

## Analysis of linear random dynamical systems with variable coefficients

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In the article a linear model of  $n$ -th order dynamical systems described by the state equation with variable coefficients is considered. This model is analysed for a random initial condition and a random force. The method of determining the probabilistic characteristics of a stochastic process, which is the solution of that equation, is shown.

KEYWORDS: moments of stochastic processes, LTV system

### 1. Introduction

Many physical phenomena can be modelled using stochastic differential equations. They are of the particular use in electronics, where each discrete element is the source of noise [1]. In addition, charges remaining in the capacitors cause, that the initial conditions should be modelled using random variables. This article presents the method of obtaining the probabilistic characteristics of the response of the system with the variable coefficients (random LTV systems).

### 2. Formalization of the problem

Let us consider a deterministic dynamical system with the variable coefficients, shown in Fig. 1.

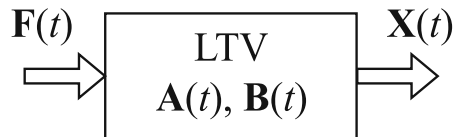


Fig. 1. Black-box model of a dynamical system

The system is described by the state equation:

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t)\mathbf{F}(t), \quad \mathbf{X}(0) = \mathbf{X}_0, \quad (1)$$

where:

- $\mathbf{A}(t), \mathbf{B}(t)$  - the continuous matrix functions,  $\mathbf{A}: \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$ ,  $\mathbf{B}: \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$ ,
  - $\mathbf{X}_0$  - the random vector of the initial conditions,  $\mathbf{X}_0 \in \mathbf{L}^n_2$ ,
  - $\mathbf{F}(t)$  - the random vector of the excitations satisfying the mean square Lipschitz condition,  $\mathbf{F}: \mathbf{R} \rightarrow \mathbf{L}^n_2$  [4],
  - $\mathbf{X}(t)$  - the random vector of the responses (the solution of the equation (1)).
- It is assumed that the vectors  $\mathbf{F}(t)$  and  $\mathbf{X}_0$  are mutually independent. Equation (1) is defined in the mean square sense.

### 3. Solution

The solution of the equation (1) exist, is unique and it is expressed by the equation:

$$\mathbf{X}(t) = \Phi(t, t_0)\mathbf{X}_0 + \int_{t_0}^t \Phi(t, s)\mathbf{B}(s)\mathbf{F}(s)ds, \quad (2)$$

where  $\Phi(t, t_0)$  is the fundamental matrix related to  $\mathbf{A}(t)$ . The above integral is understood in the mean square sense. It should be emphasized, that the analytical form of the matrix  $\Phi(t, t_0)$  is known only in a few cases [12].

Using the expected value operator  $E[\cdot]$  to equation (2) one can determine moments of the system response:

- the expected value:

$$\begin{aligned} \mu_{\mathbf{X}}(t) &= E[\mathbf{X}(t)] \\ &= \mu_{\Phi}(t, t_0)E[\mathbf{X}_0] + \int_{t_0}^t \mu_{\Phi}(t, s)\mu_{\mathbf{B}}(s)\mu_{\mathbf{F}}(s)ds, \end{aligned} \quad (3)$$

- the variance:

$$\sigma_{\mathbf{X}}^2(t) = \text{diag}(R_{\mathbf{X}}(t_1, t_2) - \mu_{\mathbf{X}}(t)\mu_{\mathbf{X}}^T(t)), \quad (4)$$

- the cross-correlation of the excitation and the response:

$$\begin{aligned} R_{\mathbf{F}\mathbf{X}}(t_1, t_2) &= E[\mathbf{F}(t_1)\mathbf{X}^T(t_2)], \\ &= \mu_{\mathbf{F}}(t_1)E[\mathbf{X}_0^T]\mu_{\Phi}^T(t_2, t_0) + \int_{t_0}^{t_2} R_{\mathbf{F}}(t_1, s)\mu_{\mathbf{B}}^T(s)\mu_{\Phi}^T(t_2, s)ds, \end{aligned} \quad (5)$$

– the correlation of the response:

$$\begin{aligned}
 R_{\mathbf{X}}(t_1, t_2) &= E[\mathbf{X}(t_1)\mathbf{X}^T(t_2)] \\
 &= E[\Phi(t_1, t_0)\mathbf{X}_0\mathbf{X}_0^T\Phi^T(t_2, t_0)] \\
 &\quad + E[\Phi(t_1, t_0)\mathbf{X}_0 \int_{t_0}^{t_2} \mathbf{F}^T(s)\mathbf{B}^T(s)\Phi^T(t_2, s)ds] \\
 &\quad + E[\int_{t_0}^{t_1} \Phi(t_1, s)\mathbf{B}(s)\mathbf{F}(s)ds \mathbf{X}_0^T\Phi^T(t_2, t_0)] \\
 &\quad + E[\int_{t_0}^{t_1} \int_{t_0}^{t_2} \Phi(t_1, s_1)\mathbf{B}(s_1)\mathbf{F}(s_1) \\
 &\quad \quad \quad \mathbf{F}^T(s_2)\mathbf{B}^T(s_2)\Phi^T(t_2, s_2) ds_2 ds_1].
 \end{aligned} \tag{6}$$

#### 4. Example

Let the matrix  $\mathbf{A}(t)$  be equal to:

$$\mathbf{A}(t) = \begin{bmatrix} 1 & t \\ 0 & D \end{bmatrix}, \tag{7}$$

where  $D$  is random variable with known probability density function (uniform distribution from 1 to 2):

$$f_D(x) = \begin{cases} 1 & x \in \langle 1, 2 \rangle \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

The fundamental matrix is equal to [2]:

$$\Phi_{\mathbf{A}}(t, t_0) = \begin{bmatrix} \exp(t - t_0) & w_1(t, t_0) \\ 0 & \exp(2D((t - t_0)/2)) \end{bmatrix}, \tag{9}$$

where:

$$w_1(t, t_0) = \frac{\exp(-t_0)(-\exp(t) + \exp(2D((t - t_0)/2) + t_0))(t + t_0)}{2D - 2}. \tag{10}$$

For simplicity, the force  $\mathbf{F}(t)$  is equal to zero and the initial value is equal to:

$$\mathbf{X}_0 = \begin{bmatrix} X_0 \\ 1 \end{bmatrix}, \tag{11}$$

where  $X_0$  is a random variable with known probability density function (uniform distribution from 0 to 2):

$$f_{X_0}(x) = \begin{cases} 0.5 & x \in \langle 0, 2 \rangle \\ 0 & \text{otherwise} \end{cases} . \quad (12)$$

The solution of the initial value problem is equal:

$$\mathbf{X}(t) = \begin{bmatrix} X_0 \exp(t) - \frac{(\exp(t) - \exp(Dt))t}{2D-2} \\ \exp(Dt) \end{bmatrix} . \quad (13)$$

Using the equations (3), (4), (5), (6) and (13) one can determine:

- the expected value (Fig. 1 and Fig. 2):
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$$\mu_{\mathbf{X}}(t) = \begin{bmatrix} \exp(t) \left( 2 - t\gamma + t\text{Chi}(t) - t \ln(t) + t\text{Shi}(t) \right) \\ \frac{\exp(t)(\exp(t) - 1)}{t} \end{bmatrix} , \quad (14)$$

where  $\gamma$  is Euler's constant, Chi is the hyperbolic cosine integral and Shi is the hyperbolic sine integral.

- the variance:

$$\sigma_{\mathbf{X}}^2(t) = \begin{bmatrix} \sigma_{X_1}^2(t) \\ \sigma_{X_2}^2(t) \end{bmatrix} , \quad (15)$$

where:

$$\sigma_{X_1}^2(t) = \frac{1}{12} \exp(2t) \left( 32 - 6t(4\gamma + (\exp(t) - 1)^2t) - 3t(4(t^2 - 2)\text{Ei}(t) + t^2(-4\text{Ei}(2t) + \ln(16)) - 4 \ln(1/t) + 4 \ln(t)) \right) , \quad (16)$$

$$\sigma_{X_2}^2(t) = \frac{\exp(3t) \sinh(t)}{t} , \quad (17)$$

where Ei is the exponential integral function,

- the cross-correlation of the excitation and the response:

$$\mathbf{R}_{\mathbf{FX}}(t_1, t_2) = [0 \quad 0] . \quad (18)$$

In the same way one can determine the correlation of the response (Fig. 3 and Fig. 4).

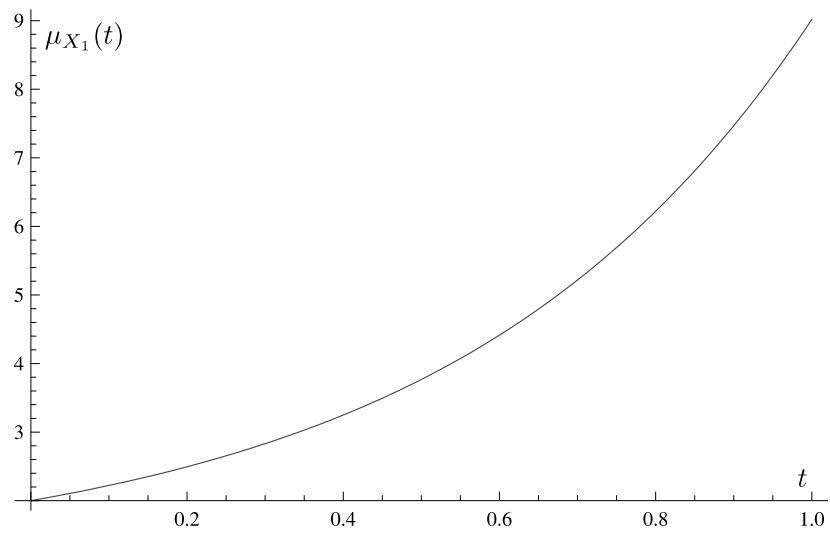


Fig. 1. Expected value  $\mu_{X_1}(t)$

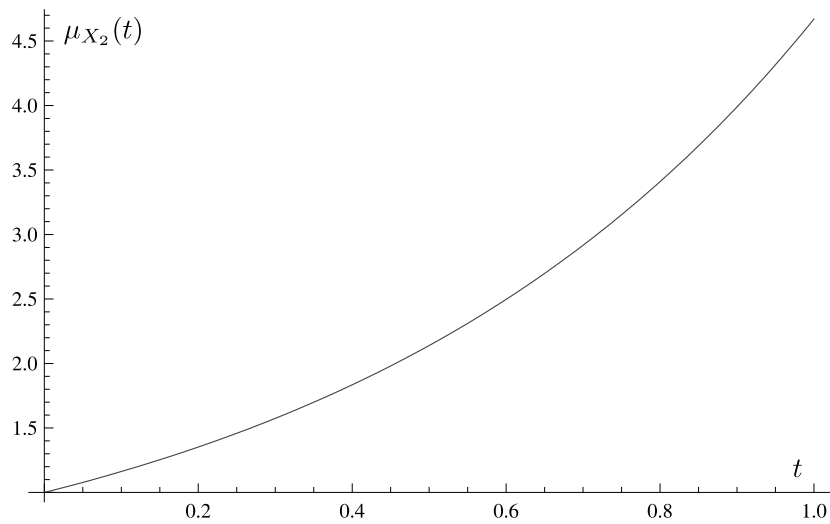


Fig. 2. Expected value  $\mu_{X_2}(t)$

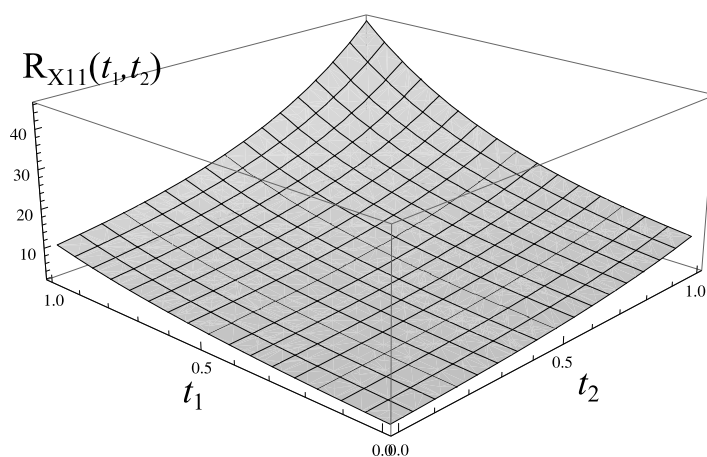


Fig. 3. Correlation of the response  $R_{X11}(t_1, t_2)$

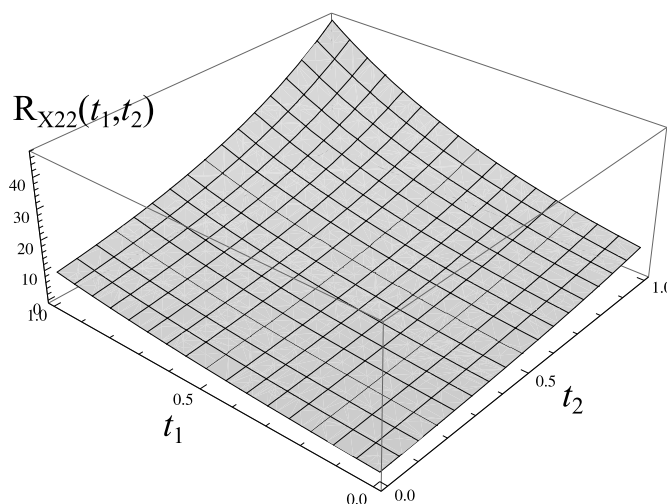


Fig. 4. Correlation of the response  $R_{X22}(t_1, t_2)$

## 5. Conclusions

In the practical engineering calculations to obtain the moments of the system response (1) the direct method should be used (the fundamental matrix of the system (1) must be known). It is impossible to apply the method of transformation of equation (1) into deterministic equations of the moments. After transformation of equation (1) into the moment equations, the structure of the equation is changed and it is practically impossible to obtain the analytical solutions for both the expected value and the correlation function.

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