FRICTIONAL HEATING OF SLIDING SEMI-SPACES WITH SIMPLE THERMAL NONLINEARITIES

Ewa OCH*

*Faculty of Mechanical Engineering, Bialystok University of Technology, ul. Wiejska 45C, Bialystok, 15-351, Poland

e.och@doktoranci.pb.edu.pl

Abstract: In the article the nonstationary thermal problem of friction for two semi-spaces with taking into account their imperfect thermal contact and thermosensitivity of materials (simple nonlinearity), has been considered. The linearization of this problem has been carried out using Kirchhoff transformation, and next using the Laplace integral transform. The analytical solution to the problem in the case of constant speed sliding, has been obtained. On the basis of the obtained solutions and using Duhamel's formula, the analytical solution to the problem for sliding with constant deceleration, has been obtained, too. The results of numerical analysis are presented for two friction pairs.

Key words: Frictional Heating, Simple Thermal Nonlinearities, Thermosensitive Materials

1. INTRODUCTION

The review of analytical and numerical methods to solve the one-dimensional heat problems of friction, has been presented in the article by Yevtushenko and Kuciej (2012). It has been established, that there are practically no solutions to contact problem involving frictional heating, which would take into account the dependence of the thermal materials properties on temperature. In articles by Abdel-Aal (1997) and Abdel-Aal and Smith (1998) the solutions to heat problems of friction in case of separate heating of the two bodies have been obtained.

The aim of this work is to obtain a solution of the contact heat problem with friction for two thermosensitive semi-spaces, which relatively slide with constant speed. It is assumed that materials properties of the bodies have a simple nonlinearity, i.e. the coefficients of thermal conductivity and specific heat are dependent on temperature and their ratio – the thermal diffusivity – is constant (Kushnir and Popovych, 2011). For many friction materials dependence of thermal properties on temperature is linear (Gundlach, 1983; Abdel-Aal, 1997; Abdel-Aal and Smith, 1998; Sok, 2006):

$$K_l(T_l) = K_{l,0}\, K_l^*(T_l)\,,\; c_l(T_l) = c_{l,0}\, c_l^*(T_l) \tag{1}$$

where:

$$K_l^*(T_l) = 1 + \lambda_l (T_l - T_0)\,,\; c_l^*(T_l) \approx K_l^*(T_l)\,,\; l = 1,2 \tag{2}$$

Here and further, all values referring to the semi-spaces will have subscripts 1 and 2, respectively.

2. STATEMENT OF THE PROBLEM

The problem of contact interaction of two bodies (semi-spaces) with different thermal-physical properties is considered. It is supposed, that the constant compressive pressure p_0 in direction of z-axis of the Cartesian system of coordinates Oxyz are applied to the infinities in semi-spaces (Fig. 1). In the initial time moment $t = 0$ the semi-spaces start to slide with a constant speed V_0 in the positive direction of the y-axis. Due to friction the heat is generated on a contact plane $z = 0$. The sum of intensities of frictional heat fluxes directed into each semi-spaces is equal to the specific power of friction $q_0 = fV_0p_0$ (Ling, 1973). Because of the thermal resistance between surfaces of the bodies, the heat transfer with a constant coefficient of thermal conductivity of contact h takes place.

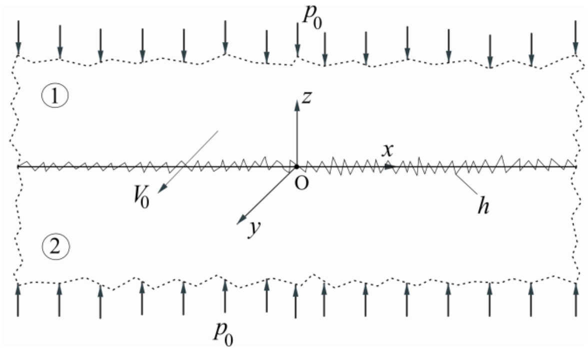

Fig. 1. Scheme of the problem

Let us find the distribution of the transient temperature fields $T_l(z,t)$, $l = 1,2$ in each of the semi-spaces from the solution to the following heat problem of friction:

$$K_1(T_1)\frac{\partial^2 T_1(z,t)}{\partial z^2} = \rho_1 c_1(T_1)\frac{\partial T_1(z,t)}{\partial t}\,,\; z > 0,\; t > 0\,, \tag{3}$$

$$K_2(T_2)\frac{\partial^2 T_2(z,t)}{\partial z^2} = \rho_2\, c_2(T_2)\frac{\partial T_2(z,t)}{\partial t}\,,\; z < 0,\; t > 0\,, \tag{4}$$

$$K_1(T_1)\frac{T_1(z,t)}{\partial z}\bigg|_{z=0} - K_2(T_2)\frac{\partial T_2(z,t)}{\partial z}\bigg|_{z=0} = -q,\ t>0, \tag{5}$$

$$K_1(T)\frac{\partial T_1(z,t)}{\partial z}\bigg|_{z=0} + K_2(T)\frac{\partial T_2(z,t)}{\partial z}\bigg|_{z=0} = \\ = h[T_1(0,t)-T_2(0,t)],\ t>0, \tag{6}$$

$$T_l(z,t) \to T_0,\ |z| \to \infty,\ l=1,2\ , \tag{7}$$

$$T_l(z,0) = T_0, |z| < \infty\ ,\ l=1,2\ . \tag{8}$$

By introducing dimensionless variables and parameters:

$$\zeta = \frac{z}{a},\ \tau = \frac{k_2 t}{a^2},\ K_0^* = \frac{K_{1,0}}{K_{2,0}},\ k^* = \frac{k_1}{k_2}, \tag{9}$$

$$Bi = \frac{h\,a}{K_{2,0}},\ T_a = \frac{q_0\,a}{K_{2,0}},\ T_{1,2}^* = \frac{T_{1,2}}{T_a},\ T_0^* = \frac{T_0}{T_a}, \tag{10}$$

the above mentioned non-linear transient boundary-value heat conduction problem of friction (3)–(8) can be represented in the dimensionless form:

$$K_1(T_1^*)\frac{\partial^2 T_1^*(\zeta,\tau)}{\partial\zeta^2} = \frac{c_1^*(T_1^*)}{k^*}\frac{\partial T_1^*(\zeta,\tau)}{\partial\tau},\ \zeta>0, \tau>0 \tag{11}$$

$$K_2(T_2^*)\frac{\partial^2 T_2^*(\zeta,\tau)}{\partial\zeta^2} = c_2^*(T_2^*)\frac{\partial T_2^*(\zeta,\tau)}{\partial\tau},\ \zeta>0,\ \tau>0 \tag{12}$$

$$K_0^* K_1^*(T_1^*)\frac{\partial T_1^*(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} - K_2^*(T_2^*)\frac{\partial T_2^*(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} = -1, \\ \tau>0, \tag{13}$$

$$K_0^* K_1^*(T_1^*)\frac{\partial T_1^*(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} + K_2^*(T_2^*)\frac{\partial T_2^*(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} = \\ = Bi[T_1^*(0,\tau)-T_2^*(0,\tau)],\ \tau>0, \tag{14}$$

$$T_l^*(\zeta,\tau) \to T_0^*, |\zeta| \to \infty,\ l=1,2 \tag{15}$$

$$T_l(\zeta,0) = T_0^*,\ |\zeta| < \infty\ l=1,2 \tag{16}$$

where, taking into account the relations (9), (10), the linear dependencies of thermal conductivity on temperature (2) express as follows:

$$K_l^*(T_l^*) = 1 + \Lambda_l(T_l^* - T_0^*),\ \Lambda_l = \lambda_l T_a\ ,\ l=1,2 \tag{17}$$

3. LINEARIZATION OF THE PROBLEM

To the linearization of the boundary-value problem (11)–(16), the Kirchhoff transform has been used (Kirchhoff, 1894):

$$\Theta_l = \int_{T_0^*}^{T_l^*} K_l^*(T_l^*)\ dT_l^*,\ l=1,2. \tag{18}$$

By applying the transformation (18) to the problem (11)–(16) and taking into account the relations $c_l^*(T_l^*) \approx K_l^*(T_l^*)$, we have:

$$\frac{\partial^2\Theta_1(\zeta,\tau)}{\partial\zeta^2} = \frac{1}{k^*}\frac{\partial\Theta_1(\zeta,\tau)}{\partial\tau},\ \zeta>0,\ \tau>0, \tag{19}$$

$$\frac{\partial^2\Theta_2(\zeta,\tau)}{\partial\zeta^2} = \frac{\partial\Theta_2(\zeta,\tau)}{\partial\tau},\ \zeta<0,\ \tau>0, \tag{20}$$

$$K_0^*\frac{\partial\Theta_1(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} - \frac{\partial\Theta_2(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} = -1,\ \tau>0, \tag{21}$$

$$K_0^*\frac{\partial\Theta_1(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\Theta_2(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} = \\ = Bi[T_1^*(0,\tau)-T_2^*(0,\tau)],\ \tau>0, \tag{22}$$

$$\Theta_l(\zeta,\tau) \to 0, |\zeta| \to \infty, \tau>0,\ l=1,2\ , \tag{23}$$

$$\Theta_l(\zeta,0) = 0, |\zeta| < \infty,\ l=1,2\ . \tag{24}$$

In the assumption of linear dependence (17), from formula (18) we find the connection between the dimensionless temperature $T_l^*(\xi,\tau)$ and the Kirchhoff function $\Theta_l(\xi,\tau)$, $l=1{,}2$ in the form:

$$T_l^*(\zeta,\tau) = T_0^* + \Lambda_l^{-1}[\sqrt{1+2\Lambda_l\Theta_l(\zeta,\tau)} - 1]\ ,\ l=1,2\ , \tag{25}$$

Using the approximate expansion (Nowinski, 1962):

$$\sqrt{1+2\Lambda_l\Theta_l(0,\tau)} \approx 1 + \Lambda_l\Theta_l(0,\tau),\ l=1,2\ . \tag{26}$$

from equation (25) we obtain the linear dependence between dimensionless temperature and the Kirchhoff function on the surface of friction:

$$T_l^*(0,\tau) = T_0^* + \Theta_l(0,\tau)\ ,\ l=1,2\ . \tag{27}$$

In accordance with relations (27) the boundary condition (22) takes the form:

$$K_0^*\frac{\partial\Theta_1(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\Theta_2(\zeta,\tau)}{\partial\zeta}\bigg|_{\zeta=0} = \\ = Bi[\Theta_1(0,\tau)-\Theta_2(0,\tau)],\ \tau>0. \tag{28}$$

Note that the linear relationship (27) between the dimensionless temperature and the Kirchhoff function takes place only on the surface of the contact. Within each semi-spaces the dependences between these variables are nonlinear and are given by equation (25).

4. THE KIRCHHOFF FUNCTIONS AT UNIFORM SLIDING

By applying the Laplace integral transform (Sneddon, 1972):

$$\overline{\Theta}_l(\zeta,p) \equiv L[\Theta_l(\zeta,\tau);p] = \int_0^\infty \Theta_l(\zeta,\tau)e^{-p\tau}d\tau\ . \tag{29}$$

to the linear boundary-value problem (19)–(21), (23), (24) and (28), we obtain the following boundary problem for two ordinary differential equations of the second order:

$$\frac{d^2\overline{\Theta}_1(\zeta,p)}{d\zeta^2} - \frac{p}{k^*}\overline{\Theta}_1(\zeta,p) = 0\ ,\ \zeta>0, \tag{30}$$

$$\frac{d^2\overline{\Theta}_2(\zeta,p)}{d\zeta^2}-p\overline{\Theta}_2(\zeta,p)=0,\ \zeta<0, \tag{31}$$

$$K_0^*\frac{d\overline{\Theta}_1(\zeta,p)}{d\zeta}\bigg|_{\zeta=0}-\frac{d\overline{\Theta}_2(\zeta,p)}{d\zeta}\bigg|_{\zeta=0}=-\frac{1}{p}, \tag{32}$$

$$K_0^*\frac{d\overline{\Theta}_1(\zeta,p)}{d\zeta}\bigg|_{\zeta=0}+\frac{d\overline{\Theta}_2(\zeta,p)}{d\zeta}\bigg|_{\zeta=0}= =Bi[\overline{\Theta}_1(0,p)-\overline{\Theta}_2(0,p)], \tag{33}$$

$$\overline{\Theta}_l(\zeta,p)\to 0,\ |\zeta|\to\infty,\ l=1,2. \tag{34}$$

The solution to the problem (30)–(34) has the form:

$$\overline{\Theta}_1(\zeta,p)=\frac{e^{-\zeta_1\sqrt{p}}}{2\varepsilon(\sqrt{p}+\beta)}\left(\frac{1}{p}+\frac{Bi}{p\sqrt{p}}\right),\ \zeta\ge 0, \tag{35}$$

$$\overline{\Theta}_2(\zeta,p)=\frac{e^{-\zeta_2\sqrt{p}}}{2\varepsilon(\sqrt{p}+\beta)}\left(\frac{\varepsilon}{p}+\frac{Bi}{p\sqrt{p}}\right),\ \zeta\le 0, \tag{36}$$

where:

$$\beta=\frac{Bi(1+\varepsilon)}{2\varepsilon},\ \varepsilon=\frac{K_0^*}{\sqrt{k^*}},\ \zeta_1=\frac{\zeta}{\sqrt{k^*}},\ \zeta_2=-\zeta. \tag{37}$$

Applying the inversion formulae (Bateman and Erdelyi, 1954):

$$L^{-1}\left[\frac{e^{-\zeta_l\sqrt{p}}}{p(\sqrt{p}+\beta)};\tau\right]=\beta^{-1}\Psi(\zeta_l,\tau), \tag{38}$$

$$L^{-1}\left[\frac{e^{-\zeta_l\sqrt{p}}}{p\sqrt{p}(\sqrt{p}+\beta)};\tau\right]=\beta^{-1}\Phi(\zeta_l,\tau)-\beta^{-2}\Psi(\zeta_l,\tau), \tag{39}$$

where:

$$\Phi(\zeta_l,\tau)=2\sqrt{\tau}\,\mathrm{ierfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right), \tag{40}$$

$$\Psi(\zeta_l,\tau)=\mathrm{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right)-e^{\beta^2\tau+\zeta_l\beta}\mathrm{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}+\beta\sqrt{\tau}\right), \tag{41}$$

to the Laplace'a transform solutions (35) and (36), we obtain the Kirchhoff functions for each semi-spaces at any time moment $\tau\ge 0$:

$$\Theta_l(\zeta,\tau)=(1+\varepsilon)^{-1}[\Phi(\zeta_l,\tau)+\gamma_l\Psi(\zeta_l,\tau)],\ l=1,2. \tag{42}$$

where:

$$\gamma_1=\frac{1-\varepsilon}{Bi(1+\varepsilon)},\ \gamma_2=\frac{\varepsilon(\varepsilon-1)}{Bi(1+\varepsilon)}. \tag{43}$$

At $\xi=0$ from solutions (40)–(42) we have:

$$\Theta_l(0,\tau)=(1+\varepsilon)^{-1}[\varphi(\tau)+\gamma_l\psi(\tau)],\ \tau\ge 0,\ l=1,2. \tag{44}$$

where:

$$\varphi(\tau)=2(\tau/\pi)^{1/2},\ \psi(\tau)=1-e^{\beta^2\tau}\mathrm{erfc}\,(\beta\sqrt{\tau}). \tag{45}$$

The dimensionless temperature of the semi-spaces we calculate using the formulae (25) and (42)–(45).

5. THE KIRCHHOFF FUNCTIONS AT SLIDING WITH UNIFORM DECELERATION

At braking with a constant deceleration the specific power of friction is equal (Kuciej, 2012):

$$q(\tau)=q_0q^*(\tau),\ q^*(\tau)=1-\tau\tau_s^{-1},\ 0\le\tau\le\tau_s, \tag{46}$$

where $\tau_s=k_2t_sa^{-2}$. In this case, the Kirchhoff functions can be found from the Duhamel formula (Ozisik, 1980) at $0\le\tau\le\tau_s$:

$$\Theta_l^*(\zeta,\tau)=\int_0^\tau q^*(s)\frac{\partial}{\partial\tau}\Theta_l^{(0)*}(\zeta,\tau-s)ds, \tag{47}$$

where the functions $\theta_l^{(0)*}(\xi,\tau)$, $l=1,2$ have the form (42) and (43). Substituting under the integral sign in formula (47) the functions (42), after integration we find at $0\le\tau\le\tau_s$:

$$\Theta_l^*(\zeta,\tau)=\Theta_l^{(0)*}(\zeta,\tau)-\tau_s^{-1}\Theta_l^{*(1)}(\zeta,\tau),\ l=1,2 \tag{48}$$

where:

$$\begin{aligned}\Theta_l^{*(1)}(\zeta,\tau)=(1+\varepsilon)^{-1}\{[\tau+6^{-1}\zeta_l^2-\gamma_l(0.5\zeta_l+\beta^{-1})]\times\\ \times\Phi(\zeta_l,\tau)-3^{-1}\tau\varphi(\tau)\exp(-0.25\zeta_l^2\tau^{-1})+\\ +\gamma_l[\beta^{-2}\Psi_l(\zeta_l,\tau)+\tau\,\mathrm{erfc}\,(0.5\zeta_l\tau^{-1/2})]\}.\end{aligned} \tag{49}$$

At $\xi=0$ from the solution (49) it follows:

$$\begin{aligned}\Theta_l^{*(1)}(0,\tau)=(1+\varepsilon)^{-1}\{[(2/3)\tau-\gamma_l\beta^{-1})]\varphi(\tau)+\\ +\gamma_l[\beta^{-2}\psi_l(\tau)+\tau]\},0\le\tau\le\tau_s.\end{aligned} \tag{50}$$

In limiting case $Bi\to\infty$ from the formulae (44) and (48), (50) we obtain the Kirchhoff functions at the perfect thermal contact between the semi-spaces in the form (Fazekas, 1953):

$$\Theta_l^*(\zeta,\tau)=\frac{1}{(1+\varepsilon)}\left[1-\frac{2\tau}{3\tau_s}\right]\varphi(\tau),\ 0\le\tau\le\tau_s. \tag{51}$$

6. NUMERICAL ANALYSIS AND CONCLUSIONS

The numerical analysis for two friction pairs gray iron - A356 and A315 - A356 for $q=1\ \mathrm{MW/m^2}$, $a=0.015\,m$ and $T_0=20°\mathrm{C}$, has been performed. The values of coefficients of thermal conductivity K_0, thermal diffusivity k of materials and values of the coefficients λ, which characterize changing of the coefficients of thermal conductivity and specific heat with tempera-ture, are shown in Tab. 1.

The evolutions of the dimensionless temperature on the contact surfaces during constant sliding for two friction pair, with and without taking into account the thermosensitive materials, are shown in Figs. 2: gray iron - A356 (Fig. 2a) and A315 - A356 (Fig. 2b).

Tab. 1. Thermo-physical materials properties used in numerical analysis

Materials	K_0 $\mathrm{Wm^{-1}{}^{\circ}C^{-1}}$	$k \times 10^5$ $\mathrm{m^2}s^{-1}$	$\lambda \times 10^5$ $\mathrm{{}^{\circ}C^{-1}}$
Gray Iron (Gundlach, 1983)	45.45	1.368	-0.253026
A315 (Overfelt, 2001)	128.65	5.9552	0.914108
A356 (Sok, 2006)	150.01	7.9	0.712619

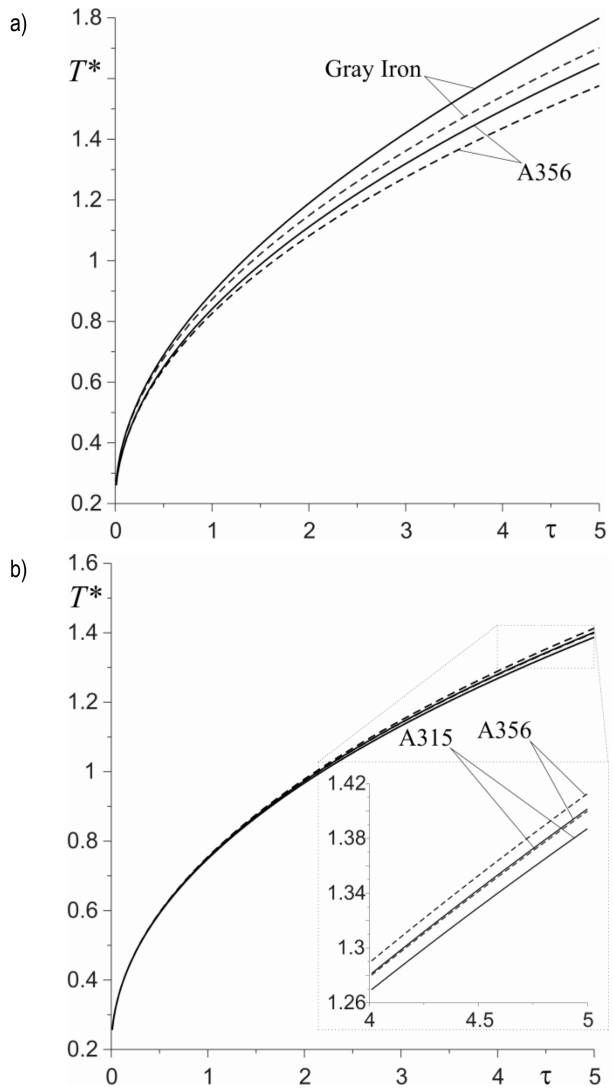

Fig. 2. Evolutions of the dimensionless temperature on the contact surface during constant sliding, for two friction pairs: a) gray iron - A356, b) A315 - A356, for *Bi* = 5. The solid curves – termosensitive materials; the dashed curves – constant properties

For this pair of friction: gray iron - A356 (Fig. 2a), taking into account the thermal sensitivity of the materials causes a slight increase of temperature in comparison with the temperature evolution calculated without taking into account thermal sensitivity of materials. Thermal conductivity of gray iron is three times less than the thermal conductivity of the material A356, which makes the temperature for this material on the contact surface is always greater, from the beginning of the friction process to the end. Decrease of the thermal conductivity of gray iron with temperature, causes increase in the temperature on the contact surface.

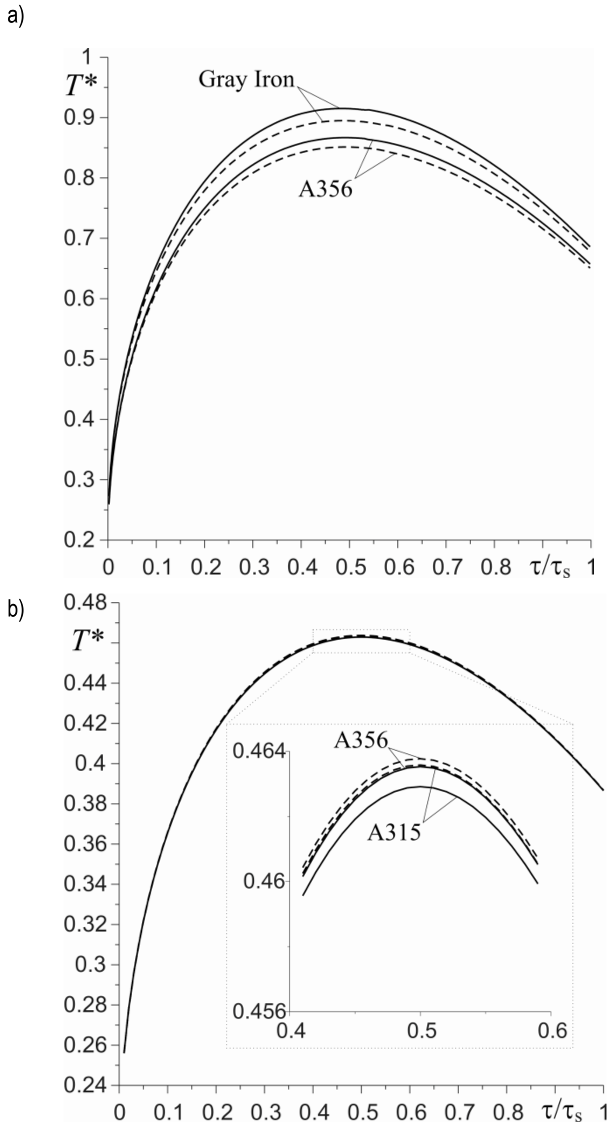

Fig. 3. Evolutions of the dimensionless temperature on the contact surface during linearly decreasing velocity of sliding for two friction pairs: a) gray iron - A356, b) A315 - A356, for *Bi* = 5. The solid curves – termosensitive materials; the dashed curves – constant properties

A different situation can be observed in Fig. 2b, where for the friction pair A315 - A356 the evolutions of temperature are shown. Because of close values of thermal conductivity of both materials (see Tab. 1), the evolutions of the temperature on the contact surface are almost the same, whether it takes into account the thermal sensitivity or not. Increase of the thermal conductivity with temperature, reduces the temperature on the contact surface.

The evolution of dimensionless temperature on the contact surface with linearly decreasing velocity from the nominal value at the start of heating to zero during a stop (braking with constant delay) is presented in Figs. 3.

For both pairs of friction, i.e. Gray iron - A356 (Fig. 3a) and A315 - A356 (Fig. 3b), with the beginning of the braking, temperature on the contact surface increases rapidly, reaching a maximum value in the middle of heating time, and then begins to decrease until it reaches a minimum value at the stop time. Taking into account changes in thermo-physical properties of materials under the influence of temperature, for both pairs of friction it causes the same changes in the evolution of the temperature as in the case of constant sliding velocity.

Using Duhamel's formula and obtained solution to the heat conduction problem of friction in the case of a constant sliding speed of two homogeneous semi-space, the mathematical model has been proposed to calculate the non-stationary temperature fields with a linear velocity sliding (braking with uniform delay). The solutions obtained in this article can be used as an introduction to further research on determining distribution of temperature in friction pairs, taking into account changes in their thermal properties under the influence of temperature.

Notations: a – characteristic dimension; Bi – Biot number; c – specific heat; $\mathrm{erf}(\ )$ – Gauss error function; $\mathrm{erfc(x)} = 1 - \mathrm{erf(x)}$; $\mathrm{ierfc(x)} = \pi^{-1/2}\mathrm{e}^{-\mathrm{x}^2} - \mathrm{xerfc(x)}$, f – coefficient of friction; h – coefficient of thermal conductivity of contact; K – coefficients of thermal conductivity; k – coefficients of thermal diffusivity; p_0 – pressure; T – temperature; T^* – dimensionless temperature; T_0 – initial temperature, t – time; t_s – breaking time; V_0 – constant speed; z – spatial coordinate.

Greek symbols: Θ – Kirchhoff's variable; τ – dimensionless time; τ_s –dimensionless breaking time; ξ – dimensionless spatial coordinate; λ – coefficient.

Subscripts: 1 – top semi-space; 2 – bottom semi-space.

REFERENCES

1. **Abdel-Aal H.A.** (1997), On the distribution of friction-induced heat in the dry sliding of metallic solid pairs, *Int. Comm. Heat Mass Trans.*, Vol. 24, No. 7, 989–998.
2. **Abdel-Aal H.A., Smith S.T.** (1998), On friction-induced temperatures of rubbing metallic pairs with temperature-dependent thermal properties, *Wear*, Vol. 216, No. 1, 41–59.
3. **Bateman H., Erdelyi A.** (1954), *Tables of integral transforms*, V. 1, McGraw-Hill, New York.
4. **Fazekas G.A.G.** (1953), Temperature gradient and heat stresses inbrakes drums, *SAE. Trans*., Vol. 61, No. 1, 279–284.
5. **Gundlach, R.B.** (1983), The effects of alloying elements on the elevated temperature properties of gray irons. (Retroactive Coverage),, *Transactions of the American Foundrymen's Society*, Vol. 91, 389–422.
6. **Kirchhoff G.R.** (1894), *Vorlesungen über die Theorie der Wärme*, B.G. Teubner, Leipzig.
7. **Kuciej M.** (2012), *Analytical models of transient frictional heating*, Oficyna Wydawnicza Politechniki Bialostockiej, Bialystok (in Polish),.
8. **Kushnir R.M., Popovych V.S.** (2011), Heat conduction problems of thermosensitive solids under complex heat exchange. In: *Heat conduction – Basic Research*, V. Vikhrenko Ed., 131–154, In Tech, Croatia.
9. **Ling F.F.** (1973), *Surface Mechanics*, John Wiley&Sons, New York.
10. **Nowinski J.** (1962), Transient thermoelastic problem for an infinite medium with a spherical cavity exhibiting temperature-dependent properties, *J. Appl. Mech*., Vol. 29, 399–407.
11. **Overfelt R.A., Taylor R.E., Bakhtiyarov S.I., Wang D.** (2001), Thermophysical properties of 319 aluminum, compacted graphite iron, and inconel 713, *AFS Transactions*, 02-115, 141–150.
12. **Ozisik M.N.** (1980), *Heat conduction*, New York, John Wiley, 687 p.
13. **Sneddon I.N.** (1972), *The use of integral transforms*, McGraw-Hill, New York.
14. **Sok W.K.** (2006), Thermophysical properties of automotive brake disk materials, *International Forum on Strategic Technology* (*IFOST*),, 18-20 Oct. 2006, 163–166, Ulsan.
15. **Yevtushenko A.A., Kuciej M.** (2012), One-dimensional thermal problem of friction during braking: The history of development and actual state*, Int. J. Heat Mass Trans*., Vol. 55, No. 15–16, 4148–4153.

Acknowledgement. The present paper is financially supported by the National Science Centre of Poland (project No 2011/01/B/ST8/07446).