Analysis of parallel resonance RLC_α circuit with supercapacitor

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The paper describes the results of analysis on the resonance phenomenon in a parallel RLC_α circuit with supercapacitor modeled as a fractional-order capacitance. A complex, more accurate fractional-order model has been taken into analysis. Formulas for frequency characteristics and resonance conditions have been derived and presented by simulation studies of the concerned fractional-order system.

KEYWORDS: phase resonance, supercapacitor, parallel RLC_α circuit

1. Supercapacitor fractional-order models

Many electrical models of supercapacitor exist in literature [1-2], derived from classic impedance models. In modern electrical engineering the concept of fractional-order reactive elements L_β, C_α has been introduced, describing real, lossy coils [3] and capacitors, eg. supercapacitors [4]. Fractional-order real inductance and supercapacitor impedance in frequency domain in the simplest, but also less accurate case is described by relations [3], [5-6]:

$$Z_L(j\omega) = R_L + (j\omega)^\beta L \qquad \beta \in R^+ \qquad (1)$$

and:

$$Z_C(j\omega) = R_C + (j\omega)^{-\alpha} C^{-1} \qquad \alpha \in R^+ \qquad (2)$$

where: R_L, R_C - internal series resistances α, β - fractional-order parameters.

The resistance R_C for supercapacitor is defined as one of their basic component - ESR - the Equivalent Series Resistance. It models energy losses on leads and is of a few dozen to fractions of ohms. The higher supercapacitor capacitance is, the lower is its ESR resistance. [7].

Model given by relation (2) is accurate only for a specific range of frequencies, up to approx. 215 mHz. Therefore a more complex fractional-order model has been introduced, called model based on fractional poles and zeros, given as [8]:

$$Z(j\omega) = R_C + k \frac{\left(1 + j\frac{\omega}{\omega_0}\right)^\alpha}{(j\omega)^\beta} \qquad (3)$$

where: k - parameter describing the inverse of the capacitance C, ω_0 - radial

frequency, at which the impedance phase changes, *α, β* - fractional-order parameters.

The above model describes well supercapacitor impedance to the frequency of about 100 Hz. It is possible to select such values of the parameters *L* and C_α to observe the resonance phenomenon in the considered circuit. The paper [9] describes resonance conditions in the ideal parallel $L_\beta C_\alpha$ circuit. Resonance phenomenon in a simple series circuit with supercapacitor, modeled as a fractional-order reactive element has been examined in [10-11]. The same problem has been analyzed with real, fractional-order coil [12]. The paper [13] is the first part of the resonance phenomena analysis in a parallel circuit. The article is a continuation of resonance phenomena analysis in parallel circuit of *RLC$_\alpha$* class with supercapacitor in frequency domain, using more complex fractional-order model, based on fractional poles and zeros, given by formula (3).

2. Model of the system

Analyzed parallel *RLC$_\alpha$* model with supercapacitor is presented in Fig. 1:
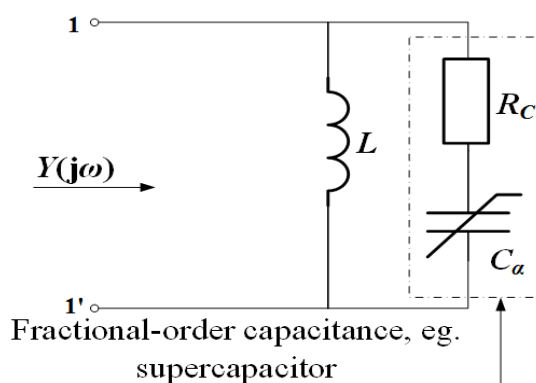

Fig. 1. Parallel *RLC$_\alpha$* circuit with supercapacitor

It consists of a lossless coil with an inductance *L* and a real, fractional - order supercapacitor C_α of impedance given by relation (3).

The equivalent admittance of the circuit from Fig. 1 seen from the terminals 1 - 1' can be written as:

$$Y(j\omega) = \frac{1}{j\omega L} + \frac{1}{R_C + k\dfrac{\left(1 + j\dfrac{\omega}{\omega_0}\right)^\alpha}{(j\omega)^\beta}} \quad .$$

(4)

Transforming formula (4), finally the equivalent admittance can be obtained in the canonical form:

$$Y(\mathrm{j}\omega) = \mathrm{Re}\{Y(\mathrm{j}\omega)\} + \mathrm{j}\cdot\mathrm{Im}\{Y(\mathrm{j}\omega)\}, \qquad (5)$$

as:

$$\mathrm{Re}\{Y(\mathrm{j}\omega)\} = \dfrac{R_C + \dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\cos\left(\alpha\,\mathrm{arctg}\left(\dfrac{\omega}{\omega_0}\right)-\dfrac{\pi}{2}\beta\right)}{R_C^2 + \dfrac{k^2}{\omega_0{}^{2\alpha}}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\alpha}}{\omega^{2\beta}} + 2R_C\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\cos\left(\alpha\,\mathrm{arctg}\left(\dfrac{\omega}{\omega_0}\right)-\dfrac{\pi}{2}\beta\right)}, \qquad (6)$$

and:

$$\mathrm{Im}\{Y(\mathrm{j}\omega)\} = -\left(\dfrac{\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\sin\left(\alpha\,\mathrm{arctg}\left(\dfrac{\omega}{\omega_0}\right)-\dfrac{\pi}{2}\beta\right)}{R_C^2 + \dfrac{k^2}{\omega_0{}^{2\alpha}}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\alpha}}{\omega^{2\beta}} + 2R_C\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\cos\left(\alpha\,\mathrm{arctg}\left(\dfrac{\omega}{\omega_0}\right)-\dfrac{\pi}{2}\beta\right)} + \dfrac{1}{\omega L}\right). \qquad (7)$$

Equivalent admittance module $|Y(\mathrm{j}\omega)|$ and phase $\varphi(\omega)$ of the parallel RLC_α circuit with supercapacitor model based on fractional poles and zeros are given by relations:

$$|Y(\mathrm{j}\omega)| = \sqrt{\dfrac{1 + 2\dfrac{1}{\omega L}\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\sin\left(\delta(\omega)\right)}{R_C^2 + \dfrac{k^2\left(\omega_0{}^2+\omega^2\right)^{\alpha}}{\omega_0{}^{2\alpha}\omega^{2\beta}} + 2R_C\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\cos\left(\delta(\omega)\right)} + \left(\dfrac{1}{\omega L}\right)^2}, \qquad (8)$$

where:

$$\delta(\omega) = \alpha\,\mathrm{arctg}\left(\dfrac{\omega}{\omega_0}\right) - \dfrac{\pi}{2}\beta, \qquad (9)$$

and:

$$\varphi(\omega) = -\mathrm{arctg}\dfrac{\left(\dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\left(\sin(\delta(\omega))+2\dfrac{1}{\omega L}R_C\cos(\delta(\omega))\right)+\dfrac{1}{\omega L}\left(R_C^2+\dfrac{k^2}{\omega_0{}^{2\alpha}}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\alpha}}{\omega^{2\beta}}\right)\right)}{R_C + \dfrac{k}{\omega_0{}^\alpha}\dfrac{\left(\omega_0{}^2+\omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta}\cos(\delta(\omega))}. \qquad (10)$$

Derived relations describing admittance of the analyzed parallel RLC_α circuit have been simulated and illustrated on graphs shown in Figs. 2 - 5, as a function of single variable - radial frequency ω with given parameter α value and as a function of two variables α and ω.
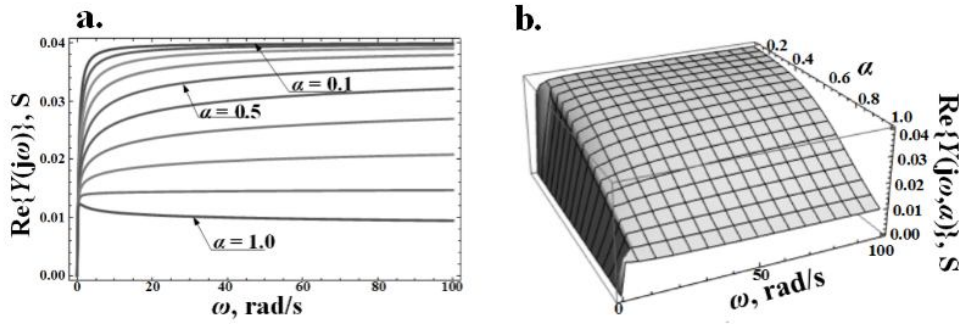
Fig. 2. Graphs of the function based on formula (6) for $\alpha \in <0,1>$: a. Re{$Y(j\omega)$}, b. Re{$Y(j\omega,\alpha)$}
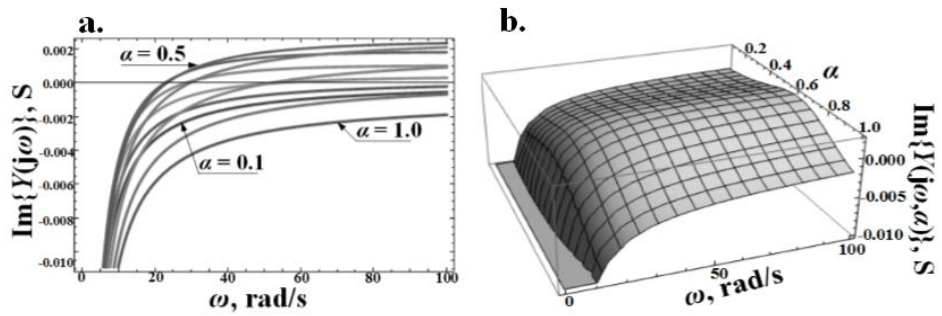
Fig. 3. Graphs of the function based on formula (7) for $\alpha \in <0,1>$: a. Im{$Y(j\omega)$}, b. Im{$Y(j\omega,\alpha)$}
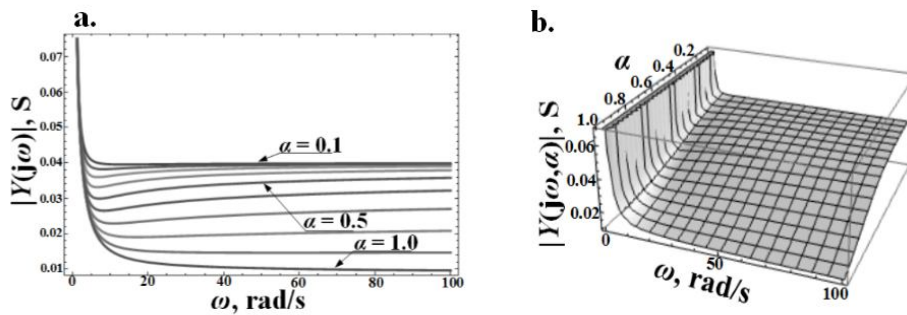

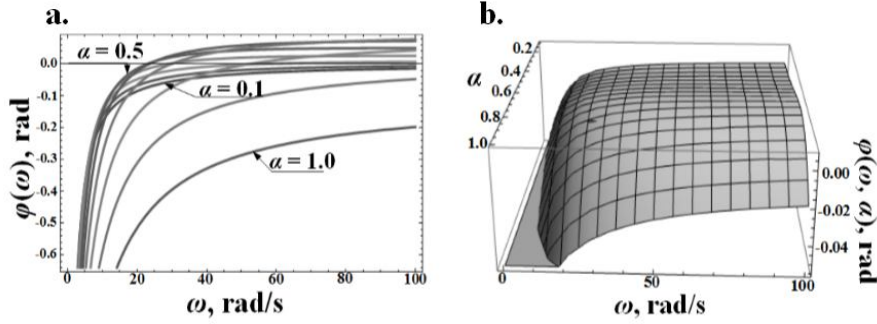Fig. 4. Graphs of the function based on formula (8) for $\alpha \in <0,1>$: a. |$Y(j\omega)$|, b. |$Y(j\omega,\alpha)$|

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Fig. 5. Graphs of the function based on formula (10) for $\alpha \in <0,1>$: a $\varphi(\omega)$, b. $\varphi(\omega,\alpha)$

Simulations have been conducted for real values of parameters of the parallel RLC_α circuit: coil of inductance $L = 10$ H, and a Panasonic supercapacitor with a nominal capacitance $C = 0.1$ F and determined parameter values: $\beta = 0.92$, $\omega_0 = 0.1958$, $k = 10.87$, $R_C = 25$ Ω [5]. Experiments on supercapacitors prove that the coefficient α changes widely within a range of $<0,1>$, but the value of β is generally more or less constant, of about 1 [8]. Therefore, a fixed value of the parameter β has been implemented for simulations.

3. Analysis of phase resonance conditions

Resonance radial frequency for the discussed circuit of RLC_α class with supercapacitor, can be determined from the general phase resonance condition $\mathrm{Im}\{Y(j\omega)\} = 0$. This condition leads to a nonlinear equation, of variable ω, given in the form of:

$$R_C^2 + \frac{k^2}{\omega_0^{2\alpha}} \frac{\left(\omega_0^{\,2} + \omega^2\right)^\alpha}{\omega^{2\beta}} + \frac{k}{\omega_0^{\,\alpha}} \frac{\left(\omega_0^{\,2} + \omega^2\right)^{\frac{\alpha}{2}}}{\omega^\beta} \left(\omega L \sin\left(\delta(\omega)\right) + 2R_C \cos\left(\delta(\omega)\right)\right) = 0 , \qquad (11)$$

where $\delta(\omega)$ is defined by formula (9).

The second general phase resonance condition $\mathrm{Im}\{Z(j\omega)\} = 0$ leads to the same equation as formula (11). It can be noticed that the determination of resonance radial frequency ω_{rez} in a closed solution, based on formula (11) is not possible, because formula (11) is a transcendental equation. Its value can be found numerically.

For the analyzed parallel RLC_α circuit with fractional-order capacitor (eg. supercapacitor), resonance radial frequency have been determined, for α coefficient changing within $\alpha \in <0,1>$, which have been presented in graphs from Figs. 6-7. Simulations have been performed for selected parameter values: the inductance L and the series resistance R_C of the circuit.
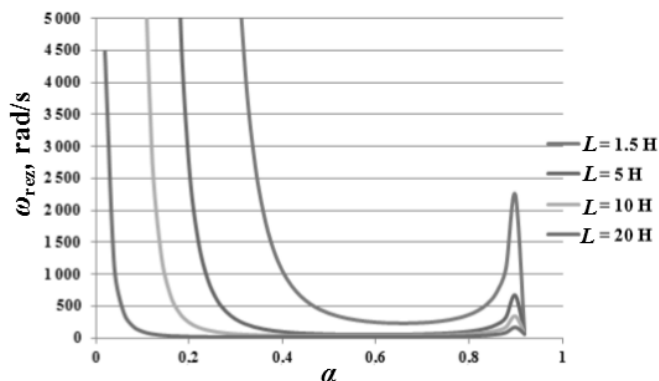
Fig. 6. Determined resonance radial frequency ω_{rez} as a function of $\alpha \in <0,1>$ for selected inductance L values in the analyzed circuit
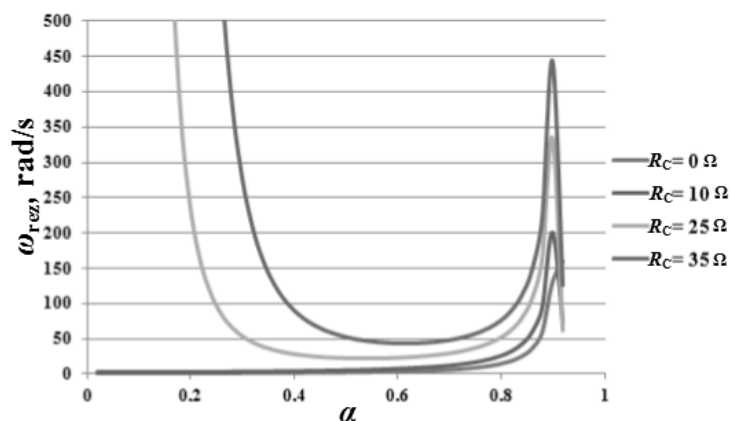

Fig. 7. Determined resonance radial frequency ω_{rez} as a function of $\alpha \in <0,1>$ for selected values of the supercapacitor internal series resistance R_C in the analyzed circuit

The maximum of resonance radial frequency can be read from Figs. 6-7, which shifts towards higher values, when the value of the inductance L is lower, or the internal series resistance R_C is higher. The maximum of resonance radial frequency appears when the fractional-order parameter equals $\alpha \approx 0.9$. The simulation also shows that for certain circuit parameter values, the resonance frequency does not exist, when the coefficient $\alpha > 0.92$. Then, equation (11) has no solution in the set of positive real numbers. The existence of resonance radial frequency is not influenced by the inductance L, or the resistance R_C, value change. These parameters have an influence only on the resonance radial frequency ω_{rez} value.

4. Summary

The article presents an analysis of resonance phenomenon in a parallel RLC_α circuit, with a capacitor modeled as a fractional-order element (eg.supercapacitor). A more complex as well as accurate fractional-order model has been used to the analysis, called model based on fractional poles and zeros. Relations for the admittance of the circuit and for the resonance frequency have been derived. It depends on six parameters: the inductance L, values of parameters: k, α, β, ω_0 and the internal series resistance R_C. Formula describing the resonance radial frequency has a transcendental form, so it can be solved effectively in a numerical way. The shape of the resonance radial frequency dependence on parameter α for the analyzed circuit with supercapacitor is not symmetrical, for small values of coefficient α the resonance frequency reaches very high values, but for $\alpha > 0.92$ it does not exist (equation (11) does not have a solution in a set of positive real numbers), and for $\alpha \approx 0.9$ it reaches the local maximum. Using more accurate fractional-order supercapacitor model with two coefficients of fractional order makes the description of phase resonance phenomenon in parallel RLC_α circuit more complex.

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