16}.

On a Boolean algebra classes Γ_i and Γ_j are isomorphic for $i, j \neq 1, 8$. Another situation occurs in the case of non-compatible random events, that is, in the case of a quantum logic:

- Γ_4 and Γ_7 are isomorphic.
- Γ_i and Γ_j are isomorphic for

$$i, j \in \{2, 3, 5, 6, 13 - 16\}.$$

- In [13] it is shown that for any $p \in \Gamma_2$ there exists a $G_p \in \Gamma_4$ induced by p. On the other side, there exists $G \in \Gamma_4$ such that the map p_G induced by G is not in Γ_2 (p_G is not an s-map).
- Γ_9 - Γ_{12} are mutually isomorphic, but their relation to other classes is not quite clear. Nevertheless, for any s-map there exists a projection, as it follows from Proposition 4.5.

6.2. Problem of Existence of G-maps on QLs.

Two principal questions related to G-maps arise in a quantum logic: existence of such map and its properties.

From the foregoing considerations it follows that the existence of a probability measure of conjunction (s-map) guarantees the existence of a probability measure of all other logical connectives. Therefore, the key question, listed as an open problem Q3 in [25], is the existence of an s-map on any quantum logic.

The existence of an s-map in the case of a separable quantum logic and additive states has been solved in [15] and [14].

Tab. 5. Results from the paper [13]

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$G(0_L, 0_L)$	0	0	0	0	1	1	1	1
$G(1_L, 0_L)$	0	0	1	1	1	0	0	1
$G(0_L, 1_L)$	0	0	1	1	1	0	0	1
$G(1_L, 1_L)$	0	1	1	0	0	0	1	1
probability of	0_L	$a \wedge b$ $a \leftrightarrow b$	$a \vee b$ $a \leftrightarrow b$	$(a \Leftrightarrow b)'$ $a \leftrightarrow b$	$a' \vee b'$ $a \leftrightarrow b$	$a' \wedge b'$ $a \leftrightarrow b$	$a \Leftrightarrow b$ $a \leftrightarrow b$	1_L

Tab. 6. Γ_9 - Γ_{16}, $G(1_L, 0_L) \neq G(0_L, 1_L)$. For $a \leftrightarrow b$: $a \Rightarrow b = a' \vee b$.

	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}	Γ_{13}	Γ_{14}	Γ_{15}	Γ_{16}
$G(0_L, 0_L)$	0	0	1	1	1	1	0	0
$G(0_L, 1_L)$	0	1	1	0	1	0	0	1
$G(1_L, 0_L)$	1	0	0	1	0	1	1	0
$G(1_L, 1_L)$	1	1	0	0	1	1	0	0
probability of	a	b	a'	b'	$a \Rightarrow b$ $a \leftrightarrow b$	$a \Leftarrow b$ $a \leftrightarrow b$	$(a \Rightarrow b)'$ $a \leftrightarrow b$	$(a \Leftarrow b)'$ $a \leftrightarrow b$

Proposition 6.1 *([15], Proposition 1.1.) Let L be an OML, let $\{a_i\}_{i=1}^n \in L$, $n \in N$ where $a_i \perp a_j$, for $i \neq j$. If for any i there exists a state α_i, such that $\alpha_i(a_i) = 1$, then there exists σ-CS such that for any $k = (k_1, \cdots, k_n)$, where $k_i \in [0,1]$ for $i \in \{1, \cdots, n\}$ with the property $\sum_i k_i = 1$, there exists a conditional state $f_k : L \times L_c \to [0,1]$, such that for any i and each $d \in L$*

$$f_k(d, a_i) = \alpha_i(d);$$

and for each a_j

$$f_k(a_j, \vee_i a_i) = k_i.$$

Proposition 6.2 *([14] Proposition 2.2.) Let L be an OML, let there be an s-map p. Then there exists a conditional state f_p such that*

$$p(a,b) = f_p(a,b) f_p(b, 1_L).$$

Let L be a QL and let $L_c = L - \{0_L\}$. If

$$f : L \times L_c \to [0,1]$$

is a conditional state, then there exists an s-map

$$p_f : L \times L \to [0,1].$$

s-maps, whose existence is guaranteed by the above cited propositions, can be constructed using techniques similar to those known for the construction of copulas. ([1,3]).

6.3. Some Differences Between G-maps on a Boolean algebra and G-maps on a QL.

1) Each probability measure on $\mathcal{B}$ induces a pseudometric. It means, that for any probability measure m, the map d_m: $d_m(a,b) = m(a \wedge b') + m(a' \wedge b)$ is a pseudometric on $\mathcal{B}$ induced by m. On a quantum logic, if $p \in \Gamma_2$ and $d_p(a,b) = p(a,b') + p(a',b)$, then $d_p \in \Gamma_4$ but it can happen that d_p is not a pseudometric.

2) Let L be a QL, m be a state on L and p be an s-map on L. The first Bell-type inequality (4) is not necessarily fulfilled for all values $a, b \in L$ while its version (5), via an s-map p is always satisfied.

$$m(a) + m(b) - m(a \wedge b) \leq 1 \qquad (4)$$

$$p(a,a) + p(b,b) - p(a,b) \leq 1 \qquad (5)$$

The second Bell-type innequality (6) is not necessarily fulfilled for all values $a, b, c \in L$ while its version (7) is fulfilled for every s-map, which induces a pseudometric on L [26].

$$m(a)+m(b)+m(c)-m(a \wedge b)-m(a \wedge c)-m(c \wedge b) \leq 1 \qquad (6)$$

$$p(a,a)+p(b,b)+p(c,c)-p(a,b)-p(a,c)-p(c,b) \leq 1 \qquad (7)$$

3) Analogically, implication (8) (Jauch-Piron state, see e.g. [4,22]) can be violated on L but implication (9) is always valid

$$m(a) = m(b) = 1 \Rightarrow m(a \wedge b) = 1 \qquad (8)$$

$$p(a,a) = p(b,b) = 1 \Rightarrow p(a,b) = 1, \qquad (9)$$

and moreover for any $c \in L$

$$p(a,c) = p(c,a) = p(c,c).$$

4) On a Boolean algebra, every projection is a pure projection. On a quantum logic, a G-map G ($G \in \Gamma_i$, $i \in \{9,10,11,12\}$) is not necessarily a pure projection, see Example 4.3.

5) Quantum logics and G-maps enable to model situations that can not occur in a Boolean algebra. The use of G-maps to model these situations on QLs is illustrated by the following considerations:

   a) Quantum logics and non-commutative s-maps (class Γ_2) enable to model stochastic causality.

Tab. 7. d-map not satisfying triangle inequality if $k > 0$

	a	b	c	a'	b'	c'	0_L	1_L
a	0	k	0	1	$1-k$	1	α	$1-\alpha$
b	k	0	0	$1-k$	1	1	β	$1-\beta$
c	0	0	0	1	1	1	γ	$1-\gamma$
a'	1	$1-k$	1	0	k	0	$1-\alpha$	α
b'	$1-k$	1	1	k	0	0	$1-\beta$	β
c'	1	1	1	0	0	0	$1-\gamma$	γ
0	α	β	γ	$1-\alpha$	$1-\beta$	$1-\gamma$	0	1
1	$1-\alpha$	$1-\beta$	$1-\gamma$	α	β	γ	1	0

Let L be a quantum logic, p an s-map on L, and $a, b \in L$. The conditional probability of some event a, given the occurrence of some other event b is

$$P(a|b) = \frac{p(a,b)}{p(b,b)}.$$

Assume that p is a non-commutative s-map. Then there are non-compatible events a, b, for which $p(a,b) \neq p(b,a)$. This situation models a stochastic causality using a non-commutative measure of conjuction p. In this case Bayes's theorem is violated ([16, 17]).

Assume moreover that the event a is independent of b, i.e. it holds

$$P(a|b) = \frac{p(a,b)}{p(b,b)} = p(a,a).$$

On the other side, the event b is not independent of a, as

$$P(b|a) = \frac{p(b,a)}{p(a,a)} = \frac{p(b,a)p(b,b)}{p(a,b)} \neq p(b,b).$$

Using a commutative s-map, we have a classical situation. A commutative s-map p_s can be obtained from an arbitrary s-map p e.g. as

$$p_s(x,y) = \frac{1}{2}\left(p(x,y) + p(y,x)\right).$$

Whether an event a is independent of b or not is determined by the measure of conjunction. Therefore it is suitable to say that a is independent of b with respect to a measure (s-map p).

b) Quantum logics and some d-maps (class Γ_4) enable to distinguish elements that are not distinguishable on a Boolean algebra.

Symmetric difference (d-map) on a Boolean algebra fulfills the triangle inequality

$$d(a,b) \leq d(a,c) + d(c,b).$$

Consequently, if a, c and b, c are indistinguishable, then a, b are also, because

$$d(a,c) = d(c,b) = 0 \Rightarrow d(a,b) = 0.$$

On a quantum logic exists a set of symmetric differencies (subclass of Γ_4), that do not fulfill the triangle inequality. Table 7 gives an example of such symmetric difference under condition $k > 0$.

For elements a, b, c it holds:

$$d(a,c) = d(c,b) = 0,$$

but $d(a,b) = k > 0$.

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