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ON GRÜNWLAD-LETINKOV FRACTIONAL OPERATOR WITH MEASURABLE ORDER ON CONTINUOUS-DISCRETE TIME SCALE

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Abstract: Considering experimental implementation control laws on digital tools that measurement cards are discharged every time unit one can see that time of simulations is partially continuous and partially discrete. This observation provides the motivation for defining the Grünwald-Letnikov fractional operator with measurable order defined on continuous-discrete time scale. Some properties of this operator are discussed. The simulation analysis of the proposed approach to the Grünwald-Letnikov operator with the measurement functional order is presented.

Keywords: Grünvald-Letnikov fractional operator, measurement order, time scale

1. INTRODUCTION

It is well known that during control system design process, one of the most important steps is to develop the proper mathematical model of an analyzed control plant. Generally, the analysis of experiment results shows that there is a large class of systems where behavior of real phenomena is not properly explained by using the classical calculus. It has been found that these systems not only contain non-local dynamics but can also be described using fractional-order operators and their properties, see for example in control engineering, signal processing, electronics and electrical engineering, (Buslowicz and Nartowicz, 2009; Djennoune et al., 2019; Kavuran et al, 2017, Ortigueira, 1997, Balaska et al., 2020). Among the other, an example of the control plant that shows that fractional calculus applied to the modelling of its behavior is better than the classical tools is the voltage – current relation of a semi-infinite lossy transmission line (Wang, 1987), the diffusion process of the heat into a semi-infinite (Podlubny et al., 1995), modeling and simulation of plant models (Alagoz et al., 2019). In automatic regulation and its industrial applications to process controlling the most popular and commonly used are PID controllers. However, it is known that controllers of fractional orders (FOPID) in many cases can provide better optimal preferences and behaves more robust than the classical ones, see (Ostalczyk et al., 2015; Patniak et al., 2002; Tepljakov et al., 2018). It follows from the fact that such controllers have more tuning freedom. However the usage of FOPID controllers usually requires some approximations which makes their applications more complex.

Since the Grünwald-Letnikov fractional order operator in automatic control and industry applications has been considered as the most useful and a proper tool for approximations in the scope of numerical solutions (see, e.g., Coimbra, 2003; Patniak et al., 2002; Alagz and Alisoy, 2018; Tepljakov, 2017 and references therein), we lay attention on it. The practical usefulness of this operator is due to the fact that its value depends on all past values of the fractionally derived function, so the history or memory of the process in naturally included in the analysis. Also, it provides a recursive solution in time and hence reduces computing time (Alagoz et al., 2019). However, taking into account the limitation of computational resources, computational complexity should be as low as possible. There are works addressed to optimization of number steps in approximation used in computing of the Grünwald-Letnikov fractional order operator (see Stanislawski and Latawiec, 2012; Alagoz et al., 2019; Tepljakov et al., 2012) and references therein.

The natural analytic extension of fractional order operators are variable order ones. In some ways, this is a natural direction, not only from mathematical point of view, but also arises from modelling of real-word phenomena (Patniak et al., 2002). The first works in that scope have already shown that it is a good approach to modelling but not easy research topic (see, e.g. Coimbra, 2003; Lorenzo and Hartley, 2002; Samko and Ross, 1993). Note that in this case, there exist four different definitions of these operator with variable order (see Sierociuk et al., 2013; Sierociuk et al., 2015; Valerio and Sa da Costa, 2001). These definitions have been used, for example, in modelling of FOPID controllers as well as in heat transfer process (Sierociuk and Macias, 2013). The influence on the shaping of the transient characteristics of a closed-loop systems has been analyzed in Ostalczyk et al. (2012, 2015). In each case, the variable order has been taken as a function defined on the set of natural numbers. As it is known from the engineering point of view in measurement process digital tools are used to test different control plants. This means that measurement cards are discharged periodically every time unit δ, so the measurment time is not only discrete, but partially continuous and partially discrete as on time scales (Bohner and Peterson, 2001).

We concentrate on the classical approach to the Grünwald-Letnikov fractional order operator. Taking into account its implementation in digital systems (Alagz and Alisoy, 2018, Koszewnik

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et al., 2016, Koszewnik et al, 2018, Pawluszewicz et al., 2019), it is natural to consider a fractional order of this operator as a function defined on continuous-discrete time scale, that is, on a model of time that extends the classical time domain of dynamical systems (Bohner and Petrson, 2002). This problem, as well as the motivation to present work, is discussed in Section 2. The maximum bandwidth of the signal occurring during discharging of the measuring card may not be a good measure of signal changes, in the case of uniform sampling, some of the samples may be unnecessary. To eliminate this redundancy, non-uniform discharging can be used. Application of the non-uniform sampling allows to reduce the amount of measured data and next decrease power consumption for computation, which is important in industrial applications. In Section 3, there is introduced the Grünwald-Letnikov operator with the discrete-continuous order following from the non-uniform process of discharging measurement cards. In Section 4, the simulation analysis of proposed approach is presented.

2. MOTIVATION

It is known that the continuous time control law of fractional order PID controller usually is expressed as $u(t) = K_p e(t) + K_i D^{\lambda} e(t) + K_d D^{\mu} e(t)$ where orders λ, μ are nonnegative, $u(t)$ denotes the control signal, $e(t)$ is the control error between the desired value and the measured value, D is a fractional operator. During experimental implementation or verification of this control law, digital tools are commonly used. This means that:

- parameters K_p, K_d, K_i are recalculated by considering particular gains of A/D and D/A (Fig. 1) converters inbuilt to measure digital tools,
- steady state error, which is strictly connected with the orders of fractional operator, is changing during the regulation process.

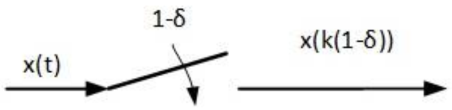

Fig. 1. Impulsator switched periodically for 1-δ time units

Furthermore, taking into account that measurement cards are discharged periodically every time unit and assuming that the discharging takes δ > 0 time units, time of simulations is partially continuous and partially discrete (not only discrete), see Fig. 1 and Fig. 2.

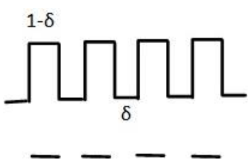

Fig. 2. Clock impulsator with duty cycle equal to 1-δ of time units

In a general case, this situation can be described using the following model of time presented on Fig. 3, see (Bohner and Petrson, 2002):

$$P_{1-\delta,\delta} = \cup_{k\in\mathbb{N}_0}[k, k+1-\delta] \quad (1)$$

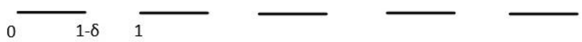

Fig. 3. Time scale $P_{1-\delta,\delta}$ of signal from clock impulsator

Then the previous time instant of $t \in P_{1-\delta,\delta}$, denoted as $\rho(t)$, is

$$\rho(t) = \begin{cases} t & \text{if } \; t \in \bigcup_{k=0}^{\infty}[k, k+1-\delta) \\ t-\delta & \text{if } \; t \in \bigcup_{k=0}^{\infty}\{k+1-\delta\} \end{cases}$$

In fact $\rho(t)$ defines the backward jump operator of t. The difference between t and its previous time instance $\rho(t)$, called *(backward) graininess function* $\upsilon : P_{1-\delta,\delta} \longrightarrow \mathbb{R}_+ \cup \{0\}$, is defined as:

$$\upsilon(t) := t - \rho(t).$$

Putting $\rho^l = \underbrace{\rho \circ \ldots . \circ \rho}_{l\ times}$ and $\rho^0(t) = t$ inductively, one can show that $\rho^n(t) = t - \nu^n(t)$ (Ortigueira et al., 2016). Then,

$$\rho^n(t) = \begin{cases} 0 & \text{if } \; t \in \bigcup_{k=0}^{\infty}[k, k+1-\delta) \\ t-n\delta & \text{if } \; t \in \bigcup_{k=0}^{\infty}\{k+1-\delta\} \end{cases} \quad (2)$$

So,

$$\nu(t) = \begin{cases} t & \text{if } \; t \in \bigcup_{k=0}^{\infty}[k, k+1-\delta) \\ t-\delta & \text{if } \; t \in \bigcup_{k=0}^{\infty}\{k+1-\delta\} \end{cases}$$

for any $t \in P_{1-\delta,\delta}$.

Similarly, as in (Ortigueira et al., 2016), on the time model given by (1), one can look as a model of time defined by a set of discrete time instants $t_n, n \in \mathbb{Z}_+$, and corresponding (backward) graininess. These instants are consecutive boundary point defining a closed interval, in which graininess is null inside the intervals starting in moment k and finishing in moment $k + 1 - \delta$. Following Ortigueira et al. (2016), one can define the *graininess interval* as the width of the considered interval. For time model (1), we have $\upsilon([k, k+1-\delta])$=1-$\delta$. Then, $t_n = t_{n-1} + \upsilon_n, n \in \mathbb{Z}_+$, is the direct graininess.

Remark 1. Such approach to time model (1) allows to consider not only a uniformly discharge periodically measurement cards but also a nonuniformly discharged measurement cards with discharge time units $\delta_k, k \in \mathbb{N}$. Nonuniform sampling in some situations reduces the required computing power and data processing. It is also possible to optimize the energy consumption of the controller, and thus save energy necessary in the control processes (see, e.g. Janczak et al., 2016; Kondratiuk et al., 2018).

3. THE GRÜNWLAD-LETNIKOV OPERATOR WITH DISCRETE-CONTINUOUS FRACTIONAL ORDER

Let us consider a function $C: P_{1-\delta,\delta} \times \mathbb{Z}_+ \cup \{0\} \longrightarrow \mathbb{R}$ defined as follows:

$$C^{\alpha(\tau)}(s)=\begin{cases}0 & \text{for } s<0\\ 1 & \text{for } s=0\\ \frac{(-1)^s}{s!}\alpha(\tau)\alpha\left(\rho(\tau)...\alpha(\rho^s(\tau)\right) & \text{for } s>0\end{cases} \quad (3)$$

where ρ is the graininess interval and $\rho^l=\underbrace{\rho\circ...\circ\rho}_{l\ times}$. From (3), it follows that the following recursive relation

$C^{\alpha(\tau)}(s)=C^{\alpha(\tau)}(s-1)\frac{\alpha(\rho^S(\tau))}{s}$ holds for given $s\geq 1$.

Proposition 2. *For any natural s and j such that $s>j$, it holds*

$$C^{\alpha(\tau)}(s)\pm C^{\alpha(\tau)}(j) = \frac{j!}{k!}C^{\alpha(\tau)}(j)\left(\prod_{i=j+1}^{k}(-1)^{k-i}\alpha(\rho^i(\tau))\pm\prod_{i=j+1}^{j}i\right)$$

Proof. Thesis follows from the fact that

$C^{\alpha(\tau)}(s)\pm C^{\alpha(\tau)}(j)=\frac{(-1)^j}{j!}\alpha(\tau)\,....\,\alpha\left(\rho^j(\tau)\right)\cdot \left[\frac{(-1)^{k-j}}{(j+k)....k}\alpha\left(\rho^{j+1}(\tau)\right)...\alpha(\rho^k(\tau))+1\right]$. □

Definition 3. The Grünwald-Letnikov – type fractional operator $\Delta^{\alpha(\tau)}$ of functional order $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$, for a function $x:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$ is defined as

$$\left(\Delta^{\alpha(\tau)}x\right)(\tau):=\sum_{s=0}^{\infty}C^{\alpha(\tau)}(s)x(\rho^s(\tau)) \quad (4)$$

In (4), weight function $C^{\alpha(\tau)}(s)$ is given by formula (3). Note that in a natural way, it contains information about the history of the process mathematically described by function $x:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$. Since $\alpha:P_{1-\delta,\delta}$, then this history strictly depends on the process of discharging a measurement card. Moreover, since the weight function $C^{\alpha(\tau)}(s)$ is defined by the graininess interval, it follows that charging/discharging of the card can be done nonuniformly. If function $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$ is unbounded, then for a fixed s, the operator (4) may not be bounded or even may not exists. We assume that for given $x:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$, function $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$ is bounded and measurable. The domain of Grünwald-Letnikov – type operator $\Delta^{\alpha(\tau)}$ is formed by two sets: a set $N_\delta=\{k\in\mathbb{Z}_+ : k+1-\delta\}$ of discrete/isolated points and a set of intervals $\{k\in\mathbb{Z}_+ : [k,k+1-\delta]\}$.

In practical implementations, instead of infinite sum in (4), there is need to use a finite one

$$\left(\Delta_J{}^{\alpha(\tau)}x\right)(\tau):=\sum_{s=0}^{J}C^{\alpha(\tau)}(s)x(\rho^s(\tau)) \quad (5)$$

where, following (Stanislawski Latawiec , 2012) and remembering that we look at time model (1) as a model of time defined by a set of discrete time instants $t_{n,}\ n\in\mathbb{Z}_+$, corresponding to the (backward) graininess,$J=\min\left(t_{n,}\bar{J}\right)$ and $\bar{J}$ is the upper bound to s when $t_{n,}>\bar{J}$. From (5), it follows that

$$\left(\Delta_J{}^{\alpha(\tau)}x\right)(\tau)= \begin{bmatrix}1 & C^{\alpha(t)}(1) & .... & C^{\alpha(t)}(J)\end{bmatrix}\begin{bmatrix}x(t)\\ x(\rho(t))\\ ....\\ x(\rho^{n-a}(t))\end{bmatrix} \quad (6)$$

Proposition 4. *If for every $t\in P_{1-\delta,\delta}$, there is a real number K such that $|x(\tau)|\leq K$ and $\alpha:P_{1-\delta,\delta}\longrightarrow[0,1]$, then*

$$\left|\left(\Delta^{\alpha(\tau)}x\right)(\tau)-\left(\Delta_J^{\alpha(\tau)}x\right)(\tau)\right|\leq K^2e.$$

Proof. Since

$$\begin{aligned}&\left|\left(\Delta^{\alpha(\tau)}x\right)(\tau)-\left(\Delta_J^{\alpha(\tau)}x\right)(\tau)\right|\\ &=\left|\sum_{s=0}^{\infty}C^{\alpha(\tau)}(s)x\left(\rho^s(\tau)\right)-\sum_{s=0}^{J}C^{\alpha(\tau)}(s)x\left(\rho^s(\tau)\right)\right|\\ &\leq\sum_{s=0}^{J}\left|C^{\alpha(\tau)}(s)x\left(\rho^s(\tau)\right)\sum_{v=J+1}^{\infty}C^{\alpha(\tau)}(v)x\left(\rho^v(\tau)\right)-1\right|\\ &\leq K\sum_{s=0}^{J}\left|C^{\alpha(\tau)}(s)\right|\left[\sum_{v=J+1}^{\infty}\left|C^{\alpha(\tau)}(v)x\left(\rho^v(\tau)\right)\right]+1\right|\\ &\leq K^2\sum_{s=0}^{J}\left|C^{\alpha(\tau)}(s)\right|\sum_{v=J+1}^{\infty}\left|C^{\alpha(\tau)}(v)\right|\end{aligned} \quad (7)$$

Since $\alpha:P_{1-\delta,\delta}\longrightarrow[0,1]$, then $\left|C^{\alpha(t)}(s)\right|\leq\frac{1}{s!}$, and from (7), it follows that

$\left|\left(\Delta^{\alpha(\tau)}x\right)(\tau)-\left(\Delta_J^{\alpha(\tau)}x\right)(\tau)\right|\leq K^2\sum_{s=0}^{\infty}\frac{1}{s!}=K^2e.$ □

4. SIMULATION ANALYSIS

In this Section, simulation analysis of the Grünwald-Letnikov operator with the discrete-continuous order $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$ is presented. To this aim, a process of switching on/off of the switch occurring in the holding circuit has been analyzed in detail based on electrical scheme shown in Fig. 4.

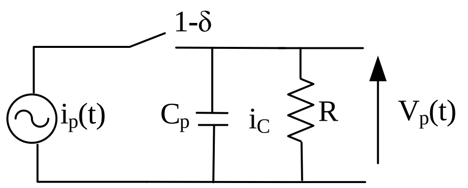

Fig. 4. Electrical circuit of RC system

Taking into account the behavior of the classical switch and the fractional operator, the capacitor has been firstly charged in the time $1-\delta$ in results of switching on/off the switch with randomly occurring delay. Next, the switch is turned off. It has led to discharging of the capacitor by time δ and also getting value of the voltage signal $x(\tau)$ from this element to further analysis. As a result, the whole process of switching on/off the switch is changeable, especially in time interval $[1-\delta,\delta]$.

The proposed approach allowed to check the influence function $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$, on fitting of the Grünwald-Letnikov – type fractional operator $\Delta^{\alpha(\tau)}$ of functional order $\alpha:P_{1-\delta,\delta}\longrightarrow\mathbb{R}$ in reference to three base signals. As the first, the sinusoidal signal $x(t)=\sin(0.04t)$, next $x(t)=H(t-a)$, where $H(\cdot)$ denotes the Heaviside's step function, and as the last one, $x(t)=e^{-0.1t}$ signal have been considered for function α, respectively. The obtained results for the given signals on time domain $P_{1-\delta,\delta}$

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presented in Figs. 5–7 showed that the signal with the Grünwald-Letnikov – type fractional operator of order $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ is best customized to the base signal $x(t)$ for ever smaller values of the function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ also of a function that has increasingly smaller values. Such behavior is especially visible for the variable value of the sampling step in the Grünwald-Letnikov – type fractional operator of order $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$, where increasing this value has led to weaker fitting to the base signals $x(t)$.

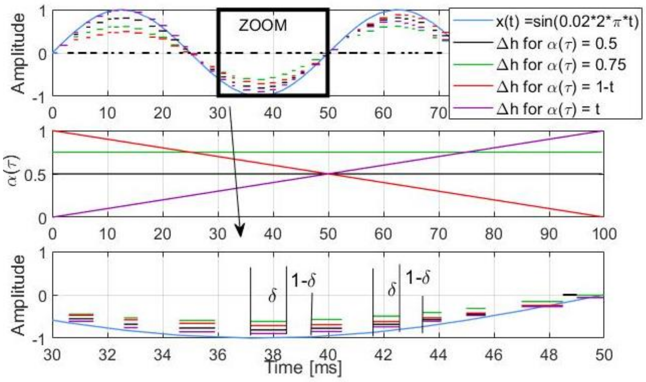

Fig. 5. The comparison of sinusoidal discrete-continuous signal with different values of function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ (for $\alpha(t) = t$, $\alpha(t) = 1 - t$, $\alpha(t) = 0.5$ respectively) to the base function $x(t) = \sin(0.04t)$

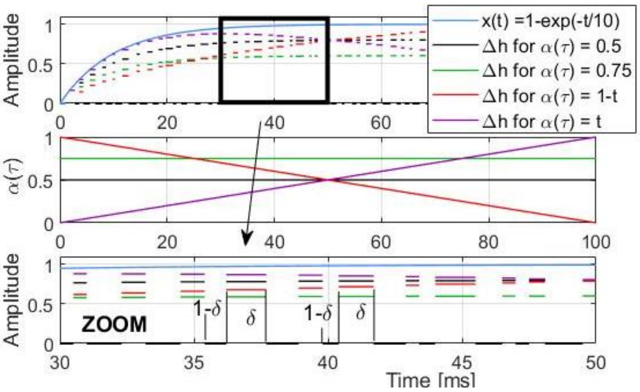

Fig. 6. The comparison of exponential discrete-continuous signal with different values of function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ (for $\alpha(t) = t$, $\alpha(t) = 1 - t$, $\alpha(t) = 0.5$ respectively) to the base function $x(t) = e^{-0.1t}$

In the second step, the influence of parameters a and b in function $x(t) = a\,sin(0.04t) + b, t \in P_{1-\delta,\delta}$, has been additionally analyzed. To this aim, again the base sinusoidal signal $x(t) = sin(0.04t)$ has been taken. The obtained result in Fig. 8 once again showed that the best customization to the base signal $x(t)$ has been achieved for the smallest values of both parameters of function $\alpha(\cdot)$. As a result, it leads to the conclusion that real industrial processes can be effectively controlled by using the discrete fractional control systems with variable sampling step and low values of the Grünwald-Letnikov – type fractional operator of functional order $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$.

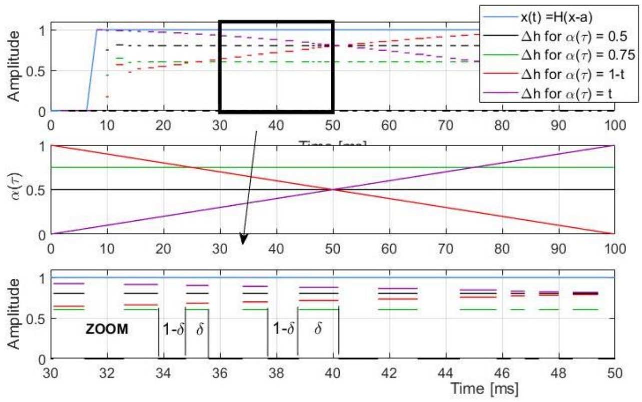

Fig. 7. The comparison of the Heaviside step function with delay discrete-continuous signal with different values of function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ (for $\alpha(t) = t$, $\alpha(t) = 1 - t$, $\alpha(t) = 0.5$ respectively) to the base function $x(t) = H(t - a)$

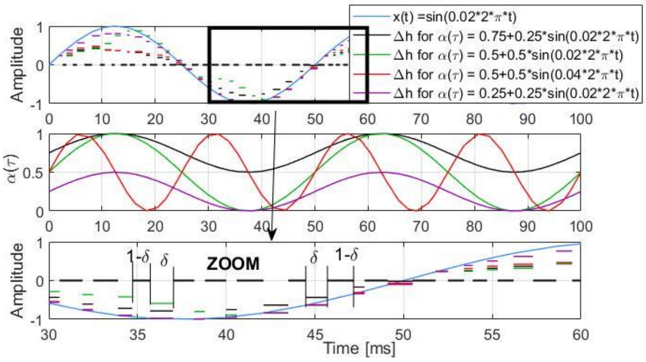

Fig. 8. The comparison of sinusoidal discrete-continuous signal with different values of function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ (for $\alpha(t) = t$, $\alpha(t) = 1 - t$, $\alpha(t) = 0.5$ respectively) to the base function $x(t) = a\sin(0.04t) + b$

Consequently, taking into account Figs. 4–7, it can be concluded that real industrial processes can be effectively controlled by using the discrete fractional control systems with variable sampling step and low values of the Grünwald-Letnikov – type fractional operator of functional order $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$.

5. CONCLUSIONS

The realization problem for Grünwld-Letnikov fractional operator with a measurable order on continuous-discrete time scale was studied. For this aim, firstly, some parameters from time scale calculus associated with sampling time (uniform and non-uniform) used in the measurement instruments like backward graininess was discussed. The proposed approach allowed to suppose that the process of charging and discharging of the capacitor inbuilt to the measurement cards of A/D and D/A converters could also be uniform. As a result, the practical implementation of the proposed approach for the measurement process can lead to reduction of consumption energy needed to control some industry processes by real time processor. Next, the proposed approach was checked in the simulation analysis. In order to do this, three base discrete-continuous signals $x(\cdot)$, such as sinusoidal signal, the Heaviside step function and exponential function for constant and changeable values of function order $\alpha(\cdot)$, were considered. The obtained results, given in Figs. 4–7 for the given signals on time domain $P_{1-\delta,\delta}$, showed that the signal Grünwald-Letnikov – type fractional operator of functional order $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ is best customized to the base function $x(\cdot)$ for even smaller values of the function $\alpha : P_{1-\delta,\delta} \longrightarrow \mathbb{R}$ also of a function that has increasingly smaller values.

Finally, it can be concluded that real industrial processes can be effectively controlled by using the discrete fractional control systems with variable sampling step and low values of the Grünwald-Letnikov type fractional operator with variable order defined on continuous-discrete time domain.

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