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On the Diophantine equation on reciprocal Fibonacci numbers

Abstract The work deals with exponential Diophantine equations of a special kind related to Fibonacci sequences. The classical Diophantine equation relates the sum of two terms of the sequence to the third. A solution to the problem of the existence of Diophantine equations relating the sum of two reciprocal terms of the Fibonacci sequence with the reciprocal of another term of this sequence is given.

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1. Introduction Let F_n be the n-th Fibonacci number defined by $F_0 = 0, F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \tag{1}$$

The problem of determining all integer solutions to Diophantine equations with Fibonacci numbers has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. In addition, these numbers show up in many areas of mathematics and in nature. Also, there is the Lucas sequence, which is as important as the Fibonacci sequence. The Lucas sequence $(L_n)_{n>0}$; follows the same recursive pattern as the Fibonacci numbers, but with initial conditions $L_0 = 2$ and $L_1 = 1$. For the beauty and rich applications of these numbers and their relatives one can see Koshy's book [2]. Many authors worked on the exponential Diophantine equations related to Fibonacci sequence . See for example [4],[3]. The Natural Diophantine equation related to Fibonacci sequence is when the sum of two terms of Fibonacci sequence is a term of Fibonacci sequence; in other words, the Diophantine equation

$$F_n + F_m = F_s, n \geq m > s$$

has infinity many non-negative solutions regarding to its definition

$$(n, m, s) = (n, n-1, n-2), n > 1$$

So, by the defining equality (1) of the Fibonacci numbers and the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$, we see that $F_n^2 + F_{n+1}^2$ $(n > 0)$ is a Fibonacci number for $s/$. Florian Luca and Carlos Alexis Gomez Ruiz [1], had shown that if the Diophantine equation

$$F_n^x + F_{n+1}^x = F_m$$

holds for all sufficiently large n, then . In the same paper, they showed that the Diophantine equation

$$F_n^x + F_{n+1}^x = F_m$$

has no positive integer solution with $n \geq 2$ and $x \geq 3$. In the contrary side, the question that can be asked is : " are there two terms of Fibonacci sequence that the sum of there reciprocal is also a reciprocal term of Fibonacci number ? In an other word, whether the Diophantine equation

$$F_n^{-1} + F_m^{-1} = F_s^{-1} \tag{2}$$

has solutions in non-negative integers, when $s < m \leq n$?. In the present work, we prove that the equation (2) has only two solutions in non-negative integers $(n, m, s) = (3, 3, 1), (3, 3, 2)$.

2. Preliminary In this section, we present the lemmas that are needed in the proof of the theorem. The first lemma is a collection of a few well-known results, we state them without proof, and in the proof of the theorem sometimes we do not refer to them.

LEMMA 2.1 *Let k and n be are arbitrary integers*

1. $gcd(F_k, F_n) = gcd(k, n)$
2. $(F_{n+1}, F_n) = 1$
3. F_k/F_n *if and only if* k/n.
4. *Assume that* $n > 1$, *then*

$$0.38\alpha^n < F_n < 0.48\alpha^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

`Proof`: Let us state the well known Binet formula for the Fibonacci sequence

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. are the characteristic roots of the characteristic polynomial $X^2 - X - 1$ of the Fibonacci sequence. Note that $\alpha = -\beta^{-1}$. The Binet formula immediately yields for $n > 1$ the inequalities

$$0.38\alpha^n < \alpha^n \frac{1 - \alpha^{-4}}{\sqrt{5}} \leq F_n$$

$$= \alpha^n \frac{1 - (-1)^n \alpha^{-2n}}{\sqrt{5}} \leq \alpha^n \frac{1 - \alpha^{-6}}{\sqrt{5}} < 0.48\alpha^n.$$

QED

LEMMA 2.2 *Assume that $n \geq m > 1$, then*

1. $0.38\alpha^n < F_n + F_m < 0.78\alpha^n$,
2. $0.14\alpha^{n+m} < F_n F_m < 0.23\alpha^{n+m}$.

Proof: Due to Lemma 2.1, we have for all $1 < m \leq n$,

$$\begin{aligned} 0.38\alpha^n < F_n < F_n + F_m & \\ &\leq 0.48\alpha^n + 0.48\alpha^{n-1} \\ &\leq 0.78\alpha^n, \end{aligned}$$

which proves the first statement. For the second statement we use the same argument we get

$$(0.38)^2 \alpha^{n+m} \leq F_n F_m \leq (0.48)^2 \alpha^{n+m}$$

then

$$0.14\alpha^{n+m} \leq F_n F_m \leq 0.23\alpha^{n+m}$$

QED

3. Main theorem In this section, we give a proof of our result, we use some properties of Fibonacci numbers cited in Lemma 1, we start by the following corollary

COROLLARY 3.1 *Assume that $1 < s < m \leq n$, such that $F_n^{-1} + F_m^{-1} = F_s^{-1}$. Then*

$$s < m \leq s + 2.$$

Proof: Assume that $1 < s < m \leq n$ and $F_n^{-1} + F_m^{-1} = F_s^{-1}$. Then,

$$F_s = \frac{F_n F_m}{F_n + F_m}.$$

But thanks to Lemma 2, we have

$$0.14\alpha^{n+m} \leq F_n F_m \leq 0.23\alpha^{n+m}$$

and

$$0.38\alpha^n < F_n + F_m < 0.78\alpha^n.$$

Then

$$0.29\alpha^m < \frac{F_n F_m}{F_n + F_m} < 0.6\alpha^m$$

it follows that

$$0.29\alpha^m < F_s < 0.6\alpha^m.$$

But we have, for all$s > 1$

$$0.38\alpha^s < F_s < 0.48\alpha^s.$$

It leads to inequalities

$$0.38\alpha^s < 0.6\alpha^m \tag{3}$$

and

$$0.29\alpha^m < 0.48\alpha^s. \tag{4}$$

For the inequality (3), take Logarithms, we get

$$s - m < 1 \implies m > s - 1.$$

For (4), take Logarithms, we have

$$0.601417\alpha^{m-s} \leq 1 \implies m - s \leq \frac{\ln 1.6552}{\ln \alpha},$$

it follows that

$$m \leq s + 2.$$

But $m > s$ then we have

$$s < m \leq s + 2.$$

and this ends the proof. QED

Now, we can announce the main result,

THEOREM 3.2 *The negative exponential Diophantine equation* (1) *has only two solutions in non-negative* $(n, m, s) = (3, 3, 1), (3, 3, 2)$ *such that* $s < m \leq n$*. In an other word, we have*

$$F_3^{-1} + F_3^{-1} = F_1^{-1}, F_3^{-1} + F_3^{-1} = F_2^{-1}.$$

Proof:. Consider the case when $s = 1$, then our equation (2) becomes

$$F_n^{-1} + F_m^{-1} = 1 \tag{5}$$

and the only integers n and m verify the equation (5) are $(n, m) = (3, 3)$. In fact, we can rewrite the equation (5) as

$$F_n F_m = F_n + F_m.$$

Using Lemma 2, we show that for all $m \leq n$, we have

$$\alpha^m \leq 5.57,$$

taking Logarithms, we get

$$m \leq 3.$$

Then $m = 2$ or $m = 3$. By a simple calculation, we find our solution. For $s > 1$, using Corollary (3.1), we can deduce that the only values can take m are $m = s + 1$ or $m = s + 2$.

Case 1: Let $m = s+1$, then the equation (2) can be expressed as

$$\frac{1}{F_n} + \frac{1}{F_{s+1}} = \frac{1}{F_s} \tag{6}$$

for some integer $s > 1$ and $n \geq s+1$. Simplifying the equation (6), we have ,

$$F_n^{-1} = \frac{F_{s+1} - F_s}{F_{s+1} F_s}$$

But $F_{s+1} - F_s = F_{s-1}$, then

$$F_n = \frac{F_{s+1} - F_s}{F_{s-1}}.$$

This implies that F_{s-1} divides the product $F_{s+1}F_s$ but F_{s+1}, F_s are coprime (Lemma 1), then F_{s-1} divides F_{s+1} or F_{s-1} divides F_s, the second statement can not be hold because F_{s-1} and F_s are coprime (Lemma 1). Then F_{s-1} divides F_{s+1}. By Lemma 1, we deduce that $s-1/s+1$, so the only values of s are $s = 2$ or $s = 3$.For $s = 2$, we have

$$\frac{1}{F_n} = \frac{1}{F_2} - \frac{1}{F_3} = 1 - \frac{1}{2} = \frac{1}{2}.$$

So that $F_n = 2$ and $n = 3$. For $s = 3$, we deduce that there is no Fibonacci number such that

$$\frac{1}{F_n} = \frac{1}{F_3} - \frac{1}{F_4} = \frac{1}{2} - \frac{1}{-} = \frac{1}{6}.$$

Case 2: Let $m = s+2$, then the equation (2) can be expressed as

$$\frac{1}{F_n} + \frac{1}{F_{s+2}} = \frac{1}{F_s} \tag{7}$$

for some integer $s > 1$ and $n \geq s+2$. Simplifying the equation (7), we get,

$$F_n^{-1} = \frac{F_{s+2} - F_s}{F_{s+2} F_s} \text{ implies } F_n = \frac{F_{s+2}F_s}{F_{s+1}}.$$

By the same argument, we know that F_{s+1} can not be a divisor of both F_s and F_{s+2} (Lemma 1), except for $s = 1$, which contradicts our assumption $s > 1$, and this completes the proof.

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O równaniu diofantycznym.

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Streszczenie Praca dotyczy wykładniczych równań diofantycznych szczególnego rodzaju związanych z ciągami Fibonacci'ego. Klasyczne równanie diofantyczne wiąże wiąże sumę dwóch wyrazów ciągu z trzecim. Podano rozwiązanie problemu istnienia równań diofantycznych wiążących sumę dwóch odwrotności wyrazów ciągu Fibonacci'ego z odwrotnością innego wyrazu tego ciągu.

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Słowa kluczowe: Równanie diofantyczne; liczby Fibonacci ego; Ujemne wykładnicze równanie diofantyczne.

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