### **Symbolic integration with respect to the Haar measure on the unitary groups** *DOI: 10.1515/bpasts-2016-00ZZ*  $S = \frac{1}{2}$  integration with respect to the unitary on the unitary measure on the unitary on the unitary on the unitary of  $S = \frac{1}{2}$

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Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, 5 Bałtycka Str., 44-100 Gliwice, Poland *DOI: 10.1515/bpasts-2016-00ZZ* BULLETIN OF THE POLISH ACADEMY OF SCIENCES

Abstract. We present IntU package for Mathematica computer algebra system. The presented package performs a symbolic integration of Abstract. We present into package for Maniematica computer algebra system. The presented package performs a symbolic integration of polynomial functions over the unitary group with respect to unique normalized Haar measure can be used to optimize the calculation speed for some classes of integrals. We also provide some examples of usage of the presented package. package.  $p_{\text{wansion}}$  group with respect to unique normalized Haar measure. We describe a number of special cases which a number of special cases which a number of special cases which cases which cases which cases which cases which

package.<br>**Key words:** unitary group, Haar measure, circular unitary ensemble, symbolic integration. <sup>1</sup> Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, Bałtycka 5, 44-100 Gliwice, Poland Key words: unitary group, Haar measure, circular unitary ensemble, symbolic integration.

# **1. Introduction** 1. Introduction

The integration over unitary group is an important subject 2.1. Basic concepts. By  $M_n$  we denote square matrices of size n. of studies in many areas of science, including mathematical The compact group of  $d \times d$  unitary matrices is denoted as U(d). 2006 Collins and Sniady [1] proved a formula for calculating sure denoted by  $dU$ . Random elements distributed with measure monomial integrals with respect to the Haar measure on the<br>unitary group physics, random matrix theory and algebraic combinatorics. In We equip the above group with unique normalized Haar mea- $\frac{2000 \text{ columns and small}}{2000 \text{ columns}}$  and  $\frac{1}{2}$  proved a formula for calculating  $\frac{1}{2}$  sure denoted integrals with respect to the Haar measure on the  $\frac{dU}{dV}$  form the The integration over unitary group is an important subject of studies in many areas of science, including mathematical The integration over unitary group is an important subject

$$
\int_{\mathrm{U}(d)} U_{IJ} \overline{U}_{I'J'} dU = \int_{\mathrm{U}(d)} U_{i_1 j_1} \dots U_{i_n j_n} \overline{U}_{i'_1 j'_1} \dots \overline{U}_{i'_n j'_n} dU. (1)
$$

Integrals of the above type, called moments of the  $U(d)$ , have been known in mathematical physics literature for a long tir was considered for the first time in the context of nuclear The problem of physics in [2]. The asymptotic behaviour of the integrals of the type  $(1)$  was considered by Weingarten in [3]. physics in [2]. The asymptotic behaviour of the integral<br>the type (1) was considered by Weingarten in [3]. integrals of the above type, called moments of the  $O(a)$ , have<br>been known in mathematical physics literature for a long time. The problem of the integration of elements of unitary matrices

In this paper we describe fine, a maintenance package  $\left[\cdot\right]$ <br>for calculating polynomial integrals over  $U(d)$  with respect to the Haar measure. We describe a number of special cases which can be used to optimize the calculation speed for some classes of integrals. We also provide some examples of usage of the presented package, including the applications in the study of the geometry of the quantum states.  $\Gamma$  from the linearity of an integral we have have have  $\Gamma$ In this paper we describe IntU, a Mathematica package [4]

This paper is organised as follows. In Section 2, we integral paper is organised as follows. auce notation and present the mathematical background of po  $\frac{1}{2}$  formal megrans over unitary group. In section s, we descri more efficiently. In Section 4 we provide the description of Intervalse in the second temple<br>Intervalse with the list of main functions. In Section 5 we<br>show some examples of the usage. In Section 6 we provide show some examples of the usage. In Section 6 we provide a summary of the presented results and conclusions. study of the geometry of the quantum states. This paper is organised as follows. In Section 2, we introduce notation and present the mathematical background of polynomial integrals over unitary group. In Section 3, we describe some special cases, in which the integration can be calculated

mial integrals over unitary group. In Section 3 we describe 3 we describe 3 we describe 3 we describe 3 we des

notation present mathematical background concerning polyno-

#### **1** Institute of Theoretical and Applied Intervention of Theoretical and Applied Intervention of Theoretical and Applied Intervention of Sciences, Polando Intervention of Sciences, Polando Intervention of Sciences, Polando Integer partition λ of a positive integer *n* is a weakly de-

We equip the above group with unique normalized Haar measure denoted by *dU*. Random elements distributed with measure *dU* form the so-called circular unitary ensemble.

Integer partition *λ* of a positive integer *n* is a weakly de-Integer partition λ of a positive integer *n* is a weakly decreasing sequence  $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$  of positive integers, such<br>the strategies of the strateg that  $\sum_{i=1}^{l} \lambda_i = |\lambda| = n$ . To denote that  $\lambda$  is a partition of *n*, we<br>that  $\sum_{i=1}^{l} \lambda_i = |\lambda| = n$ , then the second that  $\lambda$  is a partition of *n*, we write  $\lambda \vdash n$ . The length of a partition is denoted by  $l(\lambda)$ . By  $\lambda \Box \mu$  we denote a partition of  $n_1 + n_2$  obtained by joining partitions  $\lambda \vdash n_1$  and  $\mu \vdash n_2$ titions  $\lambda \vdash n_1$  and  $\mu \vdash n_2$ . meger partition  $\lambda$  of a positive integer *n* is a weakly decomposed  $\lambda = (1, 1, 1)$  of positive integers such  $\Box$  *u* we denote a partition of  $n_1 + n_2$  obtained by joining par-

Each permutation  $\sigma \in S_n$  can be uniquely decomposed into a cum of disjoint quality where the langths of the quality sum a sum of disjoint cycles, where the lengths of the cycles sum up to *n*. Thus the vector of the lengths of the cycles sum up to *n*. Thus the vector of the lengths of the cycles, after reup to *n*. Thus the vector of the lengths of the cycles, after re-<br>ordering, forms a partition  $\lambda \vdash n$ . The partition  $\lambda$  is called the  $\frac{1}{\alpha}$  ordering, forms a partition  $\lambda + n$ . The partition  $\lambda$  is called the cycle type of *σ* permutation.  $t_{\rm y}$  the cycle type of permutation.  $\alpha$  is  $\lambda + n_1$  and  $\mu + n_2$ .<br>Each nermutation  $\sigma \in S_n$  can be uniquely decomposed into  $\frac{dy}{dx}$  to  $\frac{dy}{dx}$  of  $\sigma$  permutation. U(*d*)

**2.2. Moments of the U(***d*). Let us consider a polynomial *p*. 2.2. Moments of the  $U(u)$ . Let us consider a polynomial *p*.<br>From the linearity of an integral we have *p*. The mixture of an integral we have

$$
\int_{U(d)} p(U)dU = \sum_{I,J,I',J'} c(I,J,I',J') \int_{U(d)} U_{IJ} \overline{U}_{I'J'} dU, (2)
$$

where  $I, J, I', J'$  are multi-indices and  $c$  are the coefficients of *p*. The value of such monomial integrals is given as [1] *I*, *I*, *I*, *I*, are mult *p*. The value of such monomial integrals is given as [1]

$$
\int_{U(d)} U_{IJ} \overline{U}_{I'J'} dU = \int_{U(d)} U_{i_1j_1} \dots U_{i_nj_n} \overline{U}_{i'_1j'_1} \dots \overline{U}_{i'_nj'_n} dU =
$$
\n
$$
= \sum_{\sigma,\tau \in S_n} \delta_{i_1,i'_{\sigma(1)}} \dots \delta_{i_n,i'_{\sigma(n)}} \delta_{j_1,j'_{\tau(1)}} \dots \delta_{j_n,j'_{\tau(n)}} Wg(\tau \sigma^{-1},d),
$$
\n(3)

where W*g* is the Weingarten function discussed below. In the case of the multi-indices differing in length, *i.e.*  $n \neq n'$ , we have have where  $Wg$  is the Weingarten function discussed below. In the ages of the multi-indiges differing in langth i.e.  $n \neq n'$ , we have case of the multi-indices differing in length, i.e.  $n \neq n'$ , we have

<sup>λ</sup>*<sup>n</sup> <sup>l</sup>*(λ)≤*<sup>d</sup>*

 $^*$ e-mail: z.puchala@iitis.pl  $2.$  Mathematical background

$$
\int_{U(d)} U_{IJ} \overline{U}_{I'J'} dU =
$$
\n
$$
= \int_{U(d)} U_{i_1 j_1} \dots U_{i_n j_n} \overline{U}_{i'_1 j'_1} \dots \overline{U}_{i'_{n'} j'_{n'}} dU = 0.
$$
\nThus we have

2.1. Basic concepts We denote by M*<sup>n</sup>* square matrices of size 2.3. Weingarten function The *Weingarten function* The *Weingarten function*  $\frac{1}{2}$  is defined in  $\frac{1}{2}$ fined for σ ∈ *Sn* and positive integer *d*, as The integrals of the above type are known as *moments of the* The integrals of the above type are known as moments of the The integrals of the above type are known as *moments of the* U(*d*). U(*d*).

*n*. The compact group of *d* ×*d* unitary matrices we denote as U(*d*). We equip the above group with unique normalized Haar measure denoted by *dU*. Random elements distributed with 2.5. We ingerted function. The weingation function [1] is de-<br>fined for  $\sigma \in S_n$  and positive integer d as 2.3. Weingarten function. The Weingarten function [1] is deχ<sup>λ</sup> (σ), (5) fined for σ ∈ *Sn* and positive integer *d*, as

$$
Wg(\sigma, d) = \frac{1}{(n!)^2} \sum_{\substack{\lambda \vdash n \\ l(\lambda) \le d}} \frac{\chi^{\lambda}(e)^2}{s_{\lambda, d}(1)} \chi^{\lambda}(\sigma),
$$
 (5) 3. Sp

!!<br>.. where the sum is taken over all integer partitions of *n* with where the sum is taken over an integer partitions of *n* with the<br>length  $l(\lambda) \leq d$ ,  $s_{\lambda}$ ,  $d(1)$  is the Schur polynomial  $s_{\lambda}$  evaluated under at  $(\underline{1}, \underline{1}, \dots, \underline{1})$  and  $\chi^{\lambda}$  is an irreducible character of the sym-<br>  $\chi^{\lambda}$ .  $\chi^{\lambda}$ *d* metric group **Sn** indexed by particular structure  $\frac{d}{dt}$ metric group **Sn** indexed by partition  $\frac{d}{dx}$  $I(J)$ , respectively. a d  $\frac{d}{dx}$  or  $S_n$  indexed by partition  $\lambda$ 

metric group *Sn* indexed by partition *λ*.  $\frac{1}{1}$  $\epsilon$  group  $\epsilon_n$  indexed by partition  $\lambda$ .  $\overline{a}$  is an irreducible character of the symmetric chara

 $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ 

**2.3.1. The dimension of irreducible representation of U**(*d*). The value of the Schur polynomial at the point  $(1, 1, ..., 1)$ ,  $\leftharpoondown$ *d* i.e. the dimension of irreducible representation of  $U(d)$  corresponding to partition  $\lambda$ , is equal to e.g. Theorem 6.3 in [5] 2.3.1. The dimension of irreducible representation of U(*d*) **Z.3.1.** The dimension of irreducible representation of  $U(d)$ . 2.3.1. The dimension of irreducible representation of U(*d*) The value of the Schur polynomial at the point (1,1,...,1 The value of the Schur polynomial at the point  $(1, 1, ..., 1)$ , **2.3.1.** The dimension of irreducible representation of  $U(d)$ . by The value of the Schwa palumential at the neight  $(1, 1, 1)$ The value of partition  $\lambda$ , is equal to e.g. Theorem  $0.5$  in [5] *<sup>d</sup>* 1. The dimension of include the presentation of U( $\frac{1}{2}$  is equal to e.g. Theorem 6.2  $T$  responding to partition  $\lambda$ , is equal to e.g. Theorem  $0.3$  in  $\beta$ 2.3.1. The dimension of irreducible representation of U(*d*) The value of the scheme of the Schur polynomial at the Schur polynomial at the  $T_1$ ,  $T_2$ ,  $T_3$ ,  $T_4$ ,  $T_5$ ,  $T_6$ ,  $T_7$ ,  $T_8$ ,  $T_9$ ,  $T$  *<sup>d</sup>* e. the dimension of inequation repres

$$
s_{\lambda,d}(1) = s_{\lambda}(\underbrace{1,1,\ldots,1}_{d}) = \prod_{1 \le i < j \le d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.\tag{6}
$$

2.3.2. Irreducible character of  $S_n$ . The irreducible character of **2.3.2. Irreducible character of**  $S_n$ **.** The irreducible character of  $S_n$  indexed by partition  $\lambda$ ,  $\chi^{\lambda}(\sigma)$  depends on a conjugacy class of permutation  $\sigma$ . Two permutations are in the same conjugacy.  $\lambda_n$  indexed by partition  $\lambda$ ,  $\chi$ <sup>-</sup>(*b*) depends on a conjugacy class<br>of permutation  $\sigma$ . Two permutations are in the same conjugacy of permutation b. Two permutations are in the same conjugacy<br>class if and only if they have the same cycle type, thus it is class if and only if they have the same cycle type, thus it is<br>common to write  $\chi^{\lambda}(\sigma) = \chi^{\lambda}(\mu)$ , where  $\mu$  is an integer partition common to write  $\chi(\theta) = \chi(\mu)$ , where  $\mu$  is an integer partite corresponding to the cycle type of  $\sigma$ . common to write  $\chi^{\alpha}(0) = \chi^{\alpha}(\mu)$ , where  $\mu$  is an integer partition<br>corresponding to the cycle type of  $\sigma$ 2.3.2. Irreducible character of  $S_n$ . The irreducible c<br>S indexed by partition  $\frac{1}{2}x^{\lambda}(\sigma)$  depends on a conju Class II and only II they have the same cycle type, thus I<br>common to write  $\gamma^{\lambda}(\sigma) = \gamma^{\lambda}(\mu)$ , where u is an integer partit e**ducible c**l<br>d by **newit**  $\mathcal{L}$ **3.2. Irreducible character of**  $S_n$ . The irreducible character  $\sum_{\mu}$   $\sum_{\mu}$   $\sum_{\mu}$   $\sum_{\mu}$ , where  $\mu$  is an integer partition

Exercise to the case of identity permutation the cycle type is given by a trivial partition,  $e = (1, 1, ..., 1)$ , and the character value by a trivial partition,  $e = \underbrace{(1,1,1,...,1)}_{n}$ , and the character value<br>is equal to the dimension of the irreducible representation of  $\frac{1}{n}$ by a trivial partition,  $e = (1, 1, ..., 1)$ , and the character value  $\sum_{n}$  $\sum_{n}$  $\overline{h}$  $y$  a divide parameter,  $c = (1, 1, ..., 1)$ , and the endiately value  $\frac{n}{\sqrt{n}}$ 

is equal to the dimension of the irreducible representation of *S<sub>n</sub>* indicent line is a single representation of  $S_n$  indexed by  $\lambda$ . In this case it is given by the celebrated hook<br>langth formula (see Eq. 4.12 in [5])  $L_n$  indexed by  $\lambda$ . In this case it is given by the celebrated hook<br>length formula (see Eq 4.12 in [5]) **length** formula  $\frac{1}{2}$ is equal to the dimension of the irreducible  $S_{\text{total}}$  type of  $\mathcal{L}_{\text{total}}$ a trivial partition, *<sup>e</sup>* <sup>=</sup> {1,1,...,<sup>1</sup> *<sup>n</sup>*  $\frac{1}{2}$  and  $\frac{1}{2}$ length formula (see Eq 4.12 <sub>11</sub>  $\left[5\right]$ ) is equal to the dimension of the irreducible representation rengin formula (see Eq.  $4.12 \text{ m} \left[ \nu \right]$  $\left[ \begin{matrix} 2 \end{matrix} \right]$  $S_n$  indexed by  $\lambda$ . In this contained by  $\lambda$ . In this contained by  $\lambda$ . participant to the corresponding to the medicine te Intexted by  $\kappa$ . In this case it is given by the celebrated hook a trivial partition, *exc* Eq. 1,12 *in*  $\begin{bmatrix} 5 \end{bmatrix}$  $\lambda$ 

$$
\chi^{\lambda}(e) = \frac{|\lambda|!}{\prod_{i,j} h_{i,j}^{\lambda}},\tag{7}
$$

where  $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{N+1}$  and  $h^{\lambda}$ , *is the hook length of the* where  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_{l(\lambda)}$ , and  $h_{i,j}^{\lambda}$  is the hook length of the<br>call (*i, i*) in a Ferrers diagram corresponding to partition 1.161 where  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_{l(\lambda)}$ , and  $h_{i,j}^{\lambda}$  is the hook length of the cell  $(i, j)$  in a Ferrers diagram corresponding to partition  $\lambda$  [6].

Let  $(1, 1)$  in a reflect diagram corresponding to partition  $\lambda$  [0].<br>In the case of a non-trivial partition the character of sym-In the case of a non-trivial partition the character of sym-<br>metric group  $\chi^{\lambda}(\sigma) = \chi^{\lambda}(\mu)$  can be evaluated with the use of metric group  $\chi^{\alpha}(\sigma) = \chi^{\alpha}(\mu)$  can be evaluated with the use of<br>Murnaghan-Nakayama rule (see Theorem. 4.10.2 in [7]), which  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  are  $\frac{1}{2}$  (see Theorem: 1.10.2 in [1]), which describes a combinatorial way of calculating the character; [8] is used.  $\alpha$  combinatorial way of calculating the character.  $\alpha$ see *e.g.* is used.  $\frac{1}{10}$  is used.  $\mathbf{I}$  sed.

*during the distribution* only on a cycle type of a permutation  $\sigma$  and thus it is constant on a conjugacy class represented by *σ*. From the above considerations one can deduce that the We-Thus we may define the Weingarten function as  $\frac{1}{2}$ . σ. Thus we may define the Weingarten function as

$$
Wg(\mu, d) = \frac{1}{(|\mu|!)^2} \sum_{\substack{\lambda \vdash |\mu| \\ l(\lambda) \le d}} \frac{\chi^{\lambda}(e)^2}{s_{\lambda, d}(1)} \chi^{\lambda}(\mu),
$$
 (8)

e- where  *is an integer partition, which is a cycle type of*  $σ$ *.* 

## 3. Special cases **3. Special cases**  $\mathcal{I}(\mathbf{I})$

In this section we present some operator suses of integrals with-<br>respect to the Haar measure on the unitary group. In these cases usage of (3), which requires processing of  $\prod_i^d k_i! \prod_j^d l_j!$  permutations where  $k_i(l_i)$  denotes the number of  $i(i)$  in multi-indices functions, where  $k_i(l_j)$  denotes the number of  $i(j)$  in multi-indices In this section we present some special cases of integrals with the value of the integral can be calculated without the direct  $\frac{d}{dx}$ 1- tations, where  $k_i(l_j)$  denotes the number of  $i(j)$  in multi-indices  $I(I)$  respectively.

The presented special cases have been implemented in the case of monomials of m package to increase its efficiency. This goal has been achieved package to increase its emerging, This goal has been achieved to  $U(d)$ . by minimizing the number of calculations of the Weingarten  $T(\sigma)$ , respectively.<br>The presented special cases have been implemented in the  $2$ , function. function.  $\,$ , function.  $\mathbf{u}$  minimizing the number of  $\mathbf{u}$  the Weingartens of the Wei

 $r$ - 3.1. First two moments of U(*d*). In the case of monomials of First two moments of  $C(u)$ . In the case of monomials of<br>  $[5]$  rank equal to 2 and 4, from [9] we have **5.1. FIRST two moments of U(a).** In the case of monomials of  $\frac{3.1}{2}$  First two moments of  $U(d)$ . In the case of monomials of δ*ii*δ*j j*, (9)

$$
\int_{U(d)} u_{ij} \overline{u}_{i'j'} dU = \frac{1}{d} \delta_{ii'} \delta_{jj'},
$$
\n(9)

and and and

$$
\int_{U(d)} u_{i_1 j_1} u_{i_2 j_2} \overline{u}_{i'_1 j'_1} \overline{u}_{i'_2 j'_2} dU =
$$
\n
$$
\sum_{\substack{S \text{SS} \\ S' \text{S}}} = \frac{\delta_{i_1 i'_1} \delta_{i_2 i'_2} \delta_{j_1 j'_1} \delta_{j_2 j'_2} + \delta_{i_1 i'_2} \delta_{i_2 i'_1} \delta_{j_1 j'_2} \delta_{j_2 j'_1}}{d^2 - 1} + (10)
$$
\n
$$
\sum_{\substack{S \text{S} \\ S' \text{S} \\ \overline{u}_1 \overline{u}_1 \overline{u}_1 \delta_{i_2 i'_2} \delta_{j_1 j'_2} \delta_{j_2 j'_1} + \delta_{i_1 i'_2} \delta_{i_2 i'_1} \delta_{j_1 j'_1} \delta_{j_2 j'_2}}{d(d^2 - 1)},
$$
\n
$$
\sum_{\substack{S' \text{S} \\ S' \text{S} \\ \overline{u}_1 \delta_{i_1 j'_1} \delta_{i_2 j'_2} \delta_{j_1 j'_2} \delta_{j_2 j'_1} + \delta_{i_1 i'_2} \delta_{i_2 i'_1} \delta_{j_1 j'_1} \delta_{j_2 j'_2}}
$$

llowing calculation of ciency, as in these cases the direct calculation of Weingarten fowing calculation of t allowing calculation of the values for polynomial integrals of degree less than 5 without the direct usage of (3).

This optimization gives only a minor improvement of em-<br>ciency, as in these cases the direct calculation of Weingarten that the context, as in these cases the direct calculation of *weingarten*<br>function is very fast. function is very fast to the set of the set <br>The set of the set of This optimization gives only a minor improvement of effi-<br>of ook ciency, as in these cases the direct calculation of Weingarten function is very fast. This optimization gives only a minor improvement of emfunction is very fast.

3.2. Elements from one row (column). The next optimization (7) is based on the fact that the distribution of random vector conunitary matrix distributed with the Haar measure, is uniform where U(*A*) denotes the normalized uniform measure (proporsisting of squares of absolute values of a row (or a column) of unitary matrix distributed with the Haar measure, is uniform The on a standard d-simplex  $\Delta^d$  [9] on a standard *d*-simplex ∆*<sup>d</sup>* [9, Ch. 4]

$$
\begin{aligned}\n[6].\\ \text{sym-}\\ \text{e of} \\ \text{with } \text{where } \mathbb{I}(\Lambda) \text{ denotes the normalized uniform measure (res.}\n\end{aligned}
$$

re  $U(A)$  denotes the normalized uniform measure (pro-[8] portional to Lebesgue measure) on a set  $A \subset \mathbb{R}^d$ , and standard *<sup>i</sup>*=<sup>1</sup> λ*<sup>i</sup>* = 1}.  $\mathcal{C} \cup (A)$  denotes the normalized uniform measure (provector *p* with non-negative entries *p* with a set  $A \subseteq \mathbb{R}^n$ , and standard intervalse  $A_d$  is defined as  $A_d = (1 \subset \mathbb{R}^d \cdot 1) > 0$ ,  $\nabla_d (1) = 1$ of<br>the where  $U(A)$  denotes the normalized uniform measure (prod-simplex  $\Delta^d$  is defined as,  $\Delta^d = \{\lambda \in \mathbb{R}^d : \lambda_i \geq 0, \sum_{i=1}^d \lambda_i = 1\}.$ simplex <sup>∆</sup>*<sup>d</sup>* is defined as, <sup>∆</sup>*<sup>d</sup>* <sup>=</sup> {<sup>λ</sup> <sup>∈</sup> <sup>R</sup>*<sup>d</sup>* : <sup>λ</sup>*<sup>i</sup>* <sup>≥</sup> <sup>0</sup>,∑*<sup>d</sup>*  $d$ -simplex  $\Delta^d$  is defined as,  $\Delta^d = {\lambda \in \mathbb{R}^d : \lambda_i \ge 0, \sum_{i=1}^d \lambda_i = 1}.$ simplex <sup>∆</sup>*<sup>d</sup>* is defined as, <sup>∆</sup>*<sup>d</sup>* <sup>=</sup> {<sup>λ</sup> <sup>∈</sup> <sup>R</sup>*<sup>d</sup>* : <sup>λ</sup>*<sup>i</sup>* <sup>≥</sup> <sup>0</sup>,∑*<sup>d</sup>*

Using Beta integral, one obtains that for a fixed row  $i_0$  and **Proposit** Using Beta integral, one obtains that for a fixed row  $t_0$  and **Propos** a vector p with non-negative entries  $p_j$ , we have the following tive int Murnaghan-Nakayama rule (see *e.g.* [7, Th. 4.10.2]), which  $\alpha$  vector  $p$  with non-negative entries  $p_j$ , we have the a vector p with non-negative entries  $p_j$ , we have the following  $\alpha$  vector  $\rho$  with non-negative entries  $p_j$ , we have the *<sup>i</sup>*=<sup>1</sup> λ*<sup>i</sup>* = 1}.

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\nUsing Beta integral, one obtains that for a fixed row 
$$
i_0
$$
 and **Propoa**

\na vector  $p$  with non-negative entries  $p_j$ , we have the following–itive in

\n
$$
\int_{\mathbf{U}(d)} \prod_{j=1}^d |u_{i_0,j}|^{2p_j} dU = \int_{\mathbf{U}(d)} \frac{\Gamma(p_1+1) \times \cdots \times \Gamma(p_d+1)}{\Gamma(p_1+\cdots+p_d+d)} \cdot \qquad \qquad \text{(12)}
$$
\nThe above, as a special case, gives:

 $\overline{u}$ The above, as a special case, gives: The above, as a special case, gives:

$$
\int_{U(d)} |u_{i_0,j}|^{2k} dU = \frac{(d-1)!k!}{(d-1+k)!},
$$
\n(13)

$$
\int_{\mathbf{U}(d)} |u_{i_0,j}|^2 |u_{i_0,k}|^2 dU = \frac{1}{d(d+1)},
$$
\n(14)

 $\mathbf{w}$ efficiency thanks to avoiding (∑ *pi*)!∏ *pi*! executions of Weinwhich can be found in literature  $[9, 10]$ .  $\mathbf{r}$  and  $\mathbf{r}$ . In literature  $\mathbf{r}$ which can be found in literature [9, 10]. which can be found in literature [9, 10].

the cases of the cases of the integral cases of the integral cases of the integral can be calculated with  $e$  fi  $\sin$ *<sup>j</sup> lj*! permutations, where *ki* (*lj*) denotes the number Which can be found in interature [ $\sigma$ , 10].<br>This optimization allows for an enormous improvement in efficiency thanks to avoiding  $(\sum p_i)! [p_i]$  executions of Weingarten function needed in the case of the direct usage of mula (3). formula (3). mula (3).

 $\mathcal{D}$  presented in the presente package, to increase its effects in the set of the set o 3.3. Even powers of diagonal element absolute values In 3.3. Even powers of diagonal element absolute values In **3.3. Even powers of diagonal element absolute values.** In this subsection we consider the integrals of the type  $\sigma$ 

$$
\int_{U(d)} |u_{i,j}|^{2p} |u_{k,l}|^{2q} dU,
$$
\n(15) instead of  
\nthe function, the

where p, q are non-negative integers and  $i \neq k, j \neq l$ . In the case<br>of  $n - a - 1$  this integral is known [9] where *p*, *q* are non-negative integers and  $i \neq k$ ,  $j \neq i$ . In the case of  $p = q = 1$  this integral is known [9]  $c_1 p - q$  = 1 this integral is this will  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ of  $p = q = 1$  this integral is known [9] case of *p* = *q* = 1 this integral is known [9, Prop. 4.2.3]

$$
\int_{\mathcal{U}(d)} |u_{i,j}|^2 |u_{k,l}|^2 dU = \frac{1}{d^2 - 1}.
$$
 class

For general non-negative integers  $p$ ,  $q$  the following proposition tion generalism tor gen case of *p* = *q* = 1 this integral is known [9, Prop. 4.2.3] *For general non-negative integers*  $p$ *,*  $q$  *the following is true.*  $\mathbf{1}$  $\alpha$  *p*  $\alpha$ 

**Proposition 1.** Let p, q be non-negative integers, for  $i \neq k$  and  $j \neq l$ , we have **Troposition 1.** Let p, q be non-negative integers, for  $t \neq j \neq l$ , we have

$$
\int_{\mathcal{U}(d)} |u_{i,j}|^{2p} |u_{k,l}|^{2q} dU = p!q! \sum_{\substack{\lambda \vdash p \\ \mu \vdash q}} \mathbf{k}_{\lambda} \mathbf{k}_{\mu} W g(\lambda \sqcup \mu, d), (17) \quad \text{We have the follow}
$$

where the above sum is taken over all integer partitions of  $p$ where the above sum is taken over an integer partitions of p be a c<br>and q. Symbol  $\mathbf{k}_v$  denotes a cardinality of conjugacy class for and q. Symbol  $\mathbf{K}_v$  denotes a cardinality or conjugacy class for<br>a permutation with cycle type given by partition  $v \vdash r[7]$ λ*p*  $x \mapsto r$  [7], then  $x \mapsto r$  [7],  $\frac{1}{2}$ 

$$
\mathbf{k}_{v} = \frac{r!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots r^{m_r} m_r!},
$$
 (18)

where *mi* denotes the number of *i* in partition ν. where *mi* denotes the number of *i* in partition ν. where *mi* denotes the number of *i* in partition ν. where  $m_i$  denotes the number of *i* in partition *v*. a permutation with cycle type given by partition ν *r* [7, Eq.

The above is a special case of a general fact. The above is a special case of a general fact.

> **23**. **pd**, *pol. Ac.: Tech. 65(1) 2017 F*m. Pol. Ac.: Tech.  $65(1)$  2017

**Proposition 2.** For any permutation  $\pi \in S_d$  and any non-nega**the integers**  $p_1, p_2, ..., p_d$ **, we have**  $P(X|P) = \sum_{i=1}^{d} p_i$  and any non-negative integers  $p_1, p_2, ..., p_d$ , we have  $p_1, p_2, \ldots, p_d, p_d, p_d, p_d, p_d$  $\Gamma$  **Proposition 2.** For any permutation  $n \in S_d$  and any non-integers p<sub>1</sub>, p<sub>2</sub>, ..., p<sub>d</sub>, we have negative integers *p*<sub>1</sub>, *p*<sub>2</sub>, ..., *p*<sub>*u*</sub>, we have

$$
\int_{U(d)} \prod_{j=1}^{d} |u_{j,\pi(j)}|^{2p_j} dU = \int_{U(d)} \prod_{j=1}^{d} |u_{j,j}|^{2p_j} dU =
$$
\n
$$
= \left(\prod_{j=1}^{d} p_j!\right) \sum_{\lambda_1 \vdash p_1} \sum_{\lambda_2 \vdash p_2} \dots \sum_{\lambda_d \vdash p_d} \left(\prod_{j=1}^{d} \mathbf{k}_{\lambda_j}\right) \times
$$
\n
$$
\times Wg(\lambda_1 \sqcup \lambda_2 \sqcup \dots \sqcup \lambda_d, d).
$$
\n(19)

 $a(a+1)$  permute within the blocks of sizes  $\{p_1, p_2, ..., p_d\}$ , i.e. *Proof.* If we apply the formula from Eq. (3) to integral (19), *Proof.* If we apply the formula from Eq. (3) to integral (19), **Proof.** If we apply the formula from Eq. (3) to integral (19),  $(14)$  then the non-vanishing permutations of indices are those which

$$
\sigma = \sigma_1 \oplus \sigma_2 \oplus \sigma_d,
$$

 $\frac{1}{2}$  a (3). the permutation  $\tau \sigma^{-1}$  is also in this form. Each permutation of e is obtained by joining cycle types of small permutations. Since of We-<br>where  $\sigma_j \in S_{p_j}$ , permuting indices in j<sup>th</sup> block of size  $p_j$ . The age of same situation holds in the case of the second indices. Thus the above type will be present in the sum  $\prod_{j=1}^{d} p_j!$  times. The *<sup>j</sup>*=<sup>1</sup> *pj*! In this cycle type of such permutations is given by a partition which weingarten function depends only on a cycle type of permuta-<br>tion, the size of conjugacy class is calculated for each partition, (15) instead of evaluating Weingarten function multiple times.  $\Box$ Weingarten function depends only on a cycle type of permuta-

the direct usage of  $(3)$ . One can notice that Proposition 2 allows  $\sum_{i=1}^{3}$  to avoid at reast  $\prod_{j=1}^{3} P_j$ . executions of weingarien rane the company depends also on the cardinality of conjugacy<br>since-efficiency depends also on the cardinality of conjugacy  $\frac{1}{2!}$   $T_{t,j}$  and  $T_{t,k,t}$  and  $T_{t,k,t}$  are  $T_{t,k,t}$  and  $T_{t,k,t}$  are case us to avoid at least  $\prod_{j=1}^{d} p_j!$  executions of Weingarten function comparing to the direct usage of (3). However, the exact<br> $\lim_{x \to 0} f(x) = f(x)$ • –

*x m k i m k i m k i k section we* **ns.** In thi<br>| for positi *ui*, *i*<sub>l</sub> this section<br>for positive integer *due i*, *i*ui<sub></sub>, *i*ui<sub></sub>, *i*ui<sub></sub>, *i*u<sub></sub><sup>, *i*</sup>, *i*u<sup>1</sup>, *i*ui<sup>*n*</sup>, *i*ui<sup>*n*</sup>, *i*ui<sup>1</sup>, *i*u on  $\sigma$  of  $\{1, 2, ..., m\}$  being a cycle 3.4. Cycle per mutations. In this section we consider another special type of integral for positive integer  $k$  and a permutation  $W_{\text{eff}}$  is the following proposition.  $3.4.$  Cycle permutations. In this section we consider another 3.4. Cycle permutations In this section we consider another **S.4. Cycle permutations.** In this section we consider another positive integration with respect to the unitary group special type of integral for positive integer  $\kappa$  and<br>wing proposition  $\sigma$  of  $\{1, 2, ..., m\}$  being a cycle

$$
\text{c.} \quad \text{or } i \neq k \text{ and } \quad \int_{\mathbf{U}(d)} \left( \prod_{i=1}^{m} u_{i,i} \overline{u}_{i,\sigma(i)} \right)^{k} dU. \tag{20}
$$

*<sup>k</sup>* The Mess have the following proposition.  $\frac{1}{6}$ , ii<sub>i</sub><sup>i</sup>, i<sub>i</sub> *dU*. (20)

**i**, *u***<sub><b>i**</sub>, *d*, *k* is a positive integer and  $\sigma \in S_m$ artitions of p be a cycle, *i.e.* the cycle type is given by partition  $\{m\}$ . Then **3.** Let  $m \leq d$ , k is a positive integer and  $\sigma \in S_m$ *u***<sub>2</sub>**  $i$  *d*<sub>*i*</sub>, *k*  $i$  *i s* a positive integer and  $\sigma \in S_m$ PROPOSITION 3. Let *m* ≤ *d*, *k* is a positive integer and σ ∈ **Proposition 3.** Let  $m \leq d$ ,  $k$  is a positive integer and  $\sigma \in S_m$ .  $\lim_{n \to \infty} 3$  *Let*  $m < d$  *k is a positive integer and*  $\tau \in \mathcal{S}$ .  $\lim_{\epsilon}$  3.

gugacy class for

\n
$$
\int_{U(d)} \left( \prod_{i=1}^{m} u_{i,i} \overline{u}_{i,\sigma(i)} \right)^{k} dU =
$$
\n(18)

\n
$$
= (k!)^{2m-1} \sum_{\lambda \vdash k} \mathbf{k}_{\lambda} W g(m\lambda, d).
$$

provement in efficiency by avoiding more than (*k*!)2*m*−<sup>1</sup> exe- $\frac{1}{2}$  cutions of  $\frac{1}{2}$  cuting to the case of etting must be divisi provement in efficiency by avoiding more than (*k*!)2*m*−<sup>1</sup> exe**roof.** One can notice that cycle lengths of permutations in this  $\mathbf{r}$ Proof. One can notice that cycle lengths of permutations in this  $\frac{1}{1001}$ . One can notice that eyere rengins of permaturions in any  $\frac{1}{100}$ example by the usual counting argument of the usual counting argument. external contract of the usual counting and counting argument. The usual counting argument of permutations in this counting must be divisible by *m*. The number of permutations et. setting must be divisible by *m*. The number of permutations

 $\frac{1}{2}$  implementing the main functions related to the main functions related to the theorem

implementing the main functionality, functions related to the

with cycle type given by a particular partition can be easily obtained by the usual counting argument.  $□$ 

Using  $(21)$  one obtains an enormous improvement in efficiency by avoiding more than  $(k!)^{2m-1}$  executions of Weingarten function as compared to the case of the direct usage of  $(3)$ .

## **4.** Package description

Below we describe the functions implemented in IntU package. The functions are grouped in three categories: functions implementing the main functionality, functions related to the calculation of Weingarten function and helper functions.

**4.1. Main functionality.** The main functionality of IntU med the IntegrateUnitaryHaar and package is provided be the IntegrateUnitaryHaar and IntegrateUnitaryHaarIndices functions. The first one operates directly on polynomial expressions, while the second one accepts four-tuple of indices. The examples of the usage are given in Section 5.

- es the definition on the definite integrand, {var,dim}]  $-$  gives the definite integral on unitary group with respect to the Haar measure, accepting the following arguments
	- the Haar measure, accepting the following arguments<br>• integrand the polynomial type expression of variable var with indices placed as subscripts, can contain any other symbolic expression of other variables, any other symbolic expression of other variables,  $\bullet$   $\lnot$   $\lnot$
	- $\bullet$  var the symbol of variable for integration,
	- $\dim$  the dimension of a unitary group, must be a pos-<br>itive integer itive integer.

itive integer.<br>This function is presented in the examples described in Sections  $5.1, 5.3$  and  $5.5$ .

 $\bullet$  IntegrateUnitaryHaar[f, {u,d1}, {v,d2},...] – gives multiple integral

$$
\int_{\mathcal{U}(\mathrm{d}1)} dU \int_{\mathcal{U}(\mathrm{d}2)} dV \dots \mathbf{f}.\tag{22}
$$

This function is presented in the example described in Section 5.4.

 $\dim$  – calculates the integral in (3) for given multi-indices ● IntegrateUnitaryHaarIndices[{I1,J1,I2,J2}, I1, J1, I2, J2 and the dimension  $\dim$  of the unitary group.  $\frac{1}{2}$  (3)  $\frac{1}{2}$  for the integral integral integral in the unitary group. tion 5.2.  $\text{tan } 5.2.$ and the dimension of the unitary group. The unitary group  $\mathcal{L}$  $\Box$   $\Box$   $\Box$ 

> 4.2. Weingarten function. The main functions implemented in the package, IntegrateUnitaryHaar and Integra $t$ eUnitaryHaarIndices, utilize the following functions in  $\mathcal{L}_t$ to find the value of the integral. Section 5.2. Section 5.2.

- $\bullet$  Weingarten [type, dim] returns the value of the We- $\frac{1}{\sqrt{2}}$  for  $\frac{1}{2}$  for  $\frac{1}{2}$  for  $\frac{1}{2}$  for the value of the  $\frac{1}{2}$  integral.  $r_{\text{arguments}}$
- $\bullet$  type an integer partition which corresponds to cycle  $\epsilon_{\text{TPC}}$  an integer parameter which corresponds type of permutation (see Section 2.3),
- type of permutation (see Section 2.3),<br>•  $\dim -$  the dimension of a unitary group, which must be a positive integer.  $\mathbf{r}$  – and integer partition which corresponds to cycle  $\bullet$  dim – the dimension of a unitary group, which

*P* ● CharacterSymmetricGroup[part,type] – gives the character of the symmetric group  $\chi^{\text{part}}(\text{type})$  (see Section 2.3.2).  $\mathcal{D}_{\text{deco}}(t)$  dimension of the interval  $\mathcal{D}_{\text{deco}}(t)$  symmetric representation of symmetric symmetric

Parameter type is optional. The default value is set to  $\frac{1}{2}$  at a trivial partition and in this case the function returns the dia dividi paradich did in this case are rancted reducible diindexed by part, given by  $(7)$ . If type is specified, the value of the character is calculated by Murnaghan-Nakayama rule using MNInner algorithm provided in [8].

● SchurPolynomialAt1 [part, dim] – returns the value of the Schur polynomial  $s_{\text{part}}$  at point  $(\underline{1, 1, ..., 1})$ , *i.e.* the dim sponding to , see Eq. (6). The spond

dimension of irreducible representation of U(dim) corresponding to part, see (6).

#### **4.3. Helper functions.**

- one  $\bullet$  PermutationTypePartition[perm] gives the  $\frac{1}{2}$  partition which represents the cycle type of the permutation e perm (see Section 2.1).
- MultinomialBeta[p] for a given *d*-dimensional vector of non-negative numbers  $p_1, p_2, ..., p_d$  returns the value of multinomial Beta function defined as multinomial Beta function defined as

$$
B(p) = \frac{\prod_{i=1}^{d} \Gamma(p_i)}{\Gamma(\sum_{i=1}^{d} p_i)}.
$$
 (23)  
This function is used in the optimization described in Sec-

 $tion 3.2.$ 

- ConjugatePartition[part] gives a conjugate of a partition part (see [5]). This function is used for calculating hook-length formula given by (7).
- extending that the cardinality of a conjugacy class fartition (part)<br>- gives a cardinality of a conjugacy class for the permutation a cardinality of a conjugacy class for the permutation with with cycle type given by partition part (see [7, Eq. 1.2]). This function is used in the optimization described in Sec- $\frac{1}{1}$  tion 3.3. ● CardinalityConjugacyClassPartition[part]
- BinaryPartition [part] gives a binary representation of a partition part. This function is needed for the - implementation of MNInner algorithm.

### $\frac{1}{2}$ <br>2016. Promotografiya **5. Examples of use**

 $\overline{1}$ 

In[2]:= **d 3;**

In order to present the main features of the described package, In order to present the main features of the described package In order to present the main features of the described package we provide a series of examples. we provide a series of examples. we provide a series of examples. In order to present the main features of the described package  $\sup.$ 

 $\text{total} = 5.1$ . Elementary integrals. Let us assume that  $d = 3$ . We want  $t_{\text{ca}}$  5.1. Elementary integrals. Let us assume that  $d = 3$ . We want -a<sup>-</sup> to calculate the following integrals<br>ons

$$
\int_{U(d)} |u_{1,1}|^2 dU,
$$
\n(24)

$$
\int_{\mathrm{U}(d)} |u_{1,1}|^2 |u_{2,2}|^2 dU,\tag{25}
$$

t 
$$
\int_{U(d)} u_{1,1} u_{2,2} \overline{u}_{1,2} \overline{u}_{2,1} dU.
$$
 (26)

Let us start by initializing the package which is equivalent to

 $\ln[1]:=\,\, <\,\, <\,\, \texttt{IntU}$  in

Next, we calculate the integrals.

In[2]:= **d 3;** In[2]:= **d 3;** In[2]:= **d 3;** In[3]:= **IntegrateUnitaryHaar** In[3]:= **IntegrateUnitaryHaar** In[3]:= **IntegrateUnitaryHaar**  $\frac{1}{2}$  **Abs**  $[u_{1,1}] \wedge 2$ ,  $\{u, d\}$  $\frac{1}{\text{Out}[3]_=}$ 3 In[4]:= **IntegrateUnitaryHaar**[ **Abs**  $[u_1, 1, u_2, 2] \wedge 2$  ,  $\{u, d\}$  $\frac{1}{\text{Out[4]}}$  $Out[4] = \frac{1}{8}$  $\ln[5]$ := **IntegrateUnitaryHaar** $[\mathbf{u}_{1,1} \ \mathbf{u}_{2,2}]$ **Conjugateu1,2 u2,1, u, d Conjugateu1,2 u2,1, u, d** Out[5]=  $-\frac{1}{ }$ Out[5]=  $-\frac{1}{24}$  $\sim$ Out[3 Out[4]  $Out[5] =$ In[5]:= **IntegrateUnitaryHaaru1,1 u2,2 Conjugateu1,2 u2,1, u, d**

**5.2. Operations on indices.** Let us take the following set of multi-indices multi-indices

 $I = \{1, 1, 1, 2, 2\}, \qquad J = \{2, 2, 1, 1, 1\}$  (27)

 $I' = \{1, 1, 1, 2, 2\}, \qquad J' = \{2, 1, 1, 2, 1\}$  (28)  $I^{\prime}$   $I^{\prime$  $J = \{1, 1, 1, 2, 2\}, \quad J = \{2, 1, 1, 2, 1\}$  (28)  $= \{1, 1, 1, 2, 2\}, \qquad J' = \{2, 1, 1, 2, 1\}$  (28) = {1,1,1,2,2}, *J* = {2,1,1,2,1} (28)  $\alpha$ and construction as integrand as  $\alpha$ 

and set  $d = 6$ . The above is equivalent to expression and set *d* = 6. The above is equivalent to expression

$$
u_{1,2}u_{1,2}u_{1,1}u_{2,1}u_{2,1}\overline{u}_{1,2}\overline{u}_{1,1}\overline{u}_{1,1}\overline{u}_{2,2}\overline{u}_{2,1} \tag{29}
$$

with symbolic variable  $u$ , which we aim to integrate over  $U(d)$ . After simplification the expression is equal to After simplification the expression is equal to After simplification the expression is equal to

$$
|u_{1,1}|^2 |u_{1,2}|^2 |u_{2,1}|^2 u_{1,2} u_{2,1} \overline{u}_{1,1} \overline{u}_{2,2}.
$$
 (30)

After initializing the package and defining appropriate After initializing the package and defining appropriate After initializing the package and defining appropriate in-A<br>dices In[2]:= **I1 1, 1, 1, 2, <sup>2</sup>; J1 2, 2, 1, 1, <sup>1</sup>;**

**I2 11** = {**1, 1, 1, 2, 2**}; **J1** = {**2, 2, 1, 1**, **1**}  $I2 = \{1, 1, 1, 2, 2\}; \quad J2 = \{2, 1, 1, 2, 1\};$ <br> $I2 = \{1, 1, 1, 2, 2\}; \quad J2 = \{2, 1, 1, 2, 1\};$  $d = 6;$ In[2]:= **I1 1, 1, 1, 2, <sup>2</sup>; J1 2, 2, 1, 1, <sup>1</sup>;**  $d = 6;$  $x = 0$ In[2]:= **I1 1, 1, 1, 2, <sup>2</sup>; J1 2, 2, 1, 1, <sup>1</sup>;**

we calculate the integral using provided function Integra-4 Bull. Pol. Ac.: Tech. XX(Y) 2016 teUnitaryHaarIndices as

```
In[5]:= IntegrateUnitaryHaarIndices
                                                                                              I1, J1, I2, J2, d
Out[5]= -\frac{1}{\sqrt{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{1-\frac{1}{16 200
```
.<br>المنسوب بيان which is equivalent to

$$
\begin{array}{ll}\n\ln[1]:<\nNext, we calculate the integrals.\n
$$
\begin{array}{ll}\n\ln[2]:=<
$$
$$

**5.3. Matrix expressions.** IntU package allows us to integrate matrix expressions, the package allows as to meghate<br>matrix expressions, for example let us take  $d = 2$  and integrate

$$
\int_{\mathbf{U}(d)} U^{\otimes 2} \otimes \bar{U}^{\otimes 2} dU. \tag{31}
$$

We define symbolic matrices  $U \subseteq U(d)$  and  $U2 = U \oplus U \subseteq$ We define symbolic matrices  $U \in U(d)$  and  $U2 = U \oplus U \in$ <br> $\subseteq U(d^2)$  as In[2]:= **d 2;**  $\in U(d^2)$  as matrix expressions, for example let us take *d* = 2 and integrate

 $\ln[2] = \mathbf{d} = 2;$ **<sup>U</sup> Array<sup>u</sup>1,<sup>2</sup> &, d, <sup>d</sup>; U2** = KroneckerProduct[U, U]; **<sup>U</sup> Array<sup>u</sup>1,<sup>2</sup> &, d, <sup>d</sup>;**  $U = \text{Array}[\mathbf{u}_{\text{H1}, \text{H2}} \& , \{d, d\}]$ ;<br>
U(*L*) We use a skew Product *Will* We define symbolic matrices *U* ∈ U(*d*) and *U*2 = *U* ⊗*U* ∈

In[5]:= **integrand KroneckerProduct**  $\Omega$ <sup>c</sup><sub>1</sub></sub> and construct the integrand as

```
USE Integrand = KroneckerProduct
 U2, Conjugate[U2]];
```
By using {IntegrateUnitaryHaar} function  $\mathbf{r}$  $\alpha$  integrand as integrals  $\alpha$ 8)  $By using \{IntegrateUnitaryHaar\}$ 

In[6]:= IntegrateUnitaryHaar[integrand, {u, d}] **U2, ConjugateU2;** In[6]:= IntegrateUnitaryHaar[integra

6 9) we learn that the integral in (31) is equal to <sup>3</sup> <sup>0000</sup> <sup>1</sup>



 $\frac{1}{2}$  in the surface we have  $\frac{1}{2}$ S.4. Mean value of local unitary of the limit is example we<br>calculate the mean value of a local unitary orbit of a given **5.4. Mean value of local unitary orbit.** In this example we matrix  $X \in M_{d^2}$ 

$$
\mathbb{E}[(U \otimes V)X(U \otimes V)^{\dagger}]. \tag{33}
$$

We assume that *U* and *V* are stochastically independent random we assume that  $\sigma$  and  $\gamma$  are stochastically independent random<br>unitary matrices of size  $d$  distributed with the Haar measure.<br>In this case we have In this case we have the state of the corresponding the same that the corresponding the corresponding to the corresponding of the correspo assume that  $U$  and  $V$  are stochastically independent rate of the  $U$  and  $V$  are stochastically independent rate. We<br>un:<br>In  $\overline{\phantom{0}}$ y i<br>ith We<br>un:<br>In y i:<br>ith Execution that U and V are stochastically independent as<br>tar<br>his sume that  $U$  and  $V$  are stochastically is  $\alpha$  matrices of size *d* distributed with the Haar measures we have ssu<br>ry<br>s c<br>] We assume that U and V are stochastically independent random<br> $P_X(z)$  $\frac{1}{6}$  ond  $\frac{1}{6}$  are stochastically independent random<br>matrices of size *d* distributed with the Haar measure we assume that  $\theta$  and  $\gamma$  are stochastically independent unitary matrices of size  $d$  distributed with the In this case we have we assume that  $\sigma$  and  $\gamma$  are stochastically in<br>unitary matrices of size  $d$  distributed with *PX* (*z*) =  $y$  in  $i$ the Haar measure.<br>  $\theta$  baye  $\frac{1}{6}$  bave

*W* e assume that *U* and *V* are stochastically independent random unitary matrices of size *d* distributed with the Haar measure. In this case we have\n
$$
\mathbb{E}[(U \otimes V)X(U \otimes V)^{\dagger}] = \int_{U(d)} \int_{U(d)} (U \otimes V)X(U \otimes V)^{\dagger} dU dV.
$$
\n(34) The function

In this example we take  $d = 3$ . calculate the mean value of a local unitary orbit of a given matrix *X* ∈ M*d*<sup>2</sup>

and  $\sum_{i=1}^{n} a_i$ In this example we take  $d = 3$ .<br>We define symbolic matrices X of size  $d^2$  and  $U, V \in U(d)$ <br>as  $and$ as trix *X* ∈ M*d*<sup>2</sup> In this example we take  $d = 3$ .<br>We define symbolic matrices Y of size  $d^2$  and  $U V \subset U(d)$  $\frac{1}{\sqrt{2}}$  *X* ∈ *d* trix *X* as trix *X*∈ M*d*<sup>2</sup>  $\frac{dS}{dx}$  $\frac{dS}{dx}$ we assume that  $\frac{1}{2}$  and  $\frac{1}{2}$  a we define symbolic matrices  $\Lambda$  or size  $u$  and  $\sigma$ ,  $v \in \sigma(u)$ trix *X* ∈ M*d*<sup>2</sup>

$$
\ln[2] = d = 3;
$$
  
\n
$$
X = \text{Array}[x_{\text{H1},\text{H2}} < f \, d^2, d^2], j;
$$
  
\n
$$
U = \text{Array}[u_{\text{H1},\text{H2}} < f \, d, d]; j;
$$
  
\n
$$
V = \text{Array}[v_{\text{H1},\text{H2}} < f \, d, d]; j;
$$
  
\n
$$
UV = \text{KroneckerProduct}[U, V];
$$

Using IntegrateUnitaryHaar function with two variable<br>specifications, we calculate the double integral sing integratednessly yields take the wide was variable y Haar function with two variable Using IntegrateUnitaryHaar function with two variable

$$
\text{In order to calculate (39) and (39) define symbolic matrices}
$$
\n
$$
\text{UV.X. ConjugateTranspose[UV]},
$$
\n
$$
\text{UU.A}, \{v, d\}]
$$
\n
$$
\text{UU.A}, \{v, d\}]
$$
\n
$$
\text{UU.A} \text{IV.1} \text{U.2} = \text{O.2} \text{U.3}
$$
\n
$$
\text{UU.A} \text{IV.3} \text{U.4} \text{U.5}
$$
\n
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**u, d, v, d**

and find out that the expectation value in (33) is equal to and find out that the expectation value in  $(c_0)$  is equal to If that the expectation value in (33) is equal to **u**, **d**, *d*, *d*, *d*, *d*, *d***<sub></sub>** and find out that the expectation value in

$$
\mathbb{E}[(U \otimes V)X(U \otimes V)^{\dagger}] = \frac{1}{d^2} tr X \mathbb{1}_{d \times d}.
$$
 (35) Now

Similarly one can calculate the covariance tensor of the local Similarly one can calculate the covariance tensor of the<br>local unitary orbit of a given matrix  $X \in M_d$ <sup>2</sup>. If  $z = (U \oplus V)$ <br> $V(T \oplus V)^{\dagger}$  is a given matrix  $X \in M_d$ <sup>2</sup>. If  $z = (U \oplus V)$ **IOCAL UNITARY OF DIT OF A given matrix**  $X \in M_d^2$ <br> $V(I \cap V)^{\dagger}$  then the covariance tensor is given Similarly one can calculate the covariance tensor of the  $\frac{\ln 8}{\ln 8}$  $X(U \oplus V)^{\dagger}$  then the covariance tensor is given by [11] and the analysis and the same set of  $X(U \oplus V)^{\dagger}$  and the sa  $\frac{1}{V}$  $X(U \oplus V)^{\dagger}$  then the covariance tensor is given by [11]

$$
\mathbb{E}[\{z_{ij}\overline{z}_{kl}\}_{ijkl}] =
$$
\n
$$
= \mathbb{E}[(U \otimes V)X(U \otimes V)^{\dagger} \otimes \overline{(U \otimes V)X(U \otimes V)^{\dagger}}].
$$
\n(36) After  
agree

fixed dimension, the integral agrees with the calculations pre-If *U* is a random *d* × *d* unitary matrix distributed accord- $\text{in } [11].$ Using IntegrateUnitaryHaar one can check that for the 5.5. Moments of maximally entangled numerical shadow sented in [11]. sented in [11].

5.5. Moments of maximally entangled numerical shadow. If  $\overline{6}$ , SI U is a random  $d \times d$  unitary matrix distributed according to the<br>Haar measure [12] then the nure state obtained by vectorization. We de Example 12], then the pure state obtained by vectorization We de<br>  $\ell > \frac{1}{2} \cdot \text{vec}(I)$  is maximally entangled on  $H_i \oplus H_p = \mathbb{C}^d \oplus \mathbb{C}^d$  for ca  $\ket{\zeta} = \frac{1}{\sqrt{d}}$  vec  $(\zeta)$  is maximally emalgred on  $H_A \oplus H_B = \zeta^* \oplus \zeta^*$ . To note ver, state |ς / ilas a uistribution invariant to multiplication and The numerical shadow of the numerical sha<br>Nu local unitary matrices We denote the corresponding probability measure on pure states of size  $d \times d$  as  $\mu$ . The numerical of int shadow of operator  $X \in M_d$ <sup>2</sup> with respect to this measure (max-<br>involve optimally extended numerical shadow) is defined as imally entangled numerical shadow) is defined as quant by local unitary matrices. We denote the corresponding proba-<br>bility measure on pure states of size  $d \times d$  as  $\mu$ . The numerical  $\zeta$  =  $\frac{1}{\sqrt{d}}$  vec(*C*) is maximally entangled on  $H_A \oplus H_B = C^* \oplus C^*$ . for can<br>Moreover, state  $|\xi\rangle$  has a distribution invariant to multiplication Haar  $\begin{aligned} \mathcal{L}^{\mathcal{E}}(t) &= \frac{1}{t} \text{vec}(H) \text{ is maximally entangled on } H_t \oplus H_0 = \mathbb{C} \end{aligned}$  $|\xi\rangle = \frac{1}{\sqrt{d}} \text{vec}(U)$  is maximally entangled on  $H_A \oplus H_B = \mathbb{C}^d \oplus \mathbb{C}^d$ .

1 J.A. Miszczak  
\n
$$
P_X(z) = \int_{\mathbb{C}^{d^2}} \delta(z - \langle \psi | X | \psi \rangle) d\mu(\psi).
$$
\n(37)  
\nThe corresponding probability measure is denoted by  $d\mu_X^E$ . For definition and basic facts concerning numerical shadows see

definition and basic facts concerning numerical shadows see  $[11, 13, 14]$ . The corresponding probability measure is denoted by  $d\mu_X^E$ . For definition and basic facts concerning numerical shadows see The corresponding probability measure is denoted by  $d\mu_{\tilde{Y}}$ . For<br>definition and basic facts concerning numerical shadows see<br>[11, 13, 14] HS <sup>+</sup>tr*B*(*X*)<sup>2</sup> The corresponding probability measure is denoted by  $d\mu_X^E$ . For<br>definition and basic facts concerning numerical shadows see  $\begin{bmatrix} 11, 13, 14 \end{bmatrix}$ .  $\begin{bmatrix} 11, 13, 14 \end{bmatrix}$ .  $\begin{bmatrix} 11, 13, 14 \end{bmatrix}$ .<br>
The first two moments of *d*<sub>E</sub>

3, 14].<br>he first two moments of  $d\mu_X^E$  are calculated in [11], and<br>ven by <sup>[11, 13, 14]</sup>. The first two moments of  $d\mu_X^E$  are calculated in [11], and<br>are given by  $W)X(U \otimes V)^{\dagger}dUdV$  are given by see [11, 13, 14].  $T_1$   $\sigma$  *d*<sub>*y*</sub> $\sigma$ see [11, 13, 14].  $\frac{1}{2}$   $\frac{1}{2}$ . 13, 140  $\frac{1}{2}$  given by The first two moments of  $d\mu_X^E$  are calculated in [11], and

$$
\int_{\mathbb{C}} z d\mu_X^E(z) = \frac{1}{d^2} \text{tr} X, \qquad (38)
$$
\nand

\n
$$
\int_{\mathbb{C}} z d\mu_X^E(z) = \frac{1}{d^2} \text{tr} X, \qquad (39)
$$

and

*PX* (*z*) =

$$
\begin{aligned}\n\text{ln}[2] &= \mathbf{d} = 3; \\
\mathbf{X} &= \mathbf{Array}[\mathbf{x}_{\#1, \#2} \mathbf{\&}, \{\mathbf{d} \wedge \mathbf{2}, \mathbf{d} \wedge \mathbf{2}\}]; \\
\mathbf{U} &= \mathbf{Array}[\mathbf{u}_{\#1, \#2} \mathbf{\&}, \{\mathbf{d}, \mathbf{d}\}]; \\
\mathbf{V} &= \mathbf{Array}[\mathbf{v}_{\#1, \#2} \mathbf{\&}, \{\mathbf{d}, \mathbf{d}\}]; \\
\mathbf{U} &= \mathbf{Array}[\mathbf{v}_{\#1, \#2} \mathbf{\&}, \{\mathbf{d}, \mathbf{d}\}]; \\
\mathbf{U} &= \mathbf{Error} \mathbf{c} \mathbf{K} \mathbf{Proof} \mathbf{er} \mathbf{Product}[\mathbf{U}, \mathbf{V}];\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&\int_{\mathbb{C}} z \overline{z} \, d\mu_X^E(z) = \frac{1}{d^2(d^2 - 1)} \left( \|\mathbf{tr}[X]^2 + \|X\|_{\text{HS}}^2 \right) - \\
&\quad - \frac{1}{d^3(d^2 - 1)} \left( \|\mathbf{tr}[X] \|\mathbf{H}[X] \|\mathbf{H}[X] \|\mathbf{H}[X] \|\mathbf{H}[X] \|\mathbf{H}[X] \right),\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&\text{(39)} \\
&\text{(10)} \\
&\text{(11)} \\
&\text{(11)} \\
&\text{(11)} \\
&\text{(12)} \\
&\text{(12)} \\
&\text{(13)} \\
&\text{(14)} \\
&\text{(15)} \\
&\text{(15)} \\
&\text{(16)} \\
\mathbf{F} &= \mathbf{G} \mathbf{C} \mathbf
$$

where  $tr_A$  and  $tr_B$  denote partial traces over a specified where  $\text{tr}_A$  and  $\text{tr}_B$  denote partial traces over a specified<br>th two variable sub-system and  $\|\cdot\|_{\text{HS}}$  is a Hilbert-Schmidt norm given by le sub-system and  $\|\cdot\|_{\text{HS}}$  is a Hilbert-Schmidt norm given by<br> $\|X\|_{\text{HS}} = \sqrt{\text{tr}XX^{\dagger}}$ .

 $\frac{1}{2}$  In order to calculate (39) and (39) define symbolic matrices

$$
\ln[2]:= d = 3;
$$
\n
$$
X = \text{Array}[x_{\text{H1},\text{H2}} < f(d \land 2, d \land 2)];
$$
\n
$$
U = \text{Array}[u_{\text{H1},\text{H2}} < f(d, d)];
$$
\n
$$
\xi = 1 / \text{Sqrt}[d] \text{Flatten}[U];
$$
\n
$$
z = \xi \cdot X.\text{Conjugate}[\xi];
$$
\n
$$
zz = \text{Simplify}[z \text{ Conjugate}[z]);
$$

Now we calculate the first moment

```
In[8]:= IntegrateUnitaryHaarz, u, d
```
and the second moment

```
In[9]:= int  IntegrateUnitaryHaarzz, u, d
```
After some algebraic manipulations one can see that the above agrees with Eqs. (38) and (39).

 $\ln$ IntU package has been applied successfully in the context of quantum entanglement in [15], where it was used to calculate the moments of the three-tangle.

#### **6. Summary**

We described IntU package for Mathematica computing system for calculating polynomial integrals over U(*d*) with respect to Haar measure. We described a number of special cases which can be used to optimize the calculation speed for some classes of integrals. We also provided examples of using of the presented package, including the applications in the geometry of quantum states.

Calculation time of the package strongly depends on a degree of the integrand. For polynomials of small degree, the package is able to calculate the value of integral using the direct formula (3). For polynomials of large degree, the calculation time grows rapidly and the calculation is possible only if one of the special cases (optimizations) is used.

Nevertheless, the presented package can be very useful in the investigations involving circular unitary ensemble and the geometry of quantum states and quantum entanglement.

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