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## DIFFERENTIAL EQUATION OF $y = f(y')$ AND INTERPRETATION OF THE SOLUTION IN MATHEMATICA PROGRAM

### Abstract

**Introduction and aims:** The paper presents a method of solving  $y=f(y')$  equations. The main aim of the work is to show how to solve this type of differential equations. In addition, the purpose of the discussion is to present the appropriate algorithms in *Mathematica* program, which are used to present the geometric interpretation of the obtained solutions.

**Material and methods:** The sources contain material on the subject of differential equations. The method of mathematical analysis has been used.

**Results:** In the analysis of selected examples, the method of substitution of new variable  $t$  has been used and the solution of the studied differential equation has been obtained in the form of the system of equations  $x=x(t)$  and  $y=y(t)$ .

**Conclusion** The solution of the differential equation of the type  $y=f(y')$  in the form of a system of equations  $x=x(t)$  and  $y=y(t)$  can be interpreted graphically using an appropriately used algorithm in *Mathematica* numerical program.

**Keywords:** Differential equations, equation of type  $y=f(y')$ , analytical solution, variable substitution method, geometric interpretation of the solution, *Mathematica* program.

(Received: 02.05.2019; Revised: 12.05.2019; Accepted: 22.05.2019)

## RÓWNANIE RÓŻNICZKOWE TYPU $y = f(y')$ I INTERPRETACJA ROZWIĄZANIA W PROGRAMIE MATHEMATICA

### Streszczenie

**Wstęp i cele:** W pracy przedstawiono metodę rozwiązywania równań typu  $y=f(y')$ . Głównym celem pracy jest pokazanie sposobu rozwiązywania tego typu równań różniczkowych. Ponadto celem rozważań jest przedstawienie odpowiednich algorytmów w programie *Mathematica*, które służą do przedstawienia interpretacji geometrycznej otrzymanych rozwiązań.

**Materiały i metody:** Źródła zawierają materiał dotyczący tematyki równań różniczkowych. Zastosowano metodę analizy matematycznej.

**Wyniki:** W analizie wybranych przykładów zastosowano metodę podstawienia nowej zmiennej  $t$  i otrzymano rozwiązanie badanego równania różniczkowego w postaci układu równań  $x=x(t)$  i  $y=y(t)$ .

**Wniosek:** Rozwiązanie równania różniczkowego typu  $y=f(y')$  w postaci układu równań  $x=x(t)$  i  $y=y(t)$  można zinterpretować graficznie stosując odpowiednio zastosowany algorytm w programie numerycznym *Mathematica*.

**Słowa kluczowe:** Równania różniczkowe, równanie typu  $y=f(y')$ , rozwiązanie analityczne, metoda podstawienia nowej zmiennej, interpretacja geometryczna rozwiązania, program *Mathematica*.

(Otrzymano: 02.05.2019; Zrecenzowano: 12.05.2019; Zaakceptowano: 22.05.2019)

## 1. Introduction

Let us consider the following differential equation:

$$y = f(y') \quad (1)$$

in some set A [5], [6].

The differential equation (1) can be presented in the form:

$$y = f\left(\frac{dy}{dx}\right). \quad (2)$$

To solve equation (2) we use the following substitution:

$$\frac{dy}{dx} = t. \quad (3)$$

We substitute the variable t for the equation (2):

$$y = f(t). \quad (4)$$

Equation (4) we differentiate on both sides with respect to the variable y and t respectively:

$$dy = \left(\frac{df}{dt}\right) dt. \quad (5)$$

The differential dy from equation (3) has the form:

$$dy = t dx. \quad (6)$$

From equations (5) and (6) we get:

$$\left(\frac{df}{dt}\right) dt = t dx. \quad (7)$$

From equation (7) we obtain:

$$dx = \left(\frac{1}{t} \cdot \frac{df}{dt}\right) dt. \quad (8)$$

Equation (8) we integrate on both sides with respect to the variable x and t respectively:

$$\int dx = \int \left(\frac{1}{t} \cdot \frac{df}{dt}\right) dt. \quad (9)$$

After integrating equation (9) we have:

$$x = \int \left(\frac{1}{t} \cdot \frac{df}{dt}\right) dt + C \quad (10)$$

where C is any integration constant.

Finally, we obtain the general parametric solution of the differential equation (1) in the following form:

$$\begin{cases} x(t) = \int \left(\frac{1}{t} \cdot \frac{df}{dt}\right) dt + C \\ y(t) = f(t) \end{cases} \quad (11)$$

where the integration constant C is any real number [2], [3], [6], [7].

## 2. Solving equations of type $y = f(y')$

**Example 1.** Solve the differential equation [4]:

$$y = \left( \frac{dy}{dx} - 1 \right) \exp\left( \frac{dy}{dx} \right). \quad (12)$$

### • Analytical solution

We use the substitution:

$$\frac{dy}{dx} = t. \quad (13)$$

Then

$$dy = t \, dx. \quad (14)$$

Therefore

$$dx = \frac{dy}{t}. \quad (15)$$

After introducing the substitution (13) into the equation (12) we get:

$$y = (t - 1) \cdot \exp(t). \quad (16)$$

Equation (16) we differentiate on both sides of the variable  $y$  and  $t$ :

$$dy = \exp(t) + (t - 1) \cdot \exp(t) dt. \quad (17)$$

Then

$$dy = t \cdot \exp(t) dt. \quad (18)$$

From the equations (14) and (18) we obtain:

$$t \cdot dx = t \cdot \exp(t) dt. \quad (19)$$

After simplification (19) we have:

$$dx = \exp(t) dt. \quad (20)$$

We integrate the equation (20) on both sides with respect to the variable  $x$  and  $t$ :

$$\int dx = \int \exp(t) dt. \quad (21)$$

Hence:

$$x = \exp(t) + C \quad (22)$$

where  $C$  is the integration constant.

The general parametric solution of the equation (12) makes the final form:

$$\begin{cases} x(t) = \exp(t) + C \\ y(t) = (t - 1) \cdot \exp(t) \end{cases} \quad (23)$$

where  $C$  is any real constant.

• **Graphical interpretation of solution**

In order to present the graphic interpretation of the parametric solution of equation (12), we use the following algorithm in *Mathematica* program.

**Program Mathematica** [1], [5], [9]

```
ParametricPlot[
{{Exp[t]+0.5, (t-1)*Exp[t]}, {Exp[t]+1.5, (t-1)*Exp[t]}},
{t, -3, 3}, PlotRange->{-2, 2}, Axesorigin->{0, 0},
Background->RGBColor[1, 1, 0.85],
PlotStyle->{{RGBColor[0, 0.5, 1], Thickness[0.0065]},
{RGBColor[1, 0.5, 0], Thickness[0.0065]}},
AxesStyle->Thickness[0.0035], AxelLabel->{"x", "y"},
GridLines->Automatic,
TextStyle->{FontFamily->"Arial", FontSize->12}]
```

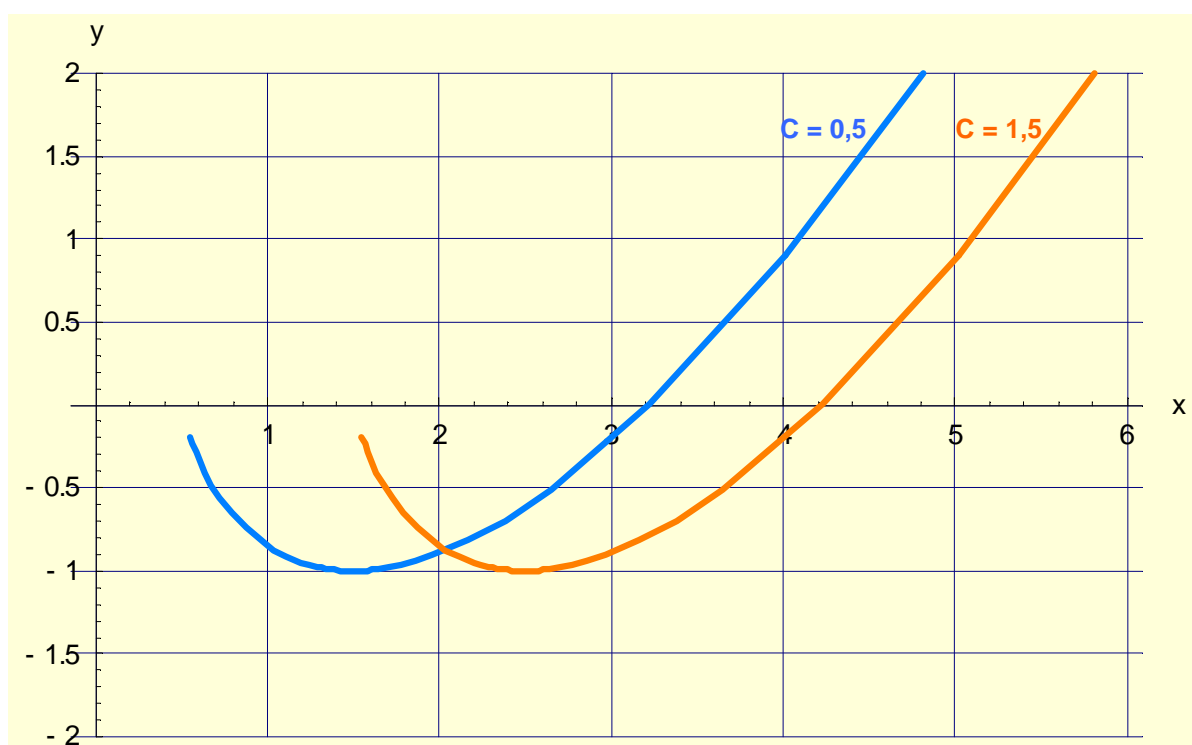


Fig. 1. Graphical interpretation of the solution (23) of differential equation (12) for parameter  $-3 < t < 3$ , values  $0 < x < 6$ ,  $-2 < y < 2$  and constant  $C = 0.5$  or  $C = 1.5$

Source: Elaboration of the Authors

**Example 2.** Solve the differential equation [4]:

$$y = \frac{dy}{dx} \cdot \ln\left(\frac{dy}{dx}\right). \tag{24}$$

• **Analytical solution**

After introducing the substitution (13) into the equation (24) we get:

$$y = t \cdot \ln(t). \tag{25}$$

The equation (25) we differentiate on both sides of the variable  $y$  and  $t$ :

$$dy = [\ln(t) + 1]dt. \tag{26}$$

From the equations (14) and (26) we get:

$$t \cdot dx = [\ln(t) + 1]dt. \quad (27)$$

After simplification (27) we have:

$$dx = \frac{\ln(t) + 1}{t} dt. \quad (28)$$

We integrate the equation (28) on both sides with respect to the variable x and t:

$$\int dx = \int \frac{\ln(t) + 1}{t} dt. \quad (29)$$

We calculate by substitution the integral on the right of the equation (29):

$$\int \frac{\ln(t) + 1}{t} dt = \left\langle \begin{array}{l} \ln(t) = p \\ \frac{dt}{t} = dp \end{array} \right\rangle = \int (p + 1) dp = \frac{1}{2} p^2 + p + C \quad (30)$$

$$\int \frac{\ln(t) + 1}{t} dt = \frac{1}{2} \ln^2(t) + \ln(t) + C \quad (31)$$

Hence:

$$x = \frac{1}{2} \ln^2(t) + \ln(t) + C \quad (32)$$

where C is the integration constant.

The general parametric solution of the equation (24) takes the final form:

$$\begin{cases} x(t) = \frac{1}{2} \ln^2(t) + \ln(t) + C \\ y(t) = t \cdot \ln(t) \end{cases} \quad (33)$$

where C is any real constant.

### • Graphical interpretation of solution

In order to present the graphic interpretation of the parametric solution of equation (24), we use the following algorithm in *Mathematica* program.

**Program Mathematica** [1], [5], [9]

```
ParametricPlot[
  {{0.5(Log[t])^2+Log[t]+1,t*Log[t]},
  {0.5(Log[t])^2+Log[t]+5,t*Log[t]}},
  {t,0.001,2},PlotRange->{-0.5,1},
  Axesorigin->{0,0},
  Background->RGBColor[1,1,0.85],
  PlotStyle->{{RGBColor[0,0.5,1],
  Thickness[0.0065]},{RGBColor[1,0.5,0], Thickness[0.0065]}},
  AxesStyle->Thickness[0.0035],
  AxelLabel->{"x","y"},
  GridLines->Automatic,
  TextStyle->{FontFamily->"Arial",FontSize->12}]
```

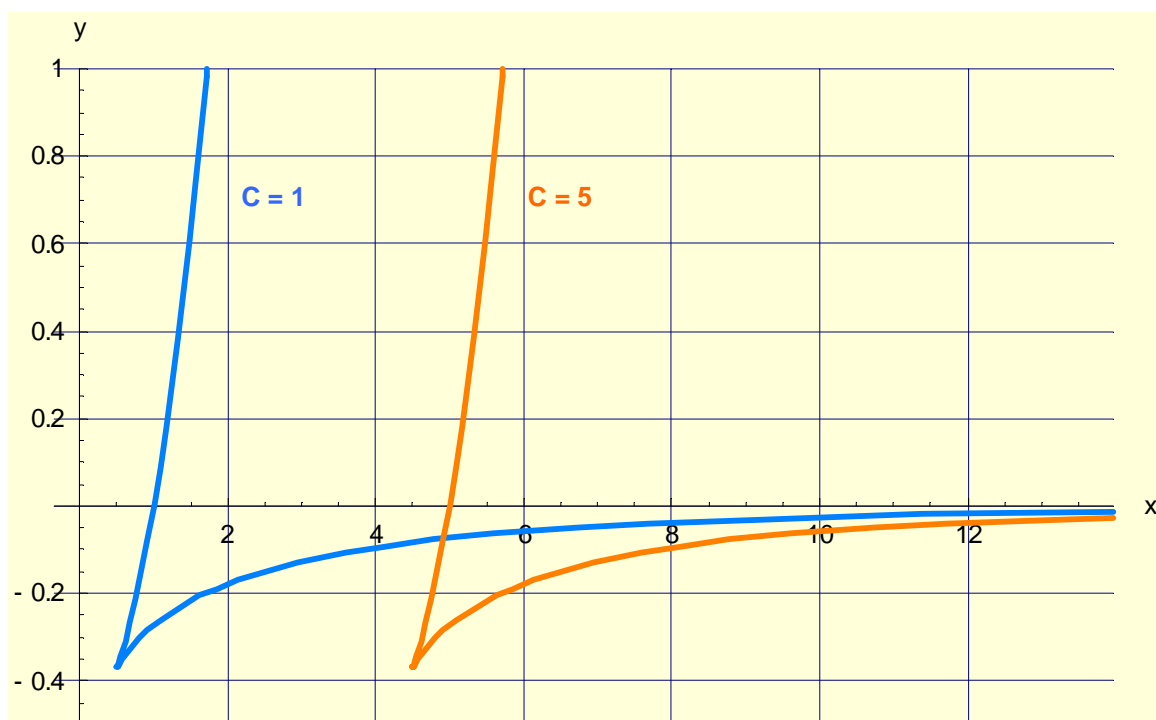


Fig. 2. Graphical interpretation of the solution (33) of differential equation (24) for parameter  $0.001 < t < 2$ , values  $0 < x < 14$ ,  $-0.4 < y < 1$  and constant  $C = 1$  or  $C = 5$

Source: Elaboration of the Authors

**Example 3.** Solve the differential equation [4]:

$$\frac{dy}{dx} = \operatorname{arctg} \left[ y \cdot \left( \frac{dy}{dx} \right)^{-2} \right]. \quad (34)$$

• **Analytical solution**

After introducing the substitution (13) into the equation (34) we obtain:

$$t = \operatorname{arctg} \left( \frac{y}{t^2} \right). \quad (35)$$

Using the inverse function for the function  $\operatorname{arctg}(\cdot)$ , we get:

$$y = t^2 \operatorname{tg}(t). \quad (36)$$

The equation (36) we differentiate on both sides of the variable  $y$  and  $t$ :

$$dy = \left[ 2t \cdot \operatorname{tg}(t) + t^2 \cdot \frac{1}{\cos^2(t)} \right] dt. \quad (37)$$

From the equations (14) and (37) we get:

$$dx = \left[ 2\operatorname{tg}(t) + \frac{t}{\cos^2(t)} \right] dt. \quad (38)$$

We integrate the equation (38) on both sides with respect to the variable  $x$  and  $t$ :

$$\int dx = \int \left[ 2\operatorname{tg}(t) + \frac{t}{\cos^2(t)} \right] dt. \quad (39)$$

We calculate the integral on the right of equation (39):

$$\int \left[ 2\operatorname{tg}(t) + \frac{t}{\cos^2(t)} \right] dt = 2 \int \operatorname{tg}(t) dt + \int \frac{t}{\cos^2(t)} dt. \quad (40)$$

Let us calculate the following integral

$$2 \int \operatorname{tg}(t) dt = 2 \int \frac{\sin(t)}{\cos(t)} dt = (-2) \int \frac{-\sin(t)}{\cos(t)} dt = (-2) \ln |\cos(t)| + C_1 \quad (41)$$

and by parts next integral

$$\begin{aligned} \int \frac{t}{\cos^2(t)} dt &= \left\langle \begin{array}{l} u = t \\ \frac{du}{dt} = 1 \\ \frac{dv}{dt} = \frac{1}{\cos^2(t)} \\ v = \int \frac{dt}{\cos^2(t)} = \operatorname{tg}(t) \end{array} \right\rangle = t \cdot \operatorname{tg}(t) - \int \operatorname{tg}(t) dt = \\ &= t \cdot \operatorname{tg}(t) - \int \frac{\sin(t)}{\cos(t)} dt = t \cdot \operatorname{tg}(t) + \int \frac{-\sin(t)}{\cos(t)} dt = t \cdot \operatorname{tg}(t) + \ln |\cos(t)| + C_2. \end{aligned} \quad (42)$$

Hence:

$$x = (-2) \ln |\cos(t)| + t \cdot \operatorname{tg}(t) + \ln |\cos(t)| + C. \quad (43)$$

The general parametric solution of the equation (34) takes the final form:

$$\begin{cases} x(t) = t \cdot \operatorname{tg}(t) - \ln |\cos(t)| + C \\ y(t) = t \cdot \ln(t) \end{cases} \quad (44)$$

where  $C$  is any real constant.

### • Graphical interpretation of solution

In order to present the graphic interpretation of the parametric solution of equation (34), we use the following algorithm in *Mathematica* program [1]-[4].

**Program Mathematica** [1], [5], [9]

```
ParametricPlot[
  {{t*Tan[t]-Log[Cos[t]]+2,t^2*Tan[t]},
  {t*Tan[t]-Log[Cos[t]]+6,t^2*Tan[t]}},
  {t,-0.5Pi,0.5Pi},
  PlotRange->{-3,3},
  Axesorigin->{0,0},
  Background->RGBColor[1,1,0.85],
  PlotStyle->{
  {RGBColor[0,0.5,1],Thickness[0.0065]},
  {RGBColor[1,0.5,0],Thickness[0.0065]}},
  AxesStyle->Thickness[0.0035],
  AxelLabel->{"x","y"},
  GridLines->Automatic,
  TextStyle->{FontFamily->"Arial",FontSize->12}]
```

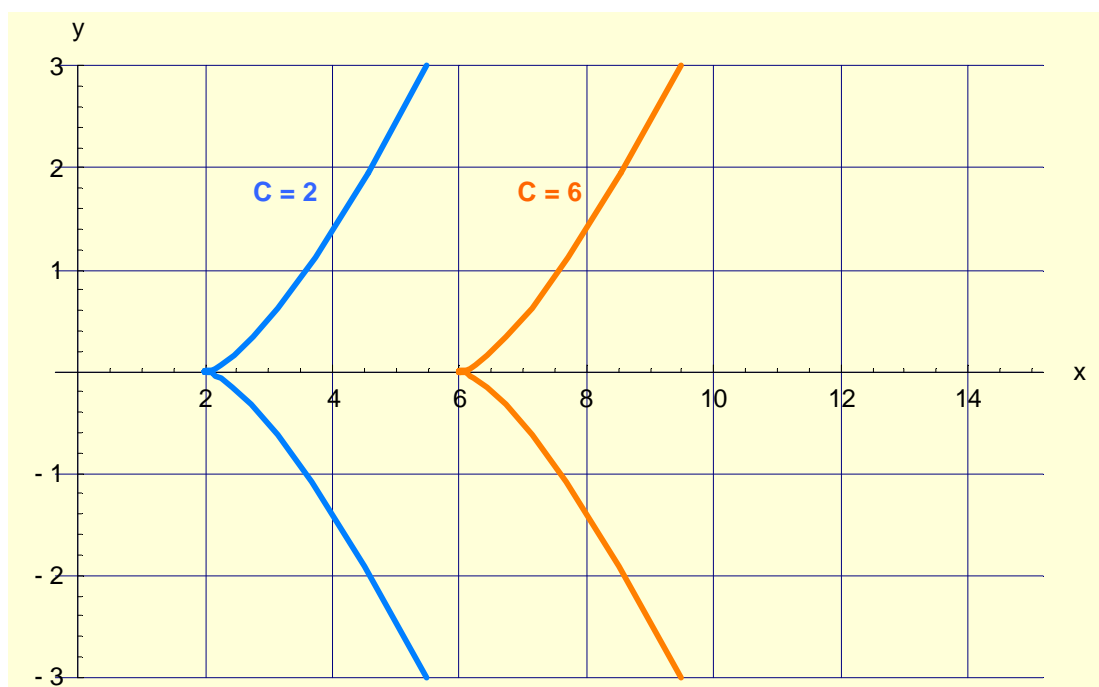


Fig. 3. Graphical interpretation of the solution (44) of differential equation (34) for parameter  $-0.5\pi < t < 0.5\pi$ , values  $0 < x < 14$ ,  $-3 < y < 3$  and constant  $C = 2$  or  $C = 6$

Source: Elaboration of the Authors

**Example 4.** Solve the differential equation [4]:

$$\frac{dy}{dx} = \arcsin \left[ x \cdot \left( \frac{dy}{dx} \right)^{-1} \right]. \quad (45)$$

• **Analytical solution**

After introducing the substitution (13) into the equation (45) we obtain:

$$t = \arcsin \left( \frac{x}{t} \right). \quad (46)$$

Using the inverse function for the function  $\arcsin(\cdot)$ , we get:

$$x = t \cdot \sin(t). \quad (47)$$

The equation (47) we differentiate on both sides of the variable  $y$  and  $t$ :

$$x = [\sin(t) + t \cdot \cos(t)]dt. \quad (48)$$

From the equations (14) and (48) we get:

$$\frac{dy}{t} = [\sin(t) + t \cdot \cos(t)]dt. \quad (49)$$

Hence

$$dy = t \cdot [\sin(t) + t \cdot \cos(t)]dt. \quad (50)$$

We integrate the equation (50) on both sides with respect to the variable  $x$  and  $t$ :

$$\int dy = \int [t \cdot \sin(t) + t^2 \cdot \cos(t)]dt. \quad (51)$$



We calculate the integral on the right of equation (51):

$$\int dy = \int t \cdot \sin(t) dt + \int t^2 \cos(t) dt. \quad (52)$$

Let us calculate by parts the following integrals:

$$\int t \cdot \sin(t) dt = \left\langle \begin{array}{l} u = t \\ \frac{dv}{dt} = \sin(t) \\ v = \int \sin(t) dt = -\cos(t) \end{array} \middle| \frac{du}{dt} = 1 \right\rangle = -t \cos(t) + \int \cos(t) dt = \quad (53)$$

$$= -t \cos(t) + \sin(t) + C_1.$$

$$\int t^2 \cos(t) dt = \left\langle \begin{array}{l} u = t^2 \\ \frac{dv}{dt} = \cos(t) \\ v = \int \cos(t) = \sin(t) \end{array} \middle| \frac{du}{dt} = 2t \right\rangle = t^2 \cdot \sin(t) - 2 \int t \cdot \sin(t) dt = \quad (54)$$

$$= \left\langle \begin{array}{l} u = t \\ \frac{dv}{dt} = \sin(t) \\ v = \int \sin(t) = -\cos(t) \end{array} \middle| \frac{du}{dt} = 1 \right\rangle = t^2 \cdot \sin(t) - 2[-t \cos(t) + \int \cos(t) dt] =$$

$$= t^2 \cdot \sin(t) + 2t \cos(t) - 2 \sin(t) + C_2.$$

Hence:

$$\begin{aligned} y &= [-t \cos(t) + \sin(t) + C_1] + [t^2 \cdot \sin(t) + 2t \cos(t) - 2 \sin(t) + C_2] = \\ &= t^2 \sin(t) + t \cos(t) - \sin(t) + C. \end{aligned} \quad (55)$$

The general parametric solution of the equation (34) takes the final form:

$$\begin{cases} x(t) = t \cdot \sin(t) \\ y(t) = t^2 \sin(t) + t \cos(t) - \sin(t) + C \end{cases} \quad (56)$$

where  $C$  is any real constant.

### • Graphical interpretation of solution

In order to present the graphic interpretation of the parametric solution of equation (34), we use the following algorithm in *Mathematica* program.

**Program Mathematica** [1], [5], [9]

```
ParametricPlot[
  {{t*Sin[t], (-1+t^2)*Sin[t]+t*Cos[t]+0},
  {t*Sin[t], (-1+t^2)*Sin[t]+t*Cos[t]+10}},
  {t, -2.55, 2.55Pi},
  PlotRange->{-25, 125},
  Axesorigin->{0, 0},
  Background->RGBColor[1, 1, 0.85],
  PlotStyle->{{RGBColor[0, 0.5, 1],
  Thickness[0.0065]}, {RGBColor[1, 0.5, 0], Thickness[0.0065]}},
  AxesStyle->Thickness[0.0035],
  AxelLabel->{"x", "y"},
  GridLines->Automatic,
  TextStyle->{FontFamily->"Arial", FontSize->12}]
```

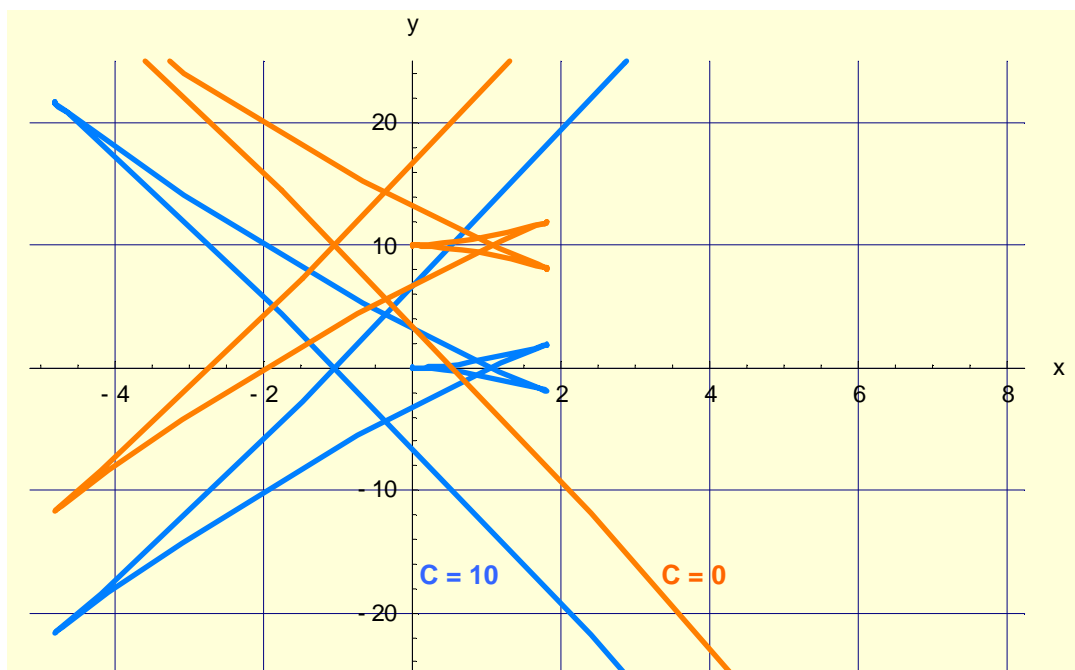


Fig. 4. Graphical interpretation of the solution (56) of differential equation (45) for parameter  $-2.55\pi < t < 2.55\pi$ , values  $-6 < x < 8$ ,  $-25 < y < 25$  and constant  $C = 0$  or  $C = 10$

Source: Elaboration of the Authors

### 3. Conclusions

- Equations of type  $y = f(y')$  can be solved by using the substitution  $y' = t$  thus obtaining a solution in the form of a system of parametric equations.
- The Mathematica numerical program can be used to interpret graphically the solution of the equation  $y = f(y')$ .

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