Journal of Polish Safety and Reliability Association
Summer Safety and Reliability Seminars, Volume 6, Number 1, 2015

Kołowrocki Krzysztof

Kuligowska Ewa

Soszyńska-Budny Joanna
Maritime University, Gdynia, Poland

Reliability assessment of an exemplary system operating at variable conditions

Keywords

system reliability, system operation process, analytical approach

Abstract

The reliability analysis of a multistate complex system subjected to varying in time its operation process is performed. A semi-Markov process is applied to construct the multistate model of the system operation process and its main characteristics are determined. Analytical linking of the system operation process model with the system multistate reliability model is proposed to get a general reliability model of the complex system operating at varying in time operation conditions and to find its reliability characteristics. The constructed integrated general model of a complex multistate system reliability, linking its reliability model and its operation process model and considering variable at different operation states its reliability structure and its components reliability parameters is applied to the reliability evaluation of an exemplary system.

1. Introduction

As the complexity of the systems' operation processes and their influence on changing in time the systems' reliability parameters are very often met in real practice then there is the practical importance and need of an approach presented in the paper and linking the system reliability model and the system operation process model into an integrated general model in reliability assessment of real technical systems is proposed.

From the point of view of more precise analysis of the reliability of complex systems the developed methods should be based on a multistate approach (Kołowrocki K. [2], Kołowrocki K., Soszyńska-Budny J. [2], Soszyńska-Budny J. [4], Xue J. [6], Yu K. et. al. [7]) to these complex systems reliability analysis instead of normally used two-state approach. This will enable different complex systems inside reliability states to be distinguished, such that they ensure a demanded level of the system operation effectiveness with accepted consequences of the dangerous accidents for the environment, population, etc.

2. System operation process

We assume that a system during its operation at the fixed moment t, $t \in \langle 0,+\infty \rangle$, may be at one of ν, $\nu \in N$, different operations states z_b, $b = 1,2,...,\nu$. Consequently, we mark by $Z(t)$, $t \in \langle 0,+\infty \rangle$, the system operation process, that is a function of a continuous variable t, taking discrete values at the set $\{z_1, z_2,..., z_\nu\}$ of the system operation states. We assume a semi-Markov model [1], [2] of the system operation process $Z(t)$ and we mark by θ_{bl} its random conditional sojourn times at the operation states z_b, when its next operation state is z_l, $b, l = 1,2,...,\nu$, $b \neq l$.

Consequently, the operation process may be described by the following parameters:

- the vector of the initial probabilities of the system operation process $Z(t)$ staying at the particular operations states at the moment $t = 0$

$$[p_b(0)]_{1\times\nu} = [p_1(0), p_2(0),..., p_\nu(0)], \quad (1)$$

where

$$p_b(0) = P(Z(0) = z_b),\ b = 1,2,...,\nu; \qquad (2)$$

- the matrix of the probabilities of the system operation process $Z(t)$ transitions between the operation states z_b and z_l, $b,l = 1,2,...,\nu,\ b \neq l$

$$[p_{bl}]_{\nu\times\nu} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\nu} \\ p_{21} & p_{22} & \cdots & p_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\nu 1} & p_{\nu 2} & \cdots & p_{\nu\nu} \end{bmatrix} \qquad (3)$$

where $p_{bb} = 0$ for $b = 1,2,...,\nu$;

- the matrix of the conditional distribution functions of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states

$$[H_{bl}(t)]_{\nu\times\nu} = \begin{bmatrix} H_{11}(t) & H_{12}(t) & \cdots & H_{1\nu}(t) \\ H_{21}(t) & H_{22}(t) & \cdots & H_{2\nu}(t) \\ \vdots & \vdots & \ddots & \vdots \\ H_{\nu 1}(t) & H_{\nu 2}(t) & \cdots & H_{\nu\nu}(t) \end{bmatrix} \qquad (4)$$

where

$$H_{bl}(t) = P(\theta_{bl} < t),\ H_{bb}(t) = 0, \qquad (5)$$

for $b,l = 1,2,...,\nu,\ b \neq l$.

Having identified the probabilities of transitions p_{bl} defined by (3) between the operation states and the distributions of conditional sojourn times θ_{bl}, the mean values M_b of the system operation process $Z(t)$ unconditional sojourn times θ_b, $b = 1,2,...,\nu$, at the particular operation states can be determined by

$$M_b = E[\theta_b] = \sum_{l=1}^{\nu} p_{bl} M_{bl},\ b = 1,2,...,\nu, \qquad (6)$$

where M_{bl} are the mean values of the conditional sojourn times θ_{bl} given by

$$M_{bl} = E[\theta_{bl}] = \int_0^{\infty} t h_{bl}(t)dt,\ b,l = 1,2,...,\nu,\ b \neq l, \qquad (7)$$

and

$$h_{bl}(t) = \frac{dH_{bl}(t)}{dt},\ b,l = 1,2,...,\nu,\ b \neq l, \qquad (8)$$

are the conditional density functions of the system operation process $Z(t)$ conditional sojourn times θ_{bl}, $b,l = 1,2,...,\nu$, $b \neq l$, at the particular operation states corresponding to the distribution functions $H_{bl}(t)$.
Further, the limit values of the system operation process $Z(t)$ transient probabilities at the particular operation states

$$p_b(t) = P(Z(t) = z_b),\ b = 1,2,...,\nu,$$

can be determined from the following relationship

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{\nu} \pi_l M_l},\ b = 1,2,...,\nu, \qquad (9)$$

where M_b are given by (6), while the steady probabilities π_b of the vector $[\pi_b]_{1x\nu}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{\nu} \pi_l = 1, \end{cases} \qquad (10)$$

where $[\pi_b] = [\pi_1, \pi_2, ..., \pi_\nu]$ and the matrix $[p_{bl}]$ is defined by (3).

3. Reliability of multistate system

In the multistate reliability analysis to define a system composed of n, $n \in N$ ageing components we assume that:

– E_i, $i = 1,2,...,n$, are components of a system,
– all components and a system under consideration have the set of reliability states $\{0,1,...,z\}$, $z \geq 1$,
– the reliability states are ordered, the state 0 is the worst and the state z is the best,
– the component and the system reliability states degrade with time t,
– $T_i(u)$, $i = 1,2,...,n$, $n \in N$, are independent random variables representing the lifetimes of components E_i in the reliability state subset $\{u,u + 1,...,z\}$, while they were in the reliability state z at the moment $t = 0$,
– $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset

$\{u, u+1, ..., z\}$, while it was in the reliability state z at the moment $t = 0$,

- $s_i(t)$ is a component E_i reliability state at the moment t, $t \in \langle 0, \infty)$, given that it was in the reliability state z at the moment $t = 0$,
- $s(t)$ is the system reliability state at the moment t, $t \in \langle 0, \infty)$, given that it was in the reliability state z at the moment $t = 0$.

The above assumptions mean that the reliability states of the ageing system and components may be changed in time only from better to worse.

Definition 1. A vector

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1), \ldots, R_i(t,z)], \tag{11}$$

for $t \in \langle 0, \infty)$, $i = 1,2,...,n$, where

$$R_i(t,u) = P(s_i(t) \geq u \mid s_i(0) = z) = P(T_i(u) > t) \tag{12}$$

for $t \in \langle 0, \infty), u = 0,1,\ldots,z,$ is the probability that the component E_i is in the reliability state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in \langle 0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multistate reliability function of a component E_i.

Definition 2. A vector

$$\boldsymbol{R}(t,\cdot) = [\boldsymbol{R}(t,0), \boldsymbol{S}(t,1), \ldots, \boldsymbol{R}(t,z)],\ t \in \langle 0, \infty), \tag{13}$$

where

$$\boldsymbol{R}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \tag{14}$$

for $t \in \langle 0, \infty)$, $u = 0,1,...,z$, is the probability that the system is in the reliability state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in \langle 0, \infty)$, while it was in the reliability state z at the moment $t = 0$, is called the multistate reliability function of a system.

The reliability functions $R_i(t,u)$ and $\boldsymbol{R}(t,u)$, $t \in \langle 0, \infty)$, $u = 0,1,...,z$, defined by (12) and (14) are called the coordinates of the components and the system multistate reliability functions $R_i(t,\cdot)$ and $\boldsymbol{R}(t,\cdot)$ given by (11) and (13) respectively. It is clear that from *Definition 1* and *Definition 2*, for $u = 0$, we have

$$R_i(t,0) = 1 \text{ and } \boldsymbol{R}(t,0) = 1.$$

Under the above definitions, the mean value of the system lifetime $T(u)$ in the reliability state subset $\{u, u+1, ..., z\}$ is given by

$$\mu(u) = \int_0^\infty \boldsymbol{R}(t,u)dt,\ u = 1,2,\ldots,z, \tag{15}$$

whereas the standard deviation of the system lifetime $T(u)$ in the reliability state subset $\{u, u+1, ..., z\}$ is given by

$$\sigma(u) = \sqrt{D[T(u)]} = \sqrt{2\int_0^\infty t \cdot \boldsymbol{R}(t,u)dt - [\mu(u)]^2}, \tag{16}$$

where $\boldsymbol{R}(t,u)$, $t \in \langle 0, \infty)$, are defined according to (14) and $\mu(u)$, $u = 1,2,\ldots,z$, are given by (15).

Now, after introducing the notion of the multistate reliability analysis, we may define basic multistate reliability structures.

Definition 3. A multistate system is called series if its lifetime $T(u)$ in the reliability state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq n}\{T_i(u)\},\ u = 1,2,...,z. \tag{17}$$

The reliability function of the multistate series system is given by the vector [2]

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), ..., \boldsymbol{R}(t,z)] \tag{18}$$

with the coordinates

$$\boldsymbol{R}(t,u) = \prod_{i=1}^{n} R_i(t,u),\ t \in <0, \infty), \tag{19}$$

for $u = 1,2,...,z$.

Definition 4. A multistate system is called series-parallel if its lifetime $T(u)$ in the reliability state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \max_{1 \leq i \leq k}\{\min_{1 \leq j \leq l_i}\{T_{ij}(u)\}\},\ u = 1,2,...,z, \tag{20}$$

where k is the number of series subsystems linked in parallel and l_i is the number of components in the i^{th} series subsystem. The reliability function of the multistate series-parallel system is given by the vector [2]

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), ..., \boldsymbol{R}(t,z)], \tag{21}$$

with the coordinates

$$\boldsymbol{R}(t,u) = 1 - \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} R_{ij}(t,u)], \quad (22)$$

for $t \in \langle 0,\infty)$, $u = 1,2,...,z$.

Definition 5. A multistate system is called series-"m out of k" system if its lifetime $T(u)$ in the reliability state subset $\{u, u+1,...,z\}$ is given by

$$T(u) = T_{(k-m+1)}(u),\ m = 1,2,...,k,\ u = 1,2,...,z, \quad (23)$$

where $T_{(k-m+1)}(u)$ is the $(k - m + 1)^{th}$ order statistic in the set of random variables

$$T_i(u) = \min_{1\le j\le l_i}\{T_{ij}(u)\},\ i = 1,2,...,k,\ u = 1,2,...,z,$$

where l_i is the number of components in the i^{th} series subsystem. The reliability function of the multistate series-"m out of k" system is given by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1),..., \boldsymbol{R}(t,z)], \quad (24)$$

with the coordinates

$$\boldsymbol{R}(t,u) =$$

$$1 - \sum_{\substack{r_1,r_2,...,r_k=0 \\ r_1+r_2+...+r_k\le m-1}}^{1} \prod_{i=1}^{k}[[\prod_{j=1}^{l_i} R_{ij}(t,u)]^{r_i}[1 - \prod_{j=1}^{l_i} R_{ij}(t,u)]^{1-r_i}, \quad (25)$$

for $t \in \langle 0,\infty)$, $u = 1,2,...,z$.

4. Reliability of multistate system at variable operation conditions

We assume that every operation state of the system operation process $Z(t)$, $t \in \langle 0,\infty)$, described in Section 2, have an influence on the system reliability [2], [4]. Therefore, the system component's reliability at the particular operation state z_b, $b = 1,2,...,\nu$, can be described using the conditional reliability function

$$[\boldsymbol{R}_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t \mid Z(t) = z_b), \quad (26)$$

for $t \in \langle 0,\infty)$, $b = 1,2,...,\nu$, that is the conditional probability that the system component's conditional lifetime $T_i^{(b)}(u)$ is greater than t, while the system operation process $Z(t)$ is at the operation state z_b, $b = 1,2,...,\nu$ [2].

Further, we denote the unconditional reliability function of the system by

$$\boldsymbol{R}(t,u) = P(T(u) > t),\ t \in \langle 0,\infty), \quad (27)$$

where $T(u)$ is the system unconditional lifetime in the reliability state subset $\{u, u + 1,...,z\}$.

In the case when the system operation time θ is large enough, the unconditional reliability function of the system is approximated by [2]

$$\boldsymbol{R}(t,u) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{R}(t,u)]^{(b)},\quad t \in \langle 0,\infty), \quad (28)$$

where p_b, $b = 1,2,...,\nu$, are the system operation process limit transient probabilities given by (9). Hence the mean value (15) of the system unconditional lifetime $T(u)$ is given by

$$\mu(u) \cong \sum_{b=1}^{\nu} p_b \mu_b(u), \quad (29)$$

where μ_b are the mean values of the system conditional lifetimes $T_i^{(b)}(u)$ at the operation state z_b, $b = 1,2,...,\nu$, given by

$$\mu_b(u) = \int_0^{+\infty} [\boldsymbol{R}_i(t,u)]^{(b)} dt,\quad b = 1,2,...,\nu, \quad (30)$$

$[\boldsymbol{R}_i(t,u)]^{(b)}$, $b = 1,2,...,\nu$, are defined by (26) and p_b are given by (9).

Whereas, the standard deviation (16) of the system unconditional lifetime $T(u)$ is given by

$$\sigma(u) = \sqrt{2\int_0^{\infty} t\ \boldsymbol{R}(t,u)dt - \mu(u)^2}, \quad (31)$$

where $\boldsymbol{R}(t,u)$ is given by (27) and $\mu(u)$ is given by (29).

Moreover, the mean values of the system lifetimes in particular reliability states are given by

$$\bar{\mu}(u) = \mu(u) - \mu(u+1),\ u = 0,1,...,z-1,$$

$$\bar{\mu}(z) = \mu(z), \quad (32)$$

where $\mu(u)$, u = 0,1,...,z are given by (29).
Further, if r is the system critical reliability state, then the system risk function is given by [2]

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t, r),\ t \in \langle 0, \infty), \tag{33}$$

and if $\boldsymbol{\tau}$ is the moment when the system risk function exceeds a permitted level δ, then if $\boldsymbol{r}^{-1}(\mathrm{t})$ exists we have

$$\boldsymbol{\tau} = \boldsymbol{r}^{-1}(\delta), \tag{34}$$

where $\boldsymbol{r}^{-1}(\mathrm{t})$ is the inverse function of the risk function $\boldsymbol{r}(\mathrm{t})$.

5. The exemplary system operation and reliability evaluation

5.1. The exemplary system description

We consider an exemplary system S consisting of subsystems S_1 and S_2 composed of the elements $E_{ij}^{(\upsilon)}$, $i = 1,2,\ldots,i^{(\upsilon)}$, $j = 1,2,\ldots,j^{(\upsilon)}$, $\upsilon = 1,2$. The parameters $i^{(\upsilon)}$ and $j^{(\upsilon)}$ are called the system structure shape parameters and are different for particular subsystems (*Figure 1*):
- subsystem S_1 is composed of the components $E_{ij}^{(1)}$, $i = 1,2$, $j = 1,2,3$,
- subsystem S_2 is composed of the components $E_{ij}^{(2)}$, $i = 1,2,3,4$, $j = 1,2$.

Moreover, we assume that the exemplary system reliability structure and its subsystems and components reliability depend on its changing in time operation states.

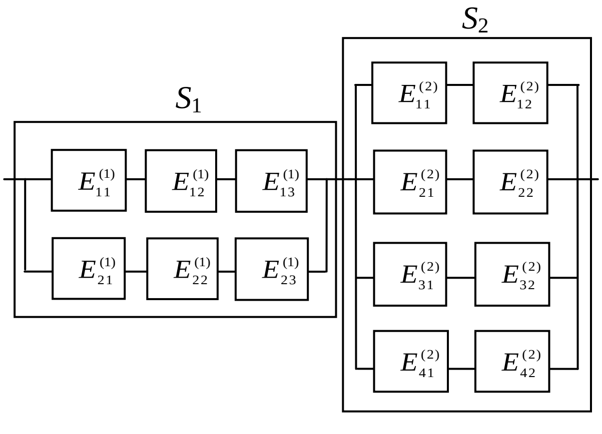

Figure 1. The scheme of the exemplary system S reliability structure

5.2. Analytical approach to an exemplary system operation process analysis

The exemplary system S illustrated in *Figure 1* is operating at $\nu = 4$ different operation states z_1, z_2, z_3 and z_4. Its structure is different at particular operation states:
- at the operation state z_1 the system is identical with subsystem S_1 having a series-parallel structure.
- at the operation state z_2 the system is identical with subsystem S_2 having a series-parallel structure.
- at the operation state z_3 the system is series composed of the subsystems S_1 and S_2 having series-parallel structure.
- at the operation state z_4 the system is series composed of the series-parallel subsystem S_1 and series-“2 out of 4” subsystem S_2.

The probabilities of the initial operation states of this system operation process $Z(t)$, $t \in \langle 0, \infty)$, are fixed arbitrarily [2] in the following way

$$[p_b(0)] = [0.21, 0.10, 0.29, 0.40]. \tag{35}$$

The probabilities of the exemplary system operation process $Z(t)$ transitions between the operation states z_b and z_l, $b, l = 1,2,3,4$, $b \neq l$ are also fixed arbitrarily [2] and given in the matrix below

$$[p_{bl}] = \begin{bmatrix} 0 & 0.22 & 0.32 & 0.46 \\ 0.20 & 0 & 0.30 & 0.50 \\ 0.12 & 0.16 & 0 & 0.72 \\ 0.48 & 0.22 & 0.30 & 0 \end{bmatrix}. \tag{36}$$

Moreover, we assume that the distribution functions of the exemplary system operation process conditional sojourn times measured in days are exponential and given as follows:

$$H_{bl}(t) = 1 - \exp[-\alpha_{bl}\, t], \tag{37}$$

where α_{bl} are given as follows:

$$[\alpha_{bl}] = \begin{bmatrix} 0 & 0.0052 & 0.0021 & 0.005 \\ 0.0104 & 0 & 0.0123 & 0.0182 \\ 0.0011 & 0.0021 & 0 & 0.0033 \\ 0.0031 & 0.0020 & 0.0023 & 0 \end{bmatrix} \tag{38}$$

for $t \in \langle 0, \infty)$.
Applying (37)-(38) and (8) to the conditional distributions given by (7), the conditional mean

values $M_{bl} = E[\theta_{bl}]$, $b,l = 1,2,3,4,$ of the exemplary system sojourn times at the particular operation states measured in days are fixed as follows:

$$M_{12} = 192, \quad M_{13} = 480, \quad M_{14} = 200,$$
$$M_{21} = 96, \quad M_{23} = 81, \quad M_{24} = 55,$$
$$M_{31} = 870, \quad M_{32} = 480, \quad M_{34} = 300,$$
$$M_{41} = 325, \quad M_{42} = 510, \quad M_{43} = 438. \qquad (39)$$

Based on the formula (6) and applying (36), (39), the system operation process unconditional mean sojourn times θ_b, $b = 1,2,...,v$, measured in days at the particular operation states are given by

$$M_1 = E[\theta_1] = 287.84, \; M_2 = E[\theta_2] = 71.00,$$
$$M_3 = E[\theta_3] = 397.20, \; M_4 = E[\theta_4] = 399.60. \quad (40)$$

Next, considering (37), the approximate solutions of the system of equations (10) are:

$$\pi_1 \cong 0.236, \quad \pi_2 \cong 0.169,$$
$$\pi_3 \cong 0.234, \quad \pi_4 \cong 0.361. \qquad (41)$$

Further, applying (9) and (41), the limit values of the system operation process transient probabilities $p_b(t)$, $b = 1,2,...,v,$ at the operations states z_b can be found after completing a few steps described in [2] and get

$$p_1 \cong 0.214, \quad p_2 \cong 0.038,$$
$$p_3 \cong 0.293, \quad p_4 \cong 0.455. \qquad (42)$$

5.3. Analytical approach to an exemplary system reliability analysis

In the reliability analysis of the considered exemplary system and its components, we arbitrarily distinguish the following four reliability states (z = 3):

- a reliability state 3 – the system operation is fully effective,
- a reliability state 2 – the system operation is less effective because of ageing,
- a reliability state 1 – the system operation is less effective because of ageing and more dangerous for the environment,
- a reliability state 0 – the system is destroyed.

We assume that the transitions between the components reliability states are possible only from better to worse ones and we fix that the system and its components critical reliability state is r = 2.

The conditional system reliability functions are given by the vector

$$[\boldsymbol{R}(t,\cdot)]^{(b)}$$
$$= [1, [\boldsymbol{R}(t,1)]^{(b)}, [\boldsymbol{R}(t,2)]^{(b)}, [\boldsymbol{R}(t,3)]^{(b)}], \qquad (1)$$

where

$$[\boldsymbol{R}(t,u)]^{(1)} = 1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{3}[R_{ij}^{(1)}(t,u)]^{(1)}], \qquad (2)$$

$$[\boldsymbol{R}(t,u)]^{(2)} = 1 - \prod_{i=1}^{4}[1 - \prod_{j=1}^{2}[R_{ij}^{(2)}(t,u)]^{(2)}] \qquad (3)$$

$$[\boldsymbol{R}(t,u)]^{(3)} = \left(1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{3}[R_{ij}^{(1)}(t,u)]^{(3)}]\right) \cdot \left(1 - \prod_{i=1}^{4}[1 - \prod_{j=1}^{2}[R_{ij}^{(2)}(t,u)]^{(3)}]\right) \qquad (4)$$

$$[\boldsymbol{R}(t,u)]^{(4)} = \left(1 - \prod_{i=1}^{2}[1 - \prod_{j=1}^{3}[R_{ij}^{(1)}(t,u)]^{(4)}]\right) \cdot \left(1 - \sum_{\substack{r_1,r_2,r_3,r_4=0 \\ r_1+r_2+r_3+r_4 \le 1}}^{1} \prod_{i=1}^{4}[\prod_{j=1}^{2}[R_{ij}^{(2)}(t,1)]^{(4)}]^{r_i} \cdot [1 - \prod_{j=1}^{2}[R_{ij}^{(2)}(t,1)]^{(4)}]^{1-r_i}\right) \qquad (5)$$

for u = 1,2,3, $t \in \langle 0,\infty)$.

Moreover, we assume that the system elements $E_{ij}^{(\upsilon)}$ having the lifetimes $T_{ij}^{(\upsilon)}(u)$, $i = 1,2,\ldots,i^{(\upsilon)}$, $j = 1,2,\ldots, j^{(\upsilon)}$, u = 1,2,3, υ = 1,2, in the reliability states subsets {1,2,3}, {2,3}, {3} respectively, have the exponential reliability functions

$$\left[R_{ij}^{(\upsilon)}(t,u)\right]^{(b)} = P\left(\left[T_{ij}^{(\upsilon)}(u)\right]^{(b)} > t\right)$$
$$= \exp\left[-\left[\lambda_{ij}^{(\upsilon)}(u)\right]^{(b)} t\right],$$

for $i = 1,2,...,i^{(\upsilon)}$, $j = 1,2,..., j^{(\upsilon)}$, $u = 1,2,3$, $\upsilon = 1,2$, $t\in\langle 0,\infty)$, with the parameters different at various operation states and presented in *Table 1*.

In the case when the system operation time is large enough its unconditional four-state reliability function is given by the vector

$$\boldsymbol{R}(t,\cdot) = [1, \boldsymbol{R}(t,1), \boldsymbol{R}(t,2), \boldsymbol{R}(t,3)],\ t\in\langle 0,\infty), \qquad (6)$$

where according to (28), the vector coordinates are given respectively by

$$\boldsymbol{R}(t,u) = \sum_{b=1}^{4} p_b [\boldsymbol{R}(t,u)]^{(b)}, \qquad (7)$$

where the coordinates $[\boldsymbol{R}(t,u)]^{(b)}$ are given by (44)-(47) and p_b are the exemplary system operation process transient probabilities at the operation states determined by (9).

According to (9) and (49), the unconditional four-state reliability function (48) coordinates are respectively

$$\begin{aligned}\boldsymbol{R}(t,1) = {} & 0.455(-3\exp[-0.0176t] + 8\exp[-0.0148t] \\ & + 6\exp[-0.0144t] - 6\exp[-0.012t] \\ & - 16\exp[-0.0116t] + 12\exp[-0.0088t]) \\ & + 0.293(\exp[-0.00148t] - 4\exp[-0.0127t] \\ & - 2\exp[-0.0116t] + 6\exp[-0.0106t] \\ & + 8\exp[-0.0095t] - 4\exp[-0.0085t] \\ & - 12\exp[-0.0074t] + 8\exp[-0.0053t]) \\ & + 0.214(-\exp[-0.006t] + 2\exp[-0.003t]) \\ & + 0.038(-\exp[-0.0112t] + 4\exp[-0.0084t] \\ & - 6\exp[-0.0056t] + 4\exp[-0.0028t])\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}(t,2) = {} & 0.455(-3\exp[-0.0188t] + 8\exp[-0.0158t] \\ & + 6\exp[-0.0154t] - 6\exp[-0.0128t] \\ & - 16\exp[-0.0124t] + 12\exp[-0.0094t]) \\ & + 0.293(\exp[-0.00156t] - 4\exp[-0.0134t] \\ & - 2\exp[-0.0122t] + 6\exp[-0.0112t] \\ & + 8\exp[-0.01t] - 4\exp[-0.009t] \\ & - 12\exp[-0.0078t] + 8\exp[-0.0056t]) \\ & + 0.214(-\exp[-0.0062t] \\ & + 2\exp[-0.0031t]) + 0.038(-\exp[-0.012t] \\ & + 4\exp[-0.009t] - 6\exp[-0.006t] + \\ & 4\exp[-0.003t])\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}(t,3) = {} & 0.455(-3\exp[-0.0202t] + 8\exp[-0.0169t] \\ & + 6\exp[-0.0167t] - 6\exp[-0.0136t] \\ & - 16\exp[-0.0134t] + 12\exp[-0.0101t]) \\ & + 0.293(\exp[-0.00162t] - 4\exp[-0.0139t] \\ & - 2\exp[-0.0127t] + 6\exp[-0.0116t] \\ & + 8\exp[-0.0104t] - 4\exp[-0.0093t] \\ & - 12\exp[-0.0081t] + 8\exp[-0.0058t]) \\ & + 0.214(-\exp[-0.0064t]+2\exp[-0.0032t]) \\ & + 0.038(-\exp[-0.0128t] + 4\exp[-0.0096t] \\ & - 6\exp[-0.0064t] + 4\exp[-0.0032t])\end{aligned}$$

The graph of the four-state exemplary system reliability function is illustrated in *Figure 2*.

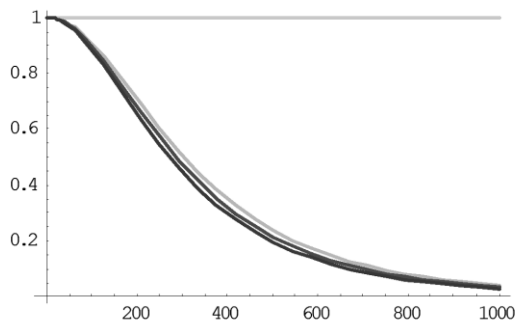

Figure 2. The graph of the exemplary system reliability function $\boldsymbol{R}(t,\cdot)$ coordinates

Table 1. Exemplary system parameters $\left[\lambda_{ij}^{(\upsilon)}(u)\right]^{(b)}$ for particular subsystems at the particular operation states

$\left[\lambda_{ij}^{(\upsilon)}(u)\right]^{(b)}$	u	1			2			3		
	j	1	2	3	1	2	3	1	2	3
S_υ	i	z_1, z_2								
S_1	{1,2}	0.000	0.001	0.001	0.000	0.001	0.001	0.000	0.0012	0.001
S_2	{1,2,3,4}	0.001	0.001	-	0.001	0.001	-	0.001	0.0017	-
z_3										
S_1	{1,2}	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.0013	0.001
S_2	{1,2,3,4}	0.000	0.001	-	0.001	0.001	-	0.001	0.0013	-
z_4										
S_1	{1,2}	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.0013	0.001
S_2	{1,2,3,4}	0.001	0.001	-	0.001	0.001	-	0.001	0.0018	-

The expected values and standard deviations of the system unconditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, according to (29), (31) and considering (30), respectively are:

$$\mu(1) \cong 377.294, \mu(2) \cong 357.542,$$

$$\mu(3) \cong 341.379, \quad (50)$$

$$\sigma(1) \cong 288.000, \sigma(2) \cong 275.011,$$

$$\sigma(3) \cong 261.614. \quad (51)$$

Further, considering (32), the mean values of the system unconditional lifetimes in the particular reliability states 1, 2, 3, respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) = 19.752,$$

$$\bar{\mu}(2) = \mu(2) - \mu(3) = 16.163,$$

$$\bar{\mu}(3) = \mu(3) = 341.379.$$

(52)

Since the critical reliability state is r =2, then the system risk function, according to (33), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{R}(t,2),\ t \in \langle 0, \infty),$$

where $\boldsymbol{R}(t,2)$ is given by (50).

Hence, by (34), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 68.572.$$

The graph of the risk function $\boldsymbol{r}(t)$ of the exemplary four-state system operating at the variable conditions is given in *Figure 3*.

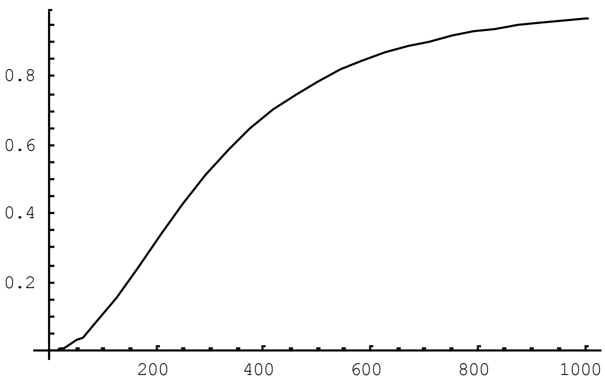

Figure 3. The graph of the exemplary system risk function $\boldsymbol{r}(t)$

6. Conclusions

Presented in this paper results are partly coming from the general analytical models of complex technical multi-state systems reliability [2]. The paper delivers the procedures and algorithms that allow to find the main an practically important reliability characteristics of the complex technical systems with independent components at the variable operation condition. The application of the presented analitical model to reliability prediction of an exemplary system yields its reliability characteristics approximate evaluation that are close to its suitable reliability characteristics obtained by Monte Carlo simulation method application presented in [3].

Acknowledgements

The paper presents the results developed in the scope of the EU-CIRCLE project titled "A pan – European framework for strengthening Critical Infrastructure resilience to climate change" that has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 653824. http://www.eu-circle.eu/

References

[1] Grabski, F. (2015). *Semi-Markov Processes: Applications in System Reliability and Maintenance.* Elsevier.

[2] Kołowrocki, K. (2014). *Reliability of large and complex systems*, Elsevier.

[3] Kołowrocki, K., Kuligowska, E. & Soszyńska-Budny, J. (2015). Monte Carlo simulation application to reliability assessment of an exemplary system operating at variable conditions. *Journal of Polish Safety and Reliability Association, Summer Safety and Reliability Seminars*, 6, 1, 2015,137-144.

[4] Kołowrocki, K. & Soszyńska-Budny, J. (2011). *Reliability and Safety of Complex Technical Systems and Processes: Modeling – Identification – Prediction – Optimization*. Springer.

[5] Soszyńska, J. (2007). Systems reliability analysis in variable operation conditions. International *Journal of Reliability, Quality and Safety Engineering.* Special Issue: System Reliability and Safety, 14, 6, 617-634.

[6] Xue, J. (1985). On multi-state system analysis. *IEEE Transactions on Reliability* 34, 329-337.

[7] Yu, K., Koren, I., Guo, Y. (1994). Generalised multistate monotone coherent systems. *IEEE Transactions on Reliability* 43, 242-250.