Opuscula Math. 36, no. 5 (2016), 603–612
http://dx.doi.org/10.7494/OpMath.2016.36.5.603 Opuscula Mathematica

M_2-EDGE COLORINGS OF DENSE GRAPHS

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Communicated by Mariusz Meszka

Abstract. An edge coloring φ of a graph G is called an M_i-edge coloring if $|\varphi(v)| \leq i$ for every vertex v of G, where $\varphi(v)$ is the set of colors of edges incident with v. Let $\mathcal{K}_i(G)$ denote the maximum number of colors used in an M_i-edge coloring of G. In this paper we establish some bounds of $\mathcal{K}_2(G)$, present some graphs achieving the bounds and determine exact values of $\mathcal{K}_2(G)$ for dense graphs.

Keywords: edge coloring, dominating set, dense graphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. The subgraph of a graph G induced by $U \subseteq V(G)$ is denoted by $G[U]$. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. The cardinality of this set, denoted $\deg_G(v)$, is called the degree of v. As usual $\Delta(G)$ and $\delta(G)$ stand for the maximum and minimum degree among vertices of G.

An edge coloring of a graph G is an assignment of colors to the edges of G, one color to each edge. So, any mapping φ from $E(G)$ onto a non-empty set is an edge coloring of G. The set of colors used in an edge coloring φ of G is denoted by $\varphi(G)$, i.e., $\varphi(G) := \{\varphi(e) : e \in E(G)\}$. For any vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors of edges incident with v, i.e., $\varphi(v) := \{\varphi(vu) : vu \in E(G)\}$. An edge coloring φ of G is an M_i-*edge coloring* if at most i colors appears at any vertex of G, i.e., $|\varphi(v)| \leq i$ for every vertex $v \in V(G)$. The maximum number of colors used in an M_i-edge coloring of G is denoted by $\mathcal{K}_i(G)$.

The concept of an M_i-edge coloring was introduced by J. Czap [2]. In [1] authors establish a tight bound of $\mathcal{K}_2(G)$ depending on the size of a maximum matching in G. In [2] and [3], the exact values of $\mathcal{K}_2(G)$ for subcubic graphs and complete graphs are determined. In [4] it is determined $\mathcal{K}_2(G)$ for cacti, trees, graph joins and complete multipartite graphs.

In this paper we establish some bounds of $\mathcal{K}_2(G)$ and determine exact values of $\mathcal{K}_2(G)$ for dense graphs.

2. AUXILIARY RESULTS

Let φ be an M_2-edge coloring of a graph G. For a set $U \subseteq V(G)$, let $\varphi(U)$ denote the set of colors of edges incident with vertices of U. Thus, $\varphi(U) := \cup_{v\in U}\varphi(v)$.

Lemma 2.1. *Let φ be an M_2-edge coloring of a graph G and let U be a non-empty subset of $V(G)$. Then the following statements hold:*

(i) $|\varphi(U)| \le |U| + c$, *where c denotes the number of connected components of $G[U]$,*
(ii) $|\{\varphi(e) : e \in E(G[U])\}| = 1$ *whenever $|\varphi(U)| = |U| + 1$ and $G[U]$ is 2-connected.*

Proof. First suppose that $G[U]$ is a connected graph. Denote the vertices of U by $u_1, u_2, \dots, u_k$ in such a way that the set $X_i := \{u_1, u_2, \dots, u_i\}$ induces a connected subgraph of G for every $i \in \{1, 2, \dots, k\}$. As $G[X_i]$ is connected for $i \ge 2$, there is j $(1 \le j < i)$ such that u_iu_j is an edge of G. Therefore, $\varphi(u_iu_j) \in \varphi(X_{i-1}) \cap \varphi(u_i)$ and

$$\begin{aligned}|\varphi(X_i)| &= |\varphi(X_{i-1}) \cup \varphi(u_i)| = |\varphi(X_{i-1})| + |\varphi(u_i))| - |\varphi(X_{i-1}) \cap \varphi(u_i)| \\ &\le |\varphi(X_{i-1})| + 2 - 1 = |\varphi(X_{i-1})| + 1.\end{aligned}$$

Clearly, $|\varphi(X_1)| = |\varphi(u_1)| \le 2 = |X_1| + 1$. Thus, by induction we get

$$|\varphi(X_i)| \le |\varphi(X_{i-1})| + 1 \le (|X_{i-1}| + 1) + 1 = |X_i| + 1$$

and consequently $|\varphi(U)| = |\varphi(X_k)| \le |X_k| + 1 = |U| + 1$.

If $G[U]$ is a disconnected graph, then the set U can be partitioned into disjoint subsets $U_1, U_2, \dots, U_c$ in such a way that $G[U_i]$ is a connected component of $G[U]$ for every $i \in \{1, 2, \dots, c\}$. Therefore,

$$|\varphi(U)| = |\varphi(\cup_{i=1}^{c} U_i)| \le \sum_{i=1}^{c} |\varphi(U_i)| \le \sum_{i=1}^{c} (|U_i| + 1) = |U| + c.$$

Now suppose that $G[U]$ is 2-connected and $|\{\varphi(e) : e \in E(G[U])\}| > 1$. Then there are edges uw and vw in $E(G[U])$ such that $\varphi(uw) \ne \varphi(vw)$. Therefore, $\varphi(w) \subseteq \varphi(u) \cup \varphi(v) \subseteq \varphi(U - \{w\})$ and consequently $\varphi(U) = \varphi(U - \{w\})$. As $G[U]$ is 2-connected, $G[U - \{w\}]$ is connected and by (i)

$$|\varphi(U - \{w\})| \le |U - \{w\}| + 1 = |U|.$$

Hence $|\varphi(U)| \le |U|$, which completes the proof. □

A *matching* in a graph is a set of pairwise nonadjacent edges. A matching is *perfect* if every vertex of the graph is incident with exactly one edge of the matching. A *maximum matching* is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$.

Lemma 2.2. *Let φ be an M_2-edge coloring of a graph G such that $\varphi(u) \cap \varphi(v) \neq \emptyset$ for all $u, v \in V(G)$. Then either $|\varphi(G)| = 3$ or $|\varphi(G)| \leq 1 + \alpha(G)$.*

Proof. Let F be a graph whose vertices are colors of the coloring φ (i.e., $V(F) = \varphi(G)$) and vertices c_1, c_2 of F (colors of φ) are adjacent whenever there is a vertex $w \in V(G)$ such that $\varphi(w) = \{c_1, c_2\}$. Clearly, any two edges of F are adjacent. Therefore, F is either a 3-cycle or a star. If F is a 3-cycle, then $|\varphi(G)| = 3$. If F is a star, then there is a color c which appears at any vertex of G. Let H be a graph obtained from G by removing all edges of color c and also removing all arisen isolated vertices. Evidently, every component of H is monochromatic. Thus, $|\varphi(G)|$ is not greater than the number of components of H increased by 1. As each component of H has at least one edge, the number of components of H is at most $\alpha(H)$. The graph H is a subgraph of G and so $\alpha(H) \leq \alpha(G)$. Hence $|\varphi(G)| \leq 1 + \alpha(G)$. □

Let φ be an M_2-edge coloring of a graph G. A set S, $S \subseteq E(G)$, is *rainbow* if no two edges of S are colored the same. Denote by $\mathcal{R}(\varphi)$ the family of all spanning subgraphs of G whose edge sets are maximal rainbow sets. Evidently, $|E(H)| = |\varphi(G)|$ and $\Delta(H) \leq 2$ whenever $H \in \mathcal{R}(\varphi)$.

Lemma 2.3. *Let φ be an M_2-edge coloring of a graph G. If $\delta(G) > 2$ then $\mathcal{R}(\varphi)$ contains an acyclic graph.*

Proof. Suppose to the contrary that any graph belonging to $\mathcal{R}(\varphi)$ contains a cycle. Let $H \in \mathcal{R}(\varphi)$ be a graph with the minimum number of cycles. Let C be a cycle of H and let u be a vertex of C. As $\delta(G) > 2$, $\deg_G(u) \geq 3$. Let e_1, e_2 and e_3 be distinct edges of G incident with u such that $\{e_1, e_2\} \subset E(C)$ and $e_3 \notin E(C)$. As $\{e_1, e_2\} \subset E(C) \subset E(H)$, $\varphi(e_1) \neq \varphi(e_2)$. Since $|\varphi(u)| \leq 2$, there is $i \in \{1, 2\}$ such that $\varphi(e_3) = \varphi(e_i)$. Clearly, if $e_3 = uv$, then $\deg_H(v) < 2$. Now, it is easy to see that the spanning subgraph of G with the edge set $\big(E(H) - \{e_i\}\big) \cup \{e_3\}$ belongs to $\mathcal{R}(\varphi)$ and contains less cycles than H, a contradiction. □

Lemma 2.4. *Let φ be an M_2-edge coloring of a graph G and let U be a non-empty subset of $V(G)$ such that $\varphi(u) \cap \varphi(v) = \emptyset$ for all distinct vertices $u, v \in U$. Then*

$$|\varphi(G)| \leq |U| + |V(G)| - \frac{1}{2} \sum_{u \in U} \deg_G(u).$$

Proof. As $\varphi(u) \cap \varphi(v) = \emptyset$ for all distinct vertices $u, v \in U$, the set U is an independent set of vertices in G. Thus, there is a spanning subgraph H of G belonging to $\mathcal{R}(\varphi)$ such that $\deg_H(u) = |\varphi(u)|$ for each vertex $u \in U$. For any non-negative integer i put $N_i := \{w \in V(G) - U : |N_G(w) \cap U| = i\}$. Since $|\varphi(w)| \leq 2$ and $\varphi(u) \cap \varphi(w) \neq \emptyset$ for any edge $uw \in E(G)$, $N_i = \emptyset$ for every $i \geq 3$. Thus the sets U, N_0, N_1 and N_2 form a partition of $V(G)$. So, $|V(G)| = |U| + |N_0| + |N_1| + |N_2|$. Moreover, $\sum_{u \in U} \deg_G(u) = 1 \cdot |N_1| + 2 \cdot |N_2|$. Combining these equations we get

$$2|N_0| + |N_1| = 2|V(G)| - 2|U| - \sum_{u \in U} \deg_G(u).$$

Let H' be subgraph of H induced by $V(H) - U$. Then, $\deg_{H'}(w) \leq 2 - i$ for any vertex $w \in N_i$, $0 \leq i \leq 2$, because $\varphi(w)$ contains at least i colors belonging to $\varphi(U)$. Therefore,

$$\begin{aligned} 2|E(H')| &= \sum_{v \in V(H')} \deg_{H'}(v) \leq 2 \cdot |N_0| + 1 \cdot |N_1| + 0 \cdot |N_2| \\ &= 2|V(G)| - 2|U| - \sum_{u \in U} \deg_G(u). \end{aligned}$$

Hence

$$\begin{aligned} |\varphi(G)| = |E(H)| &= \sum_{u \in U} |\varphi(u)| + |E(H')| \leq 2|U| + |E(H')| \\ &\leq 2|U| + \Big(|V(G)| - |U| - \frac{1}{2} \sum_{u \in U} \deg_G(u)\Big) \\ &= |U| + |V(G)| - \frac{1}{2} \sum_{u \in U} \deg_G(u), \end{aligned}$$

which completes the proof. □

Lemma 2.5. *Let G be a connected graph of order at least 2. If any vertex in $U \subset V(G)$ is a cut-vertex of G then there is an M_2-edge coloring φ of G such that $|\varphi(G)| = 1 + |U|$ and $|\varphi(v)| = 2$ if and only if $v \in U$.*

Proof. Denote by $u_1, u_2, \ldots, u_k$ vertices of the set U. Put $U_0 := \emptyset$ and $U_i := U_{i-1} \cup \{u_i\}$, for $i \in \{1, 2, \ldots k\}$.

Let φ_0 be a mapping from $E(G)$ to $\{0\}$. As $\varphi_0(e) = 0$, for every edge $e \in E(G)$, $|\varphi_0(G)| = 1 = 1 + |U_0|$ and $|\varphi_0(v)| = 2$ if and only if $v \in U_0$.

Now suppose that a mapping φ_i from $E(G)$ onto $\{0, 1, \ldots, i\}$ is an M_2-edge coloring of G such that $|\varphi_i(G)| = 1 + |U_i|$ and $|\varphi_i(v)| = 2$ if and only if $v \in U_i$. As $u_{i+1} \notin U_i$, $|\varphi_i(u_{i+1})| = 1$. Since u_{i+1} is a cut-vertex of G, the graph $G - u_{i+1}$ is disconnected. Let C be a chosen connected component of $G - u_{i+1}$ and H be a subgraph of G induced by $V(C) \cup \{u_{i+1}\}$. Consider a mapping φ_{i+1} from $E(G)$ onto $\{0, 1, \ldots, i+1\}$ given by

$$\varphi_{i+1}(e) = \begin{cases} i+1 & \text{if } \varphi_i(e) \in \varphi_i(u_{i+1}) \text{ and } e \in E(H), \\ \varphi_i(e) & \text{otherwise.} \end{cases}$$

Evidently, $|\varphi_{i+1}(v)| = |\varphi_i(v)|$ for every vertex $v \in V(G) - \{u_{i+1}\}$, and $|\varphi_{i+1}(u_{i+1})| = 1 + |\varphi_i(u_{i+1})|$. Therefore, φ_{i+1} is an M_2-edge coloring of G such that $|\varphi_{i+1}(G)| = 1 + |U_{i+1}|$ and $|\varphi_{i+1}(v)| = 2$ if and only if $v \in U_{i+1}$. Thus, by induction we get a desired coloring. □

Lemma 2.6. *Let G be a graph with $\delta(G) \geq 2$. Then the line graph $L(G)$ of G satisfies*

$$\mathcal{K}_2(L(G)) \geq |V(G)|.$$

Proof. All edges of G incident with a vertex v induce a subgraph $K(v)$ of $L(G)$, which is isomorphic to a complete graph of order $\deg_G(v)$. Subgraphs $K(v)$, for all $v \in V(G)$, form a decomposition of $L(G)$, where any edge of G (i.e., vertex of $L(G)$) belongs to precisely two distinct subgraphs. For every vertex $v \in V(G)$ color all edges of $K(v)$ with color c_v. It is easy to see that this gives an M_2-edge coloring of $L(G)$ using $|V(G)|$ colors, so $\mathcal{K}_2(L(G)) \geq |V(G)|$. □

3. BOUNDS

It is easy to see that for disjoint graphs G and H we have

$$\mathcal{K}_2(G \cup H) = \mathcal{K}_2(G) + \mathcal{K}_2(H).$$

Therefore, it is sufficient to consider connected graphs.

A *vertex cover* of a graph G is a subset U of $V(G)$ such that every edge of G is incident with a vertex in U and it is said to be *connected* if the subgraph of G induced by U is connected. The smallest number of vertices in any connected vertex cover of G is denoted by $\beta_c(G)$. Note that the set $V(G) - U$ is independent if and only if U is a vertex cover of G.

Theorem 3.1. *Let G be a connected graph. Then $\mathcal{K}_2(G) \leq 1 + \beta_c(G)$.*

Proof. Let U be a connected vertex cover of a graph G such that $|U| = \beta_c(G)$. Let φ be an M_2-edge coloring of G which uses $\mathcal{K}_2(G)$ colors, i.e., $|\varphi(G)| = \mathcal{K}_2(G)$. Since U is a vertex cover of G, $\varphi(G) = \varphi(U)$, and the desired inequality follows from Lemma 2.1. □

The following result presents some graphs achieving the bound $1 + \beta_c(G)$.

Theorem 3.2. *Let I be an independent set of a connected graph G satisfying*

(i) $|V(G) - I| \geq 2$,
(ii) $G[V(G) - I]$ *is a connected subgraph of* G,
(iii) *if* $u \in V(G) - I$ *is not a cut vertex of* $G[V(G) - I]$, *then there is a vertex* $v \in I$ *adjacent to* u,
(iv) $\deg_G(v) \leq 2$ *for all* $v \in I$.

Then

$$\mathcal{K}_2(G) = 1 + |V(G)| - |I|.$$

Proof. According to (ii), $V(G)-I$ is a connected vertex cover of G, and by Theorem 3.1, $\mathcal{K}_2(G) \leq 1 + |V(G) - I| = 1 + |V(G)| - |I|$.

On the other hand, let U be a subset of $V(G) - I$ containing vertices that have no neighbor in I. According to Lemma 2.5, there is an M_2-edge coloring φ of $G[V(G) - I]$ such that $|\varphi(G[V(G) - I])| = 1 + |U|$ and $|\varphi(v)| = 2$ if and only if $v \in U$. For every vertex $w \in V(G) - (I \cup U)$ let c_w be a new color, i.e., c_w does not belong to $\varphi(G[V(G) - I])$. The edge coloring ψ of G is defined in the following way

$$\psi(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G[V(G) - I]), \\ c_w & \text{if } e \notin E(G[V(G) - I]) \text{ and } e \text{ is incident with } w. \end{cases}$$

It is easy to see that ψ is an M_2-edge coloring of G which uses $1+|U|+|V(G)-(I\cup U)| = 1+|V(G)|-|I|$ colors, i.e., $\mathcal{K}_2(G)\geq 1+|V(G)|-|I|$. □

Let $V_j(G)$ denote the set of vertices of degree j in G.

If T is a tree different from a star, then the set $V_1(T)$ satisfies the conditions of Theorem 3.2. Thus, we immediately have the following assertion (the assertion is evident for stars).

Corollary 3.3. *If T is a tree on at least two vertices, then*

$$\mathcal{K}_2(T) = 1+|V(T)|-|V_1(T)|.$$

Similarly, $V_2(O)$ is an independent set of a maximal outerplanar graph O on at least four vertices. So, we have

Corollary 3.4. *Let O be a maximal outerplanar graph on at least four vertices. If every vertex of degree at least 3 is adjacent to a vertex of degree 2, then*

$$\mathcal{K}_2(O) = 1+|V(O)|-|V_2(O)|.$$

One can see that a maximal outerplanar graph in the previous result may be replaced by a maximal 2-degenerate graph (including a 2-tree).

We are also able to prove the following result.

Corollary 3.5. *Let G be a Hamiltonian graph with $\Delta(G)\leq 3$. Then the line graph $L(G)$ of G satisfies*

$$\mathcal{K}_2(L(G)) = |V(G)|.$$

Proof. If $\Delta(G)<3$, then G is a cycle and $L(G)$ is isomorphic to G. Clearly, the claim holds in this case. So, we will suppose that $\Delta(G)=3$.

Let C be a Hamilton cycle of G. Set $U=E(C)$ and $I=E(G)-U$. Evidently, $|U|=|V(G)|$, $1\leq|I|\leq|V(G)|/2$ and the vertex set of $L(G)$ is partitioned into disjoint non-empty subsets U and I. Let H be a subgraph of $L(G)$ induced by U. As H is a cycle and I is an independent set of vertices in $L(G)$ (a matching of G), U is a connected vertex cover of $L(G)$. According to Theorem 3.1, $\mathcal{K}_2(L(G))\leq 1+|U|$.

Suppose that there is an M_2-edge coloring φ of $L(G)$ which uses $1+|U|$ colors. In this case, $\varphi(U)=1+|U|$ and the subgraph H of $L(G)$ induced by U is 2-connected. By Lemma 2.1, all edges of H have the same color c. Since any edge of $L(G)$ not belonging to H is incident with a vertex of I, $|\varphi(I)|\geq|U|$. As φ is an M_2-edge coloring and $|I|\leq|V(G)|/2=|U|/2$, $|\varphi(x)|=2$ for each $x\in I$, $c\notin\varphi(I)$, $\varphi(x)\cap\varphi(y)=\emptyset$ for $x\neq y$, $\{x,y\}\subset I$, and $|I|=|V(G)|/2$ (i.e., G is regular of degree 3). Let e be an edge of C (an element of U) and let e_1, e_2 be distinct edges of I adjacent to e. Clearly, e and e_i, $i\in\{1,2\}$, are adjacent vertices of $L(G)$ and so $\varphi(e)\cap\varphi(e_1)\neq\emptyset$, $\varphi(e)\cap\varphi(e_2)\neq\emptyset$, and moreover the color c belongs to $\varphi(e)$. This implies $|\varphi(e)|\geq 3$, a contradiction. Therefore, $\mathcal{K}_2(L(G))\leq|U|=|V(G)|$.

The opposite inequality follows from Lemma 2.6. □

A set $D\subseteq V(G)$ is called *dominating* in G, if for each $v\in V(G)-D$ there exists a vertex $u\in D$ adjacent to v.

Theorem 3.6. *Let D be a dominating set of a graph G. If c denotes the number of connected components of $G[D]$, then*

$$\mathcal{K}_2(G) \leq c + |D| + \alpha(G[V(G) - D]).$$

Proof. Let φ be an M_2-edge coloring of G which uses $\mathcal{K}_2(G)$ colors, i.e., $|\varphi(G)| = \mathcal{K}_2(G)$. Let H be a graph obtained from G by removing all edges of colors belonging to $\varphi(D)$ and also removing all arisen isolated vertices. Evidently, every component of H is monochromatic. Thus, the number of components of H increased by $|\varphi(D)|$ is equal to $\mathcal{K}_2(G)$. As each component of H has at least one edge, the number of components of H is at most $\alpha(H)$. Since H is a subgraph of $G[V(G) - D]$, $\alpha(H) \leq \alpha(G[V(G) - D])$. Therefore,

$$\mathcal{K}_2(G) \leq |\varphi(D)| + \alpha(H) \leq |\varphi(D)| + \alpha(G[V(G) - D]).$$

By Lemma 2.1, $|\varphi(D)| \leq c + |D|$ and the desired inequality follows. □

Let $G \cup H$ denote the disjoint union of graphs G and H. Let h be a mapping from $V(H)$ to $V(G)$. By $G \cup_h H$ we denote the graph $G \cup H$ together with all edges joining each vertex $u \in V(H)$ and $h(u) \in V(G)$.

Let G be a graph of order n. Let nH denote the disjoint union of n copies of a graph H. If $h : V(nH) \to V(G)$ is a mapping such that the image of any vertex of ith copy of H is the ith vertex of G, then $G \cup_h nH$ is called a *corona* of G with H and it is denoted by $G \odot H$.

Theorem 3.7. *Let G be a graph without isolated vertices. Let h be a surjective mapping from the vertex set of a graph H onto $V(G)$ such that there exists a maximum matching M of H satisfying: $h(u) = h(v)$ for every edge $uv \in E(H) - M$. If c denotes the number of connected components of G, then*

$$\mathcal{K}_2(G \cup_h H) = c + |V(G)| + \alpha(H).$$

Proof. Clearly, $V(G)$ is a dominating set of $G \cup_h H$. According to Theorem 3.6, $\mathcal{K}_2(G \cup_h H) \leq c + |V(G)| + \alpha(H)$.

On the other hand, let $G_1, G_2, \ldots, G_c$ be connected components of G. For every vertex $v \in V(G)$, let H_v be a subgraph of H induced by $\{u \in V(H) : h(u) = v\}$. Consider an edge coloring ψ of $G \cup_h H$ defined by

$$\psi(e) = \begin{cases} i & \text{if } e \in E(G_i), \\ e & \text{if } e \in M, \\ v & \text{if } e \text{ is incident with a vertex of } H_v \text{ and } e \notin M. \end{cases}$$

It is easy to see that ψ is an M_2-edge coloring of $G \cup_h H$ which uses $c + \alpha(H) + |V(G)|$ colors, i.e., $\mathcal{K}_2(G \cup_h H) \geq c + |V(G)| + \alpha(H)$. □

The corona of a connected graph G on at least two vertices with a non-empty graph H satisfies conditions of Theorem 3.7, therefore, we have the following corollary.

Corollary 3.8. *Let G be a connected graph on at least two vertices. For any non-empty graph H, the corona of G with H satisfies*

$$\mathcal{K}_2(G \odot H) = 1 + |V(G)|\big(1 + \alpha(H)\big).$$

The *Cartesian product* $G_1 \square G_2$ of graphs G_1, G_2 is a graph whose vertices are all ordered pairs $[v_1, v_2]$, where $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and two vertices $[v_1, v_2]$, $[u_1, u_2]$ are joined by an edge in $G_1 \square G_2$ if and only if either $v_1 = u_1$ and v_2, u_2 are adjacent in G_2, or v_1, u_1 are adjacent in G_1 and $v_2 = u_2$.

Corollary 3.9. *Let G be a graph with $\Delta(G) = |V(G)| - 1$. If $|V(G)| > 1$ then*

$$\mathcal{K}_2(G \square K_2) = 2 + |V(G)|.$$

Proof. Let u be a saturated vertex of a graph G (i.e., $\deg_G(u) = |V(G)| - 1$). Let v_1 and v_2 be vertices of K_2. Denote by G^* the subgraph of $G \square K_2$ induced by $\{[u, v_1], [u, v_2]\}$. Similarly, the subgraph of $G \square K_2$ induced by $V(G \square K_2) - \{[u, v_1], [u, v_2]\}$ denote by H. Evidently, G^* is isomorphic to K_2 and $\alpha(H) = |V(G)| - 1$. Let h be a mapping from $V(H)$ onto $V(G^*)$ given by $h([w, v_i]) = [u, v_i]$, $i \in \{1, 2\}$. Clearly, the graph $G^* \cup_h H$ is isomorphic to $G \square K_2$. As $G^* \cup_h H$ satisfies the conditions of Theorem 3.7, we immediately get the assertion. □

It is easy to see that $\mathcal{K}_2(G) \leq |V(G)|$. For graphs with $\delta(G) > 2$ we can establish the better bound.

Theorem 3.10. *Let G be a graph with $\delta(G) \geq 3$. Then*

$$\mathcal{K}_2(G) \leq \left\lceil \frac{2}{\delta(G)} \left(1 + \left\lceil \frac{|V(G)|}{2} \right\rceil\right) \right\rceil + \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1.$$

Proof. Suppose that G is a counterexample of order n. Then there is an M_2-edge coloring φ of G using $k + \lfloor n/2 \rfloor$ colors, where $k = \lceil 2(1 + \lceil n/2 \rceil)/\delta(G) \rceil$. According to Lemma 2.3, there is an acyclic graph $H \in \mathcal{R}(\varphi)$. Then

$$2|E(H)| = 2|V_2(H)| + |V_1(H)| \leq 2|V_2(H)| + (n - |V_2(H)|) = |V_2(H)| + n.$$

Therefore

$$|V_2(H)| \geq 2|E(H)| - n = 2(k + \lfloor n/2 \rfloor) - n \geq 2k - 1.$$

As $\Delta(H) \leq 2$ and H is acyclic, any component of H is a path. Thus, there is an independent set U of H such that $U \subset V_2(H)$ and $|U| = k$. Since $E(H)$ is a rainbow set of φ, $\varphi(u) \cap \varphi(v) = \emptyset$ for all distinct vertices $u, v \in U$. By Lemma 2.4, we have

$$|\varphi(G)| \leq |U| + n - \frac{1}{2} \sum_{u \in U} \deg_G(u) \leq k + n - \frac{1}{2} k\delta(G).$$

As $k = \lceil 2(1 + \lceil n/2 \rceil)/\delta(G) \rceil$, $k\delta(G) \geq 2(1 + \lceil n/2 \rceil)$ and

$$|\varphi(G)| \leq k + n - (1 + \lceil n/2 \rceil) \leq k + \lfloor n/2 \rfloor - 1,$$

a contradiction. □

Some line graphs achieve the bound established in previous theorem.

Corollary 3.11. *Let G be a bipartite graph with parts A and B such that $\deg_G(u) = p \geq 3$, for every vertex $u \in A$, and $\deg_G(v) = 2$, for every vertex $v \in B$ except for a vertex $w \in B$ when $\deg_G(w) = 3$ if $p|A|$ is odd. Then the line graph $L(G)$ of G satisfies*

$$\mathcal{K}_2(L(G)) = |V(G)|.$$

Proof. Set $t = |A|$. Then $|V(L(G))| = |E(G)| = pt$ and $|B| = \lfloor |E(G)|/2 \rfloor = \lfloor pt/2 \rfloor$. As $\delta(L(G)) = p$ and $\lceil 2(1 + \lceil pt/2 \rceil)/p \rceil = t + 1$, according to Theorem 3.10, we have

$$\mathcal{K}_2(L(G)) \leq t + \lfloor pt/2 \rfloor = |A| + |B| = |V(G)|.$$

The opposite inequality follows from Lemma 2.6. □

For dense graphs we have the following result.

Theorem 3.12. *Let G be a graph with $\delta(G) > \left\lceil \frac{|V(G)|}{2} \right\rceil$. Then*

$$\mathcal{K}_2(G) = 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Proof. According to Theorem 3.10, $\mathcal{K}_2(G) \leq 1 + \lfloor |V(G)|/2 \rfloor$.

On the other hand, G is Hamiltonian because $\delta(G) > \lceil |V(G)|/2 \rceil$. Thus, $\alpha(G) = \lfloor |V(G)|/2 \rfloor$. Let M be a maximum matching of G. Color all edges of $E(G) - M$ with one color and every edge of M with a different color. It is easy to see that this gives an M_2-edge coloring of G using $1 + |M|$ colors. Therefore, $\mathcal{K}_2(G) \geq 1 + |M| = 1 + \lfloor |V(G)|/2 \rfloor$, as required. □

Acknowledgments

This work was supported by the Slovak VEGA Grant 1/0652/12.

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Received: June 19, 2015.
Revised: February 24, 2016.
Accepted: February 24, 2016.