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On Maximum Induced Matching Numbers of Special Grids

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ABSTRACT: A subset M of the edge set of a graph G is an induced matching of G if given any two edges $e_1, e_2 \in M$, none of the vertices on e_1 is adjacent to any of the vertices on e_2. Suppose that $Max(G)$, a positive integer, denotes the maximum size of M in G, then, M is the maximum induced matching of G and $Max(G)$ is the maximum induced matching number of G. In this work, we obtain upper bounds for the maximum induced matching number of grid $G = G_{n,m}$, $n \geq 9, m \equiv 3 \mod 4, m \geq 7$, and nm odd.

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1. Introduction

For a graph G, let $V(G), E(G)$ be vertex and edge sets respectively and let $e \in E(G)$. We define $e = uv$, where $u, v \in V(G)$ and the respective order and size of $V(G)$ and $E(G)$ are $|V(G)|$ and $|E(G)|$. For some $M \subseteq E(G)$, M is an induced matching of G if for all $e_1 = u_iu_j$ and $e_2 = v_iv_j$ in M, $u_kv_l \notin M$, where k and l are from $\{i, j\}$. Induced matching, a variant of the matching problem, was introduced in 1982 by Stockmeyer and Vazirani [10] and has also been studied under the names strong matching [7] and "risk free" marriage problem [8]. It has found theoretical and practical applications in a lot of areas including network problems and cryptology [3]. For more on induced matching and its applications, see [2], [3], [4], [5] and [11].

The size $|M|$ of an induced matching M of G is a positive integer and translates as the maximum induced matching number $Max(G)$ (or strong matching number) of

G if $|M|$ is maximum. Obtaining $Max(G)$ is $NP-$hard, even for regular bipartite graphs [4]. However, $Max(G)$ of some graphs have been found in polynomial time such as the cases in [3], [6].

A grid $G_{n,m}$ is the Cartesian product of two paths P_n and P_m, resulting in n-rows and m-columns. Marinescu-Ghemaci in [9], obtained the $Max(G)$ for $G_{n,m}$, grid where both n, m are even; either of n and m is even and for quite a number of grids $G_{n,m}$ where nm is odd, which is called the odd grid in [1]. Marinescu-Ghemaci [9] also gave useful lower and upper bounds and conjectured that the $Max(G)$ of grids can be found in polynomial time and also by combining the maximum induced numbers of partitions of odd grids, Marinescu-Ghemaci confirmed that for any odd grid $G \equiv G_{n,m}$, $Max(G) \leq \lfloor \frac{nm+1}{4} \rfloor$. This bound was improved on in [1] for the case where $n \geq 9$ and $m \equiv 1 \mod 4$.

In this paper, the Marinescu-Ghemaci's bound for the case where $n \geq 9$ and $m \equiv 3$ mod 4 is considered and more compact values are obtained. The results in this work, combined with some of the results in [9], confirm the maximum induced matching numbers of certain graphs, whose lower bounds were established in [9].

2. Definitions and Preliminary Results

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths P_n and P_m with $V(P_n) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_m) = \{v_1, v_2, \cdots, v_m\}$. We adopt the following notations which are similar to those in [1]:

$$V_i = \{u_1v_i, u_2v_i, \cdots, u_nv_i\} \subset V(G_{n,m}), \quad i \in [1, m],$$

$$U_i = \{u_iv_1, u_iv_2, \cdots, u_iv_m\} \subset V(G_{n,m}), \quad i \in [1, n].$$

For edge set $E(G_{n,m})$ of $G_{n,m}$, if $(u_iv_j\ u_kv_j) \in E(G_{n,m})$ and $(u_iv_j\ u_iv_k) \in E(G_{n,m})$, we write $u_{(i,k)}v_j \in E(G_{n,m})$ and $u_iv_{(j,k)} \in E(G_{n,m})$ respectively.

A saturated vertex v is any vertex on some edge in M, otherwise, v is unsaturated, cf. [1]. We define v as saturable if it can be saturated relative to the nearest saturated vertex. Any vertex that is at least distant-2 from the nearest saturated vertex is saturable. By this definition, therefore, it is clear that a saturated vertex is at first saturable. However, not every saturable vertex is saturated. The set of all saturable vertices on a graph G is denoted by $V_{sb}(G)$ while the set of saturated vertices is $V_{st}(G)$. Clearly, $|V_{st}(G)|$ is even and $V_{st}(G) \subseteq V_{sb}(G)$. Free saturable vertex set (FSV) is the set of saturable vertices which can not be on any members of M. In other words, $v \in FSV$ is a saturable vertex of graph G, which is not adjacent to some saturable vertex $u \in G$. Note that $FSV = V_{sb} \backslash V_{st}$. Let G be a $G_{n,m}$ grid. We define $G^{|k|}$ as a $G_{n,k}$ subgraph of G induced by $\{V_{i+1}, V_{i+2}, \cdots, V_{i+k}\}$. An unsaturated vertex $v \in G$ is unsaturable if $v \notin FSV$ and $v \notin V_{sb}(G)$. Furthermore, for positive integers a and b, $a < b$, $[a, b] := \{a, a + 1, \cdots, b\}$.

The following results from [9] on G, a $G_{n,m}$ grid, are useful in this work:

Lemma 2.1. *Let $m, n \geq 2$ be two positive integers and let G be a $G_{n,m}$ grid. Then,* *(a) If* $m \equiv 2 \mod 4$ *and* n *odd then* $|V_{sb}(G)| = \frac{mn+2}{2}$*; and* $|V_{sb}(G)| = \frac{mn}{2}$ *otherwise;*

(b) for $m \geq 3$*,* m *odd,* $|V_{sb}(G)| = \frac{nm+1}{2}$*, for* $n \in \{3, 5\}$.

Theorem 2.2. *Let* G *be a* $G_{n,m}$ *grid with* $2 \leq n \leq m$*. Then,*

(a) if n *even and* m *even or odd, then* $Max(G) = \lceil \frac{mn}{4} \rceil$*;*

(b) if $n \in \{3, 5\}$ *then for*

(i) $m \equiv 1 \mod 4$, $Max(G) = \frac{n(m-1)}{4} + 1$,

(ii) $m \equiv 3 \mod 4$, $Max(G) = \frac{n(m-1)+2}{4}$.

The following theorem is the statement of the bound given by Marinescu-Ghemaci [9].

Theorem 2.3. *Let* G *be a* $G_{n,m}$ *grid,* $m, n \geq 2$*,* mn *odd. Then* $Max(G) \leq \lfloor \frac{mn+1}{4} \rfloor$.

3. Maximum Induced Matching Number of Odd Grids

The following lemma and the remark describe the importance of the saturation status of certain vertices in $G_{5,p}$ grid, where $p \equiv 2 \mod 4$.

Lemma 3.1. *Let* G *be a* $G_{n,m}$ *grid and let* $\{V_{i+1}, V_{i+2}, \cdots, V_{i+p}\} \subset G$ *induce* $G^{|p|}$*, a* $G_{5,p}$ *subgrid of* G*, where* $p \equiv 2 \mod 4$*. Suppose that* M_1*, is an induced matching of* $G^{|p|}$ *and that for* $u_3v_{i+1} \in V_{i+1} \subset V(G^{|p|})$*,* $u_3v_{i+1} \notin V_{st}(G^{|p|})$*. Then,* $V_{st}(G^{|p|}) \leq 10k + 4$*, for positive integer* k*, where* $p = 4k + 2$ *and* M_1 *is not a maximum induced matching of* $G^{|p|}$.

Proof. For a positive integer k, let $p = 4k + 2$, $G^{|2|}$ and $G^{|p-2|}$ be partitions of G_1, induced by $\{V_{i+1}, V_{i+2}\}$ and $\{V_{i+3}, V_{i+4}, \cdots, V_{i+p}\}$, respectively. Since u_3v_{i+1} is not saturated in $G^{|2|}$, it easy to check that $|V_{sb}(G^{|2|})| = 5$. From [9], $|V_{sb}(G^{|p-2|})| = |V_{st}(G^{|p-2|})| = 10k$. Thus $|V_{sb}(G^{|p|})| \leq |V_{sb}(G^{|2|})| + |V_{sb}(G^{|p-2|})| \leq 10k + 5$ and therefore, $|V_{st}(G^{|p|})| \leq 10k + 4$ since $|V_{st}(G)|$ is even, for any graph G. This is a contradiction since by [9], $|V_{st}(G^{|p|})| = 10k + 6$. □

Remark 3.2. It should be noted that M_1 in Lemma 3.1 will still not be a maximum induced matching of $G^{|p|}$ if for the vertex set $A = \{u_1v_{i+1}, u_5v_{i+1}, u_1v_{i+p}, u_3v_{i+p}, u_5v_{i+p}\} \subset V(G^{|p|})$, any member of A is unsaturated.

Lemma 3.3. *Suppose* $u_{(1,2)}v_i$*,* $u_5v_{(i-1,i)} \in M$ *or* $u_{(1,2)}v_i$*,* $u_5v_{(i,i+1)} \in M$*, where* M *is an induced matching of* G*, a* $G_{5,m}$ *grid,* $m \equiv 3 \mod 4$*,* $m \geq 23$ *and* $1 < i < m$*,* $i \notin \{4, m-3\}$*. Then* M *is not a maximum induced matching of* G.

Proof. Let G be partitioned into $G^{|m(1)|}$ and $G^{|m(2)|}$, which are induced respectively by $A = \{V_1, V_2, \cdots, V_i\}$ and $B = \{V_{i+1}, V_{i+2}, \cdots, V_m\}$. Suppose that M is a maximum induced matching of G.

Case 1: $i \equiv 1 \mod 4$.

Let $m = 4k + 3$ and set $i = 4t + 1$, where $k \geq 5$ and $t > 0$. Then, $|m(1)| \equiv 1 \mod 4$ and $|m(2)| \equiv 2 \mod 4$. Since u_1v_i, u_2v_i, u_5v_i and u_5v_{i-1} are saturated vertices in V_i and V_{i-1}, then the only FSV member on V_{i-1} is u_3v_{i-1}. Suppose that u_3v_{i-1} remains unsaturated. Let $G^{|m(3)|} \subset G^{|m(1)|}$ be induced by $\{V_1, V_2, \cdots, V_{i-2}\}$, where $|m(3)| \equiv 3 \mod 4$. By [9], $|V_{st}(G^{|m(3)|})| = 10t-4$. Thus, $|V_{st}(G^{|m(1)|})| \leq 10t$. Suppose that u_3v_{i-1} is saturated, then, $u_3v_{(i-1,i-2)} \in M$. Thus, $u_3v_{i-3} \in V_{i-3} \subset G^{|m(4)|}$ is unsaturable, where $G^{|m(4)|}$ is $G^{|m(3)|} \backslash V_{i-2}$. Note that $|m(4)| \equiv 2 \mod 4$. From Lemma 3.1, therefore, $|V_{st}(G^{|m(4)|})| \leq 10t - 6$ and thus, $|V_{st}G^{|m(1)|}| \leq 10t - 6 + 6 = 10t$. Now, since u_1v_i, u_2v_i and u_5v_i are saturated vertices in V_i, then, $u_3v_{i+1}, u_4v_{i+1} \in V(G^{|m(2)|})$ are saturable vertices in $G^{|m(2)|}$.

Claim: Edge $u_{(3,4)}v_{i+1}$ belongs to M.

Reason: Suppose that both u_3v_{i+1} and u_4v_{i+1} are not saturated, then V_{i+1} contains no saturable vertices. Let $G^{|m(2)|} \backslash \{V_{i+1}\} = G^{|m(5)|}$, where $|m(5)| \equiv 1 \mod 4$. Thus, $|V_{st}(G)| \leq |V_{st}G^{|(m(1))|}| + |V_{st}(G^{|m(5)|})| = 10k + 2$, which is less than the required saturated vertices by 4 and hence the claim. Now, $u_{(3,4)}v_{i+1}$ belongs to M. Clearly for $G^{|m(5)|}$ defined above, $|V_{sb}(G^{|m(5)|})| = 10(k - t) + 3$ and suppose $u_3v_{i+1}, u_4v_{i+1} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10k + 5$. In fact, $|V_{st}(G)| = 10k + 4$. Thus establishing the first part of the case that with $u_{(1,2)}v_i$,$u_5v_{(i-1,i)} \in M$, $M \neq Max(G)$.

For the second part of the case, suppose that $u_{(1,2)}v_i$, $u_5v_{(i,i+1)} \in M$. Let $G^{|n(1)|} = G^{|m(1)|} \backslash \{V_i\}$ and $G^{|n(2)|} = G^{|m(2)|} \cup \{V_i\}$. Now, $|n(1)| \equiv 0 \mod 4$ and $|n(2)| \equiv 3 \mod 4$. Consequently, $|V_{st}(G^{|n(2)|})| = 10(k - t) + 6$. Now, on $V_{i-1} \subset G^{|n(1)|}$, only vertices u_3v_{i-1} and u_4v_{i-1} are saturable. Suppose they are both not saturated after all. Let $G^{|n(3)|} \subset G^{|n(1)|}$ be induced by $\{V_1, V_2, \cdots, V_{i-2}\}$, where $|n(3)| \equiv 3 \mod 4$. $|V_{st}(G^{|n(3)|})| = 10t-4$. Thus $|V_{st}(G)| = 10k+2$. Therefore, M requires four saturated vertices to be a maximum induced matching of G. Now, $|V_{sb}(G^{|n(3)|})| = 10t - 2$, and thus, $V(G^{|n(3)|})$ contains two extra FSV vertices, say, v_1, v_2 which are not adjacent. Thus, the maximum number of saturable vertices from the vertex set $\{v_1, v_2, u_3v_{i-1}, u_4v_{i-1}\}$ is 2. Therefore, $|V_{st}(G)| \leq 10k + 4$, which is a contradiction.

Case 2: $i \equiv 2 \mod 4$.

Let $G^{|p(1)|}$ and $G^{|p(2)|}$ be partitions of G induced by $\{V_1, V_2, \cdots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$, with $m = 4k + 3$ and $i = 4t + 2$. Let $u_{(1,2)}v_i$ and $u_5v_{(i-1,i)} \in M$. Since $u_{(1,2)}v_i$ belongs in M of G, then u_3v_i cannot be saturated. Thus, $|V_{st}(G^{|p(2)|})| \geq 10(k - t) + 2$ for M to be maximal. It can be seen that $|p(2)| \equiv 1 \mod 4$. Now, u_3v_{i+1} and u_4v_{i+1} are saturable vertices in V_{i+1}. Suppose both of them are not saturated, then for $G^{|p(3)|}$ induced by $\{V_{i+2}, V_{i+3}, \cdots, V_m\}$, where $|p(3)| \equiv 0 \mod 4$, $|V_{st}(G^{|p(3)|})| \leq 10(k - t)$. Thus u_3v_{i+1} and v_4v_{i+1} are saturable vertices and in fact, $u_{(3,4)}v_{i+1} \in M$. On V_{i+2}, therefore, there exists three saturable vertices u_1v_{i+1}, u_2v_{i+2} and u_5v_{i+5}. Suppose none of these three vertices are saturated. Then, $|V_{st}(G^{|p(3)|})| \leq |V_{st}(G^{|p(4)|})| + 2$, with $G^{|p(4)|}$ induced by $\{V_{i+3}, \cdots, V_m\}$ and $|p(4)| \equiv 3 \mod 4$ and thus, $|V_{st}(G^{|p(2)|})| \leq 10(t - k) - 2$. Therefore it requires extra four saturated vertices for M to be maximum. There exist two other saturable vertices, $v_1, v_2 \in V(G^{|p(4)|})$ (since $V_{st}(G^{|p(4)|}) = 10(k-t) - 4$ and $V_{sb}(G^{|p(4)|}) = 10(k-t) - 2$). Clearly, v_1, v_2 are not adjacent, else they would have formed an edge in M. Suppose $v_1, v_2 \in V_{i+3}$. For v_1 and v_2 to be saturated, they have to be u_5v_{i+3} and one of u_1v_{i+3} and u_2v_{i+3}. Thus, $u_5v_{i+2,i+3} \in M$ and one of $u_1v_{(i+2,i+3)}$ $u_2v_{(i+2,i+3)}$ or $u_{(1,2)}v_{i+2}$ belongs to M. Let $G^{|p(5)|}$ be induced by $\{V_{i+4}, \cdots, V_m\}$, where $|p(5)| \equiv 2 \mod 4$. Now, since $v_5v_{(i+2,i+3)} \in M$, then $u_5v_{i+5} \in V_{i+4}$ is unsaturable and therefore, by Remark 3.2, $|V_{st}(G^{|p(5)|})| = 10(k-t-1) + 4$ and thus, $|V_{st}(G^{|p(2)|})| = 10(k-t)$, which is less than required. The case of $u_5v_{(i,i+1)} \in M$ is the same as the case of $u_5v_{(i-1,i)} \in M$ for $i \equiv 2 \mod 4$.

Case 3: $i \equiv 0 \mod 4$, $i \geq 6$ or $i \leq m-5$, with $u_{(1,2)}v_i, u_5v_{(i-1,i)} \in M$. Let $G^{|r(1)|}$ and $G^{|r(2)|}$ be partitions of G which are induced respectively by $\{V_1, V_2, \cdots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$. Since $i \equiv 0 \mod 4$, then $|r(1)| \equiv 0 \mod 4$, while $|r(2)| \equiv 3 \mod 4$. Also, $u_5v_{(i-1,i)} \in M$, implies u_5v_{i-1} is unsaturable. Since $i-2 \equiv 2 \mod 4$, then by Lemma 3.1 and Remark 3.2, $|V_{st}(G^{|r(1)|})| \leq 10t - 2$, implying that for M to be maximal, $|V_{st}(G^{|r(2)|})| \geq 10(k-t) + 8$. It can be seen that V_{i+1} has two only saturable vertices u_3v_{i+1}, u_4v_{i+2} left. It should also be noted that if any of u_3v_{i+1} and u_4v_{i+2} is saturated, then u_3v_{i+3} can not be saturated in $G^{|r(3)|}$, a subgrid of $G^{|r(2)|}$ induced by $\{V_{i+2}, V_{i+3}, \cdots, V_m\}$, with $|r(3)| \equiv 2 \mod 4$. Thus suppose $u_3v_{i+1}, u_4v_{i+2} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10(k-t) + 4$. Likewise, if $u_3v_{i+1}, u_4v_{i+2} \notin V_{st}(G)$, $|V_{st}(G)| \leq 10t - 2 + 10(k-t) + 6$. The case of $u_5v_{(i,i+1)} \in M$ follows the same argument as the case of $u_5v_{(i-1,i)} \in M$. □

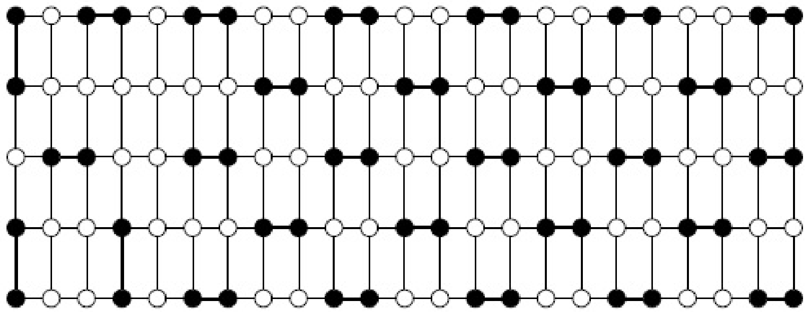

Figure 1: A Grid $G \equiv G_{5,23}$ with $Max(G) = 28$, $u_{(1,2)}v_1$, $u_{(1,2)}v_4 \in M$ of G

Remark 3.4.

(a) In the case of $i \equiv 0 \mod 4$ in Lemma 3.3, M remains a maximum induced matching when $i = 4$ or when $i = m-3$ as seen in Figure 1 of $Max(G) = 28$ of $G_{5,23}$.

(b) It should be noted that the case of $i \equiv 3 \mod 4$ has been taken care of by the case of $i \equiv 1 \mod 4$ by 'flipping' the grid from right to left or vice versa.

(c) From Lemma 3.3, we note that if for some induced matching M of $G_{5,m}$, $m \equiv 3 \mod 4$, $u_{(1,2)}v_i$ and $u_5v_{(i-1,i)}$ (or $u_5v_{(i,i+2)}$) $\in M$, then M is not a maximal induced matching of G for any $1 < i < m$.

Next we investigate some induced matching M of $G_{5,m}$ if it contains $u_{(1,2)}v_i$ and $u_{(4,5)}v_i$.

Lemma 3.5. *Suppose $G = G_{5,m}$, where $m \geq 23$ and $m \equiv 3 \mod 4$. Let $u_{(1,2)}v_i$,-$u_{(4,5)}v_i \in M$, an induced matching of G and $1 < i < m$, $i \not\equiv 0 \mod 4$ then M is not a maximum induced matching of G.*

Proof. Let M be an induced matching of $G = G_{5,m}$. Suppose that $i \equiv 2 \mod 4$. Let $G^{|m(1)|}$ and $G^{|m(2)|}$ be partitions of G induced by $\{V_1, V_2, \cdots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$. Since $u_{(1,2)}v_i, u_{(4,5)}v_1 \in M$, then, u_3v_i is unsaturated. Let $i = 4t + 2$, for some positive integer t, by Lemma 3.3, $|V_{st}(G^{|m(1)|})| = 10t + 4$. Now, only u_3v_{i+1} is saturable on V_{i+1}. Let $G^{|m(3)|} \subset G^{|m(2)|}$, induced by $\{V_{i+2}, \cdots, V_m\}$. Clearly $|m(3)| = |m(2)| - 1 = 4(k - t)$. Therefore, $|V_{st}(G^{|m(3)|} \cup u_3v_i)| \leq 10(k - t) + 1$, which, in fact, is $10(k - t)$. Thus, $|V_{st}(G)| = 10k + 4$.

Now, suppose $i \equiv 1 \mod 4$. Let $G^{|n(1)|}$ be induced by $\{V_1, V_2, \cdots, V_i\}$ and let $G^{|n(2)|}$ be induced by $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$. Since $|n(1)| = 4t + 1$, it is easy to see that $|n(2)| \equiv 2 \mod 4$ and hence, $|n(2)| = 4(k - t) + 2$.

Claim: For M to be maximum, both u_3v_{i-1} and u_3v_{i+1} must be saturated.

Reason: Suppose, say u_3v_{i-1} is not saturated. Then, no vertex on V_{i-1} is saturable. Now, let $\{V_1, V_2, \cdots, V_{i-2}\}$ induce grid $G^{|n(3)|}$, with $|n(3)| \equiv 3 \mod 4$. Then, $|V_{st}(G^{|n(3)|})| = 10t - 4$, and thus, $G^{|n(1)|} = 10t$. Also, let $G^{|n(4)|}$ be induced by $\{V_{i+2}, V_{i+3}, \cdots, V_m\}$. Since $|n(4)| = 4(k - t) + 1$, then for $G^{|n(4)|} + u_5v_{i+1}$, $|V_{sb}[(G^{|n(4)|}) \cup u_3v_{i+1}]| = 10(k - t) + 4$. Therefore, $|V_{st}(G)| \leq 10k + 4$. Now suppose $u_3v_{(i-2,i-1)} \in M$ and let $G^{|n(5)|}$ be induced by $\{V_1, V_2, \cdots, V_{i-3}\}$, with $|n(5)| \equiv 2 \mod 4$. By Lemma 3.1, $|V_{st}(G^{|n(5)|})| = 10t - 6$. Thus, $|V_{st}(G^{|n(1)|})| = 10t$ and therefore, $|V_{st}(G)| \leq 10k + 4$, which is less than required number by at least 2. Hence, $M \neq Max(G)$. □

Remark 3.6. Like in Remark 3.4, for $i \equiv 0 \mod 4$, it can be seen that $u_{(1,2)}v_1, u_{(1,2)}v_4$ or $u_{(1,2)}v_{m-3}, u_{(1,2)}v_m$ can be in M if M is a maximum induced matching of G. Also given $i \equiv 0 \mod 4$ and $4 < i < m - 3$, for at most one i in $[4, m - 3]$ for which $u_{(1,2)}v_i$ can be a member of maximal M.

Next we investigate the maximality of the induced matching of $G = G_{5,m}$, $m \equiv 3 \mod 4$.

Lemma 3.7. *Let $u_{(1,2)}v_i$, $u_4v_{(i-1,i)} \in M$ or $u_{(1,2)}v_i$, $u_4v_{(i,i+1)} \in M$, where M is an induced matching of G, a $G_{5,m}$ grid, $m \equiv 3 \mod 4$, $m \geq 23$ and $1 < i < m$, $i \not\equiv 0 \mod 4$. Then M is not a maximum induced matching of G.*

Proof. **Case 1**: $i \equiv 1 \mod 4$.
Suppose that $m = 4k + 3$ and $i = 4t + 1$, $t \geq 1$. Let $G^{|m(1)|}$ and $G^{|m(2)|}$ be two partitions of G, induced by $\{V_1, V_2, \cdots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$, respectively. Since $u_{(1,2)}v_i, u_4v_{(i-1,i)} \in M$, then there is no other saturated vertex on both of V_{i-1} and V_i. Let $G^{|m(3)|} \subset G^{|m(1)|}$ be a grid induced by $\{V_1, V_2, \cdots, V_{i-2}\}$. Now, $n(3) \equiv 3 \mod 4$. Therefore, $|V_{st}(G^{|m(3)|})| = 10t - 4$ and hence, $|V_{st}(G^{|m(1)|})| = 10t$. Now, $|m(2)| \equiv 2 \mod 4$, since $u_{(1,2)}v_i \in M$, then $u_1v_{i+1} \in V_{i+1}$ is unsaturable. From a previous result, $|V_{st}(G^{|n(2)|})| = 10(k - t) + 4$ and thus, $|V_{st}(G)| = 10k + 4$. For $u_4v_{(i,i+1)} \in M$, let $G^{|n(1)|}$ and $G^{|n(2)|}$ be induced by $G^{|m(1)|}\backslash V_i$ and $G^{|m(2)|} \cup V_i$. Then, $|n(1)| \equiv 0 \mod 4$ and $|n(2)| = 4(k - t) + 3$. It can be seen that on V_{i-1}, only u_3v_{i-1} and u_5v_{i-1} are saturable vertices.
Claim: Vertices u_3v_{i-1} and u_5v_{i-1} are not saturable for M to be maximal.
Reason: Suppose without loss of generality, that any of u_3v_{i-1} and u_5v_{i-1} is saturated, say u_5v_{i-1}. Then $u_5v_{(i-2,i-1)} \in M$. This implies that v_5v_{i-3} is not saturable in V_{i-3}. Now $\{V_1, V_2, \cdots, V_{i-3}\}$ induces a grid $G^{(|n(4)|)}$ and $|n(4)| \equiv 2 \mod 4$. Then, $|V_{st}(G^{|m(4)|})| = 10t-6$ and thus, $|V_{st}(G^{|n(1)|})| = 10t-4$. Now, since $|n(2)| = 4(k-t)+3$, $|V_{st}(G^{|m(2)|})| = 10(k - t) + 6$ and therefore, $|V_{st}(G)| = 10k + 2$.
Case 2: $i \equiv 2 \mod 4$.
Let $G^{|n(1)|}$ and $G^{|n(2)|}$ be two partitions of G, induced by $\{V_1, V_2, \cdots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \cdots, V_m\}$ respectively. Since $u_{(1,2)}v_i$ and $u_4v_{(i-1,i)} \in M$, vertex $u_5v_i \in V_{sb}(G^{|n(1)|})$, and therefore, $|V_{st}G^{|n(1)|}| = 10t + 4$, where $|n(1)| = 4t + 2$. Also, only u_3v_{i+1} and u_5v_{i+1} are saturable on V_{i+1}. Suppose without loss of generality, that both u_3v_{i+1} and u_5v_{i+1} are saturated and thus, $u_3v_{(i+1,i+2)}, u_5v_{(i+1,i+2)} \in M$. Now, suppose that $G^{|n(4)|}$ is induced by $\{V_{i+3}, V_{i+4}, \cdots, V_m\}$, with $|n(4)| = 4(k - t - 1) + 3$. By following the techniques employed earlier, it can be shown that $|V_{st}(G)| \leq |V_{st}(G^{|n(1)|})| + |V_{st}(G^{|n(2)|})| \leq 10k + 4$. The $u_4v_{(i,i+4)}$ case, has the same proof as the $u_4v_{(i-1,i)}$ case. □

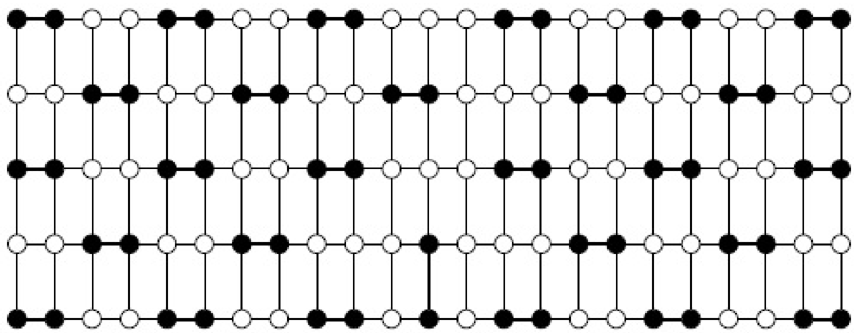

Figure 2: A $G \equiv G_{5,23}$ Grid with $Max(G) = 28$, $u_{1,2}v_i \in M, i \equiv 0 \mod 4$

Remark 3.8.

(a) There can be only one edge $u_{(1,2)}v_i \in M$ for which M is the maximum induced matching of $G_{5,m}$, if M contains $u_{(1,2)}v_i$ and $u_4v_{(i-1,i)}$ (or $u_4v_{(i,i+1)}$), and in this case, $i \equiv 0 \mod 4$ as shown in Figure 2.

(b) It should be noted that the proof of the case $i \equiv 1 \mod 4$ in Lemma 3.7 will hold for $i \equiv 3 \mod 4$ by flipping the grid from right to left.

The previous results and remarks yield the following conclusion.

Corollary 3.9. *Suppose that $m \geq 23$ and M is the maximum induced matching of G, some $G_{5,m}$ grid. Then, if for at most some positive integer i, $1 < i < m$, $u_{(1,2)}v_i \in M$, then, $i \equiv 0 \mod 4$.*

Lemma 3.10. *Let M be a matching of $G_{5,m}$ with $m \equiv 3 \mod 4$ and let $u_{(1,2)}v_i$, $u_{(1,2)}v_j \in M$, $1 < i < j < m$, such that $i \equiv 0 \mod 4$ and $j \equiv 0 \mod 4$, then M is not a maximum induced matching of G.*

The claim in Lemma 3.10 can easily be proved using earlier techniques and Lemma 3.1 and Remark 3.2.

Remark 3.11. It should be noted from the previous results and from Corollary 3.9 that if M is the maximum induced matching of $G_{5,m}$, $m \equiv 3 \mod 4$, then at most, M contains two edges of the form $u_{(1,2)}v_i$, $u_{(1,2)}v_j$ and j can only be 4 when $i = 1$ or i can only be $m - 3$ when $j = m$.

Theorem 3.12. *Let M be the maximum induced matching of G, a $G_{5,m}$ grid, with $m \geq 7$, $m = 4k + 3$ and $k \geq 1$. Let M contain $u_{(1,2)}v_1$ and $u_{(1,2)}v_4$ (or $u_{(1,2)}v_{m-3}$ and $u_{(1,2)}v_m$). Then there are at least $2k + 2$ saturated vertices on $U_1 \subset G$.*

Proof. For $u_{(1,2)}v_1$ and $u_{(1,2)}v_4$ to be in M, either $u_{(4,5)}v_4 \in M$ or $u_5v_{(3,4)} \in M$. Now, let $\{V_6, V_7, \cdots, V_m\}$ induce $G^{|m(1)|} \subset G$. Clearly, $|m(1)| \equiv 2 \mod 4$ and $|V_{st}(G^{|m(1)|})| = 10k - 4$.

Let $G^{|m(1)|} \setminus \{u_1v_6, u_1v_7, \cdots, u_1v_m\}$ induce $G^{|m(2)|} \subset G^{|m(1)|}$. Then, $G^{|m(2)|}$ is a $G_{4,m-5}$ subgraph of $G^{|m(1)|}$. Now, $|V_{st}(G^{|m(2)|})| \leq 8k - 4$. Thus for $V(U_1) \subset V(G^{|m(1)|})$, $|V(U)| \geq 2k$. Thus, U_1 contains at least $2k + 2$ (i.e. $\frac{m-1}{2}$) saturated vertices. □

Next we investigate $G_{3,m}$, where $m \equiv 3 \mod 4$.

Lemma 3.13. *Suppose that G is a $G_{3,m}$ grid with $m \equiv 3 \mod 4$ and M is an induced matching of $G_{3,m}$, with $\{u_{(1,2)}v_i, u_{(1,2)}v_{i+2}, u_{(1,2)}v_j, u_{(1,2)}v_{j+2}\} \in M$ and $i + 2 \geq j$. Then M is not a maximum induced matching of G.*

Proof. Suppose $i + 2 \geq j$. Since $m = 4k + 3$, $|V_{sb}(G)| = 6k + 5$ and $|V_{st}(G)| = 6k + 4$. Thus, G contains at most one FSV vertex. Now from the conditions in the hypothesis, it is clear that u_3v_{i+1} and u_3v_{j+1} are FSV members in G, which is a contradiction. Same argument hold if $i + 2 = j$ since both u_3v_{i+1} and u_3v_{i+3} are FSV vertexes in G. □

Remark 3.14. Suppose that G_n is $G_{3,n}$, a subgrid of $G_{3,m}$ and induced by $\{V_{i+1}, V_{i+2}, \cdots, V_{i+n}\}$ and G' is a subgraph of G, with $G' = G_n + \{u_3v_i, u_3v_{i+n+1}\}$, then the following are easy to verify. For

(a) $n \equiv 0 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$.

(b) $n \equiv 1 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$.

(c) $n \equiv 2 \mod 4$, $|V_{st}(G')| = |V_{sb}(G_n)|$.

(d) $n \equiv 3 \mod 4$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 1$.

Lemma 3.15. *Let $u_{(1,2)}v_j, u_{(1,2)}v_{j+3}, u_{(1,2)}v_k, u_{(1,2)}v_{k+3}, u_{(1,2)}v_l, u_{(1,2)}v_{l+3}$ be in M an induced matching of G a $G_{3,m}$ grid and $m \equiv 3 \mod 4$. Then M is not maximum induced matching of G.*

Proof. Case 1: $j + 3 = k$ and $l = k + 3$.
Suppose $m = 4p + 3$ and $G^{|m(1)|}$ is a subgraph of G, induced by $\{V_{j-1}, V_j, \cdots, V_{i+4}\}$. Then $|m(1)| = 12$ and $u_3v_{j-1}, u_3v_{i+4} \in FSV$. For one of u_3v_{j-1} and u_3v_{i+4} to be relevant for M to be a maximum induced matching of G, say u_3v_{j-1}, then for $G^{|m(2)|}$, induced by $\{V_1, V_2, \cdots, V_{j-2}\}$, $|V_{sb}(G^{|m(2)|})|$ must be odd, which can only be if $j-2 \equiv 3 \mod 4$. Suppose $j - 2 \equiv 3 \mod 4$, then $|V_{st}(G^{|m(2)|}) + u_3v_{j-1}| \leq |V_{sb}(G^{|m(2)|})| + 1 = 6q + 6$, where $|m(2)| = 4q + 3$, for $q \geq 1$, since $|m(1)| = 12$ and $|n(2)| \equiv 3 \mod 4$. Now let $G^{|m(3)|} = G^{|m(1)|} \cup G^{|m(2)|}$, where $|m(3)| = |m(1)| + |m(2)| \equiv 3 \mod 4$ and $G^{|m(4)|} \subset G$ be defined as a subgrid of G induced by $\{V_{i+5}, V_{i+6}, \cdots, V_m\}$. Clearly, $|m(4)| \equiv 0 \mod 4$. Since $|V_{sb}(G^{|m(4)|})| = |V_{st}(G^{|m(4)|})|$, which is even, then $|V_{st}(G^{|m(4)|} \cup u_3v_{i+4})| = |V_{st}(G^{|m(4)|})| = 6p - 6q - 18$. It can be seen that $|V_{st}(G^{|m(1)|}) \setminus \{u_3v_{j-1}, u_3v_{l+4}\}| = 14$. Therefore, $|V_{st}(G)| \leq 6p + 2$ instead of $6p + 4$, and hence a contradiction.
Case 2: $j + 3 < k$ and $k + 3 < l$.
As in Case 1 and without loss of generality, let $j - 2 \equiv 3 \mod 4$ and let $G^{|m(2)|}$ still be induced by $\{V_1, V_2, \cdots, V_{j-2}\}$. Also, let $G^{|m(4)|}$ be induced by $\{V_{l+5}, V_{l+6}, \cdots, V_m\}$, and set $|m(4)| \equiv 3 \mod 4$. Thus, u_3v_{j-1} and u_3v_{i+4} are both relevant for M to be a maximum induced matching of G, $|V_{st}(G^{|m(2)|} \cup V_{j-1})| = |V_{sb}(G^{|m(2)|})| + 1$ and $|V_{st}(G^{|m(4)|} \cup V_{l+4})| = |V_{sb}(G^{|m(4)|})| + 1$. Set $G^{|m(2)|} \cup V_{j-1} = G^{|m(2^+)|}$ and $G^{|m(4)|} \cup V_{i+4} = G^{|m(4^+)|}$ also let $\{V_j, V_{j+1}, V_{j+2}, V_{j+3}\}$ and $\{V_i, V_{i+1}, V_{i+2}, V_{i+3}\}$ induce $G^{|m(5)|}$ and $G^{|m(6)|}$, respectively. Furthermore, let $G^{|m(5^+)|} = G^{|m(5)|} \cup V_{j+4}$ and $G^{|m(6^+)|}$ contain, say, h columns of V_i in all, where $h \equiv 2 \mod 4$. Therefore, for $G^{|(m(7))|} = G \setminus \{G^{|m(2^+)|} \cup G^{|m(4^+)|} \cup G^{|m(5^+)|} \cup G^{|m(6^+)|}\}$, $|m(7)| = m - h = b \equiv 1 \mod 4$. Let $b = 4a + 1$, for some positive integer a and let $G^{|m(4)|} \subset G^{|m(7)|}$, where $G^{|m(7)|}$ is induced by $\{V_k, V_{k+1}, V_{k_2}, V_{k+3}\}$. Certainly, $u_3v_{k-1}, u_3v_{k+4}, u_3v_{j+4}, u_3v_{l-1} \in V_{sb}(G)$. Now, let $G^{|(4)|}$ be induced by $\{V_k, V_{k+1}, V_{k+2}, V_{k+3}\}$ and $G^{|4^{++}|}$ be induced by $G^{|(4)|} \cup \{V_{k-1}, V_{k+4}\}$, with $|4++| = 6$. So, $b - 6 \equiv 3 \mod 4$, which is odd and thus can only be the sum of an even and an odd positive integer. Therefore, let $G^{|m(8)|}$ and $G^{|m(9)|}$ be induced by $\{V_{j+5}, V_{j+6}, \cdots, V_{k-2}\}$ and $\{V_{j+5}, V_{j+6}, \cdots, V_{l-2}\}$, respectively, with $|m(8)| + |m(9)| = b$. Suppose thus, that $|m(8)| \equiv 0 \mod 4$, then, $|m(9)| \equiv 3 \mod 4$ and suppose $|m(8)| \equiv 1 \mod 4$, then $|m(9)| \equiv 2 \mod 4$. For $|m(8)| \equiv 0 \mod 4$, let

$G^{|m(10)|} = G^{|m(2^+)|+|m(5^+)|}$ be $G^{|m(2^+)|} \cup G^{|m(5^+)|}$ and $G^{|m(11)|} = G^{|m(6^+)|+|m(4^+)|}$ be $G^{|m(6^+)|} \cup G^{|m(4^+)|}$, where $|m(2^+)| + |m(5^+)| = 4q+9$ and $|m(4^+)| + |m(6^+)| = 4r+9$, where $|m(4)| = 4r+3$. Therefore, as defined, $b = |m(7)| = 4p - 4q - 4r - 15$ and thus $b-6 = 4(p-q-r-6)+3$. Set $p-q-r-6 = f$. Now, for $|m(8)|$ and $|m(9)|$, if $|m(8)| = 4g$, for some positive integer g, then $|m(9)| = 4(f-g)+3$. The maximal values of the subgrid of G is: $|V_{st}(G)| \leq |V_{st}(G^{|m(2^+)|} \cup G^{|m(5)|})| + |V_{st}(G^{|m(8)|} + \{u_3v_{j+4}, u_3v_{k-1}\})|$ $+ |V_{st}(G^{|m(4)|})| + |V_{st}(G^{|m(9)|} + \{u_3v_{k+4}, u_3v_{l-1}\})| + |V_{st}(G^{|m(6)|} \cup G^{|m(4^+)|})| \leq 6p+2$, which is less than $6p+4$ and hence a contradiction. For $|m(8)| \equiv 1 \mod 4$, and $|m(9)| \equiv 2 \mod 4$, we have $|m(8)| = 4g+1$ and hence $|m(9)| = 4(f-g)+2$ and $|V_{st}(G^{|m(9)|} \cup \{u_3v_{k+4}, u_3v_{l-1}\})| = 6(f-g)+4$ and thus, $|V_{st}(G)| \leq 6p+2$.

Case 3: $j+3 = k$ or $k+3 = i$.

Suppose as in Case 2, $j-2 \equiv 3 \mod 4$ and $m-(i+4) \equiv 3 \mod 4$. Let $G^{|n(1)|} \subset G$, a $G_{3,9}$ subgrid of G be induced by $\{V_{j-1}, v_j, \cdots, V_{j+7}\}$. Then for $G^{|n(2)|} = G^{|m(2)|} \cup G^{|n(1)|}$, $|n(2)| = |m(2)| + |n(1)|$, $|n(2)| \equiv 0 \mod 4$. Likewise, suppose $\{V_{i-1}, V_i, \cdots, V_m\}$ induces $G^{|n(3)|}$, for which $|n(3)| \equiv 1 \mod 4$. If $|n(2)|$ and $|n(3)|$ are $4q$ and $4r+1$ respectively, then $|n(4)| \equiv 2 \mod 4$. So far, $G^{|n(4)|}$, is induced by $\{V_{i+8}, V_{i+9}, \cdots, V_{l-2}\}$ and by Remark 3.14, $|V_{st}(G^{|n(4)|}) + \{u_3v_{j+7}, u_3v_{l-1}\}| = |V_{sb}(G^{|n(4)|})|$. By a summation similar to the one at the end of Case 2, $|V_{st}(G)| \leq |V_{st}G^{|n(2)|}| + |V_{st}(G^{|n(4)|})| + |V_{st}(G^{|n(3)|})| \leq 6p+2$. □

Remark 3.16.

(a) By following the technique employed in Lemma 3.15, it can be established that given $u_{(1,2)}v_i, u_{(1,2)}v_{i+2} \in M$ and $u_{(1,2)}v_j, u_{(1,2)}v_{j+2} \in M$ of G, a $G_{3,m}$ grid, $m \equiv 3 \mod 4$, $i+2 \leq j$, then M is not a maximum induced matching of G.

(b) Let M be an induced matching of G, a $G_{3,m}$ grid, and i be some fixed positive integer. Suppose $u_{(1_2)}v, i+8(n) \in M$, for all non-negative integer n for which $1 \leq i+8(n) \leq m$. Let M be the maximum induced matching of G. Then,

  (i) if $i > 1$, then $i-1$ is either $2, 3, 4$ or 6;

  (ii) if $i+8(n) < m$, for the maximum value of n, then $m-(i+8(n))$ is either $2, 3, 4$ or 6.

Based on the results so far, we note that if M is the maximum induced matching of G, a $G_{3,m}$ grid, $m \equiv 3 \mod 4, m \geq 11$, the maximum number of edges of the type $u_{(1,2)}v_k$ that is contained in M, k, a positive integer, is $k+2$ when $m = 8k+3$ and $k+3$ when $m = 8k+7$.

It can be easily established that for H that is a $G_{k,m}$ grid, with $k \equiv 0 \mod 4$ and $m \equiv 3 \mod 4$, which is induced by $\{U_1, U_2, \cdots, U_k\}$, if M_1 is a maximum induced matching of H, then, the least saturated vertices in U_k is $\frac{m-1}{2}$. The next result describes the positions of the members of M_1 in $E(H)$ if U_k contains $\frac{m-1}{2}$ saturated vertices.

Lemma 3.17. *Let H be a $G_{k,m}$ grid with $k \equiv 0 \mod 4$ and $m \equiv 3 \mod 4$ and let U_k contain the least possible, $\frac{m-1}{2}$, saturated vertices for which N remains maximum induced matching of H. Then, for any adjacent vertices $v', v'' \in U_k$, edge $v'v'' \notin M$.*

Proof. Induced by $\{U_1, U_2, \cdots, U_{k-2}\}$ and $\{U_{k-1}, U_k\}$ respectively, let $G_1^{|m|}$ and $G_2^{|m|}$ be partitions of H with $k-2 \equiv 2 \mod 4$. It can be seen that $|V_{st}(G_1^{|m|})| = |V_{sb}(G_1^{|m|})| = \frac{km-2m+2}{2}$. Since $|V_{st}(H)| = \frac{km}{2}$, then $|V_{st}(G_2^{|m|})| \leq m-1$. Now, let $G_3^{|m|}$ be a $G_{1,m}$ subgrid (a P_m path) of H, induced by U_k. By the hypothesis, U_k contains maximum of $\frac{m-1}{2}$ saturated vertices. Now, let u_kv_i, u_kv_{i+1} be adjacent and saturated vertices of $G_3^{|m|}$. Then there are $\frac{m-5}{2}$ other saturated vertices on $G_3^{|m|}$. Without loss of generality, suppose that each of the remaining $\frac{m-5}{2}$ saturated vertices in $G_3^{|m|}$ is adjacent to some saturated vertex in U_{k-1}. Now, suppose $u_{k-1}v_j$ is a saturable vertex in U_{k-1} and that $v \in V(H)$, such that $u_{k-1}v_j\ v \in M_1$. Now, $v \notin U_k$, since all the saturable vertices in U_k is saturated. Likewise, suppose $v \in U_{k-1}$ and then $u_{k-1}v_j\ v \in E(G_4^{|m|})$, where $G_4^{|m|}$ is a $G_{1,m}$ subgraph of H induced by U_{k-1}. Then, clearly, at least one of $u_{k-1}v_j$ and v is adjacent to a saturated vertex in $V_{st}(G_1^{|m|})$. Also, suppose that $v \in U_{k-2}$, since $|V_{sb}(G_1^{|m|})| = |V_{st}(G_1^{|m|})|$, then $|V_{st}(G_1^{|m|})| = |V_{st}(G_1^{|m|} + u_{k-1}u_j)|$. Hence $v \in FSV$ in $G_1^{|m|}$. Therefore, $|V_{st}H| \leq |V_{st}G_1^{|m|}| + |V_{st}G_2^{|m|}| \leq \frac{km-4}{2}$, which is a contradiction since $|V_{st}(H)| = \frac{km}{2}$, by [9]. □

Remark 3.18. The implication of Lemma 3.17 is that for a grid $H' \subset H$, which is induced by $\{U_1, U_2, \cdots, U_{k-2}\} \subset V(H)$, $k-2 \equiv 2 \mod 4$, suppose U_k contains the least possible number of saturated vertices, $\frac{m-1}{2}$, then $u_kv_2, u_kv_4, \cdots, u_kv_{m-1}$ are saturated as shown in the example in Figure 3, for which $k = 4$ and $m = 7$.

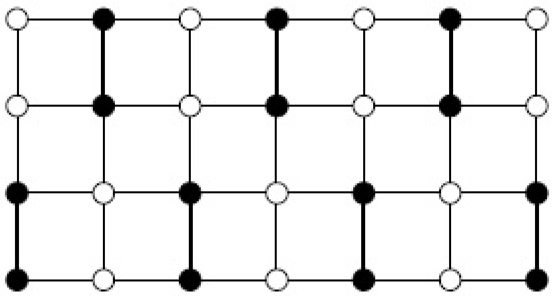

Figure 3: A $G_{4,7}$ Grid with $Max(G) = 7$

Lemma 3.19. *Let G be a $G_{3,m}$ with an induced matching M and $G^{|(9)|}$, induced by $\{V_i, V_{i+2}, \cdots, V_{i+8}\}$ be a $G_{3,9}$ subgrid of G. Suppose that $M' \subset M$ is an induced matching of $G^{|(9)|}$ such that $u_{(1,2)}v_i, u_{(1,2)}v_{i+8} \in M'$. No other edge $u_{(1,2)}v_{i+t}, 1 < t < i+7$ is contained in M'. Then for $G'^{|(9)|} \subset G^{|(9)|}$, defined as $G^{|(9)|} \backslash U_1$, $|V_{sb}(G'^{|(9)|})| \leq 8$.*

Proof. Let $G^{|(7)|} = G^{|(9)|} \backslash \{\{u_1v_{i+1}, u_iv_{i+2}, \cdots, u_1v_{i+7}\}, V_i, V_{i+8}\}$. It can be seen that $G^{|(7)|}$ is a $G_{2,7}$ subgrid of $G^{|(9)|}$. Clearly also, $G^{|(7)|} \subset G'^{|(9)|}$. Since

$u_{(1,2)}v_i, u_{(1,2)}v_{i+8} \in M'$, then, u_2v_{i+1} and u_2v_{i+7} can not be saturated. Let G_y be subgraph of $G^{|(7)|}$, defined as $G^{|(7)|} \setminus \{u_2v_{i+1}, u_2v_{i+7}\}$. Now, $|V(G_y)| = 12$ and $|V_{sb}(G_y)|$ can be seen to be at most 6. Thus $|V_{sb}(G'^{|(9)|})| = |V_{sb}(G_y)| + 2 = 8$, since u_2v_i and u_2v_{i+8} are saturated in M'. □

Remark 3.20. For $U_1 \subset G^{|(9)|}$ as defined in Lemma 3.19, U_1 contains at least 6 saturated vertices, implying that M' contains two edges whose four vertices are from U_1.

Corollary 3.21. *Let G be a $G_{3,m}$ grid with $m \geq 11$ and $m \equiv 3 \mod 4$. If M' is a maximum induced matching of G. Then M' contains at least $2k'$ edges from U_1, where $m = 8k' + 3$ or $m = 8k' + 7$.*

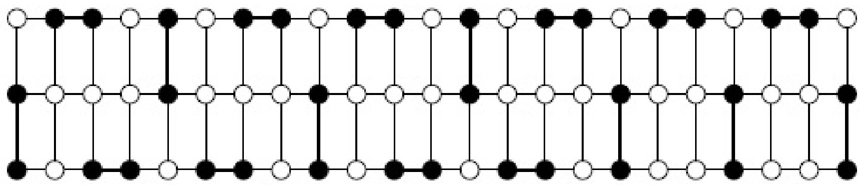

Figure 4: A $G \equiv G_{3,23}$ Grid with $Max(G) = 17$

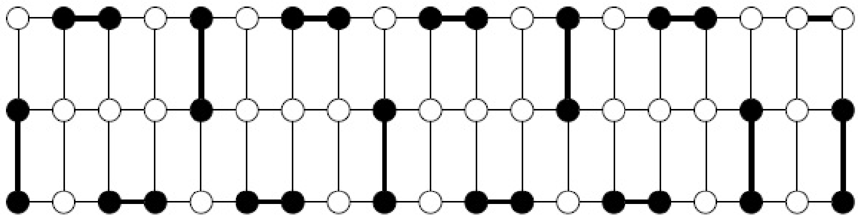

Figure 5: A $G \equiv G_{3,19}$ Grid with $Max(G) = 14$

Theorem 3.22. *Let G be a $G_{n,m}$ grid, with $m \geq 23$. Then for $n \equiv 1 \mod 4$, $Max(G) \leq \lfloor \frac{2mn-m-3}{8} \rfloor$.*

Proof. For $n \equiv 1 \mod 4$, $n - 5 \equiv 0 \mod 4$. Let G_1 and G_2 be partitions of G induced by $\{U_1, U_2, \cdots, U_{n-5}\}$ and $\{U_{n-4}, U_{n-3}, U_{n-2}, U_{n-1}, U_n\}$ respectively. Also, let M', M'' be maximum induced matching of G_1 and G_2 respectively.

Suppose, U_{n-5} contains at least $\frac{m-1}{2}$ saturated vertices, the least U_{n-5} can contain for M' to remain maximum induced matching of G_1. By Theorem 3.12, $U_1 \subset G_2$ (the U_{n-4} of G) contains at least $2k + 2$ saturated vertices with $k = \frac{m-3}{4}$. Following the proof of Theorem 3.12, it is shown that M'' contains $\frac{m-3}{4}$ edges of $U_1 \subset G_2$ and either of $u_{(1,2)}v_4$ and $u_{(1,2)}v_{m-3}$. Now, with $G = G' \cup G''$ and hence, $|M| \leq |M'| + |M''|$, it is obvious therefore, that for each edge $u_\alpha u_\beta \in U_{n-4}$ contained in M'', either u_α or u_β is adjacent to a saturated vertex on U_{n-5} and also, $u_{n-4}v_4$ (or $u_{n-4}v_{m-3}$) is adjacent to saturated $u_{n-5}v_4$ (or to saturated $u_{n-4}v_{m-3}$). Hence, $|V_{st}(G)| \leq \frac{2mn-m-7}{4}$ and thus, $Max(G) \leq \lfloor \frac{2mn-m-7}{8} \rfloor$. □

Theorem 3.23. *Let G be a $G_{n,m}$ grid with $n \equiv 3 \mod 4$ and $m \equiv 3 \mod 4$, $m \geq 11$. Then $Max(G) \leq \lfloor \frac{2mn-m+1}{8} \rfloor$ and $Max(G) \leq \lfloor \frac{2mn-m+5}{8} \rfloor$ for $m = 8k' + 3$ and $m = 8k' + 7$ respectively.*

Proof. The proof follows similar techniques as in Theorem 3.22. □

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