TransNav
http://www.transnav.eu
the International Journal
on Marine Navigation
and Safety of Sea Transportation
Volume 9
Number 1
March 2015

DOI: 10.12716/1001.09.01.11

# On Modelling of Nonlinear Systems and Phenomena with the Use of Volterra and Wiener Series

A. Borys
*Gdynia Maritime University, Gdynia, Poland*

ABSTRACT: This is a short tutorial on Volterra and Wiener series applications to modelling of nonlinear systems and phenomena, and also a survey of the recent achievements in this area. In particular, we show here how the philosophies standing behind each of the above theories differ from each other. On the other hand, we discuss also mathematical relationships between Volterra and Wiener kernels and operators. Also, the problem of a best approximation of large-scale nonlinear systems using Volterra operators in weighted Fock spaces is described. Examples of applications considered are the following: Volterra series use in description of nonlinear distortions in satellite systems and their equalization or compensation, exploiting Wiener kernels to modelling of biological systems, the use of both Volterra and Wiener theories in description of ocean waves and in magnetic resonance spectroscopy. Moreover, connections between Volterra series and neural network models, and also input-output descriptions of quantum systems by Volterra series are discussed. Finally, we consider application of Volterra series to solving some nonlinear problems occurring in hydrology, navigation, and transportation.

## 1 INTRODUCTION

The objective of this paper is to show universality of the Volterra and Wiener series in description of nonlinear systems and phenomena, and in solving numerous nonlinear problems occurring in diverse engineering disciplines, ranging from electronics and telecommunications to such ones as navigation and transportation. This is possible because the Volterra series is a natural extension of the convolution integral description for linear systems to the nonlinear case, but the Wiener series exploits the powerful orthogonality principle applied to the Volterra series to describe nonlinear systems with stochastic inputs. It follows from the material presented in this paper how powerful are these two mathematical tools in consideration of nonlinear problems of engineering.

## 2 NONLINEAR SYSTEMS AND PHENOMENA

What are the nonlinear systems and phenomena ? The simplest answer to this question is the following: these are the ones that are not linear. In other words, their description (model) cannot be formulated with the use of one or a set of linear algebraic equations, or linear operators, or ordinary or partial differential equations, or combinations of them. One very useful and, on the other hand, also fundamental criterion for recognition whether a given system or phenomenon behaves linearly is investigation of its response to an amplified or attenuated sum of two external signals (excitations) applied at its input. If this response is a sum of two output signals (responses) received in the case of applying them separately to the system, and amplified or attenuated exactly in the same way as were the input signals. Mathematically, using system description by operators, we can express the above as follows

$$H\left(\alpha \cdot x_1 + \beta \cdot x_2\right) = \alpha \cdot H\left(x_1\right) + \beta \cdot H\left(x_2\right) \tag{1}$$

where $H$ denotes an operator describing the system. This operator works on a set of admissible input signals, producing responses at the system output. In (1), $x_1$ and $x_2$ mean some input signals, members of the above set. Usually in applications, they are functions of time or position, or both of them. Moreover, $\alpha$ and $\beta$ are real numbers expressing amplification or attenuation factors mentioned above. Note further that the condition (1) assumes the same form when $H$, $x_1$, and $x_2$ are assumed to be vectors. Then, $\alpha$ and $\beta$ remain scalars.

In (1), we assumed the usage of ordinary algebra with the common understanding of addition operation "+" and multiplication operation " · ". However, in this context, note there are some other algebras in which the condition (1), with another understanding of the aforementioned algebraic operations, is fulfilled. Examples of such systems of interest in the areas of signal processing and networking are considered in (Oppenheim, A. V. 1965) and (Boudec, J.-Y. & Thiran P. 2004), respectively. Obviously, then, these systems linear in new algebras behave non-linearly in ordinary one.

In this paper, we do not study dynamics of nonlinear systems or phenomena, which, by the way, are very interesting because getting richer than those of linear ones. Here, rather, we focus on searching for descriptions of their steady states, having in mind the input-output relations. For this purpose, the Volterra series (Volterra V. 1959), named so in honor of its founder an Italian mathematician Vito Volterra, turned out to be very useful in solving many nonlinear engineering problems. However, among advantages, it has also some drawbacks. These are the following: convergence problems occurring for signals of higher amplitudes (similarly as in a Taylor series) and problems with measuring its kernels. For circumventing this, Norbert Wiener devised a related mathematical tool by orthogonalization of components of the Volterra series leading to an expansion named after him a Wiener expansion (Wiener N. 1942, Wiener N. 1958).

This paper is organized as follows. In sections 2 and 3, respectively, the Volterra series and Wiener series are presented. The next section describes shortly the problem of a best approximation of large-scale nonlinear systems using Volterra operators in weighted Fock spaces. Finally, the last section 5 presents a list of interesting applications of the Volterra and Wiener theories in different engineering disciplines.

## 3 VOLTERRA SERIES

### 3.1 *Basics of Volterra series for time-invariant systems with memory*

Let us begin with consideration of a Volterra series of continuous time for description of nonlinear time-invariant (stationary) systems with memory. To this end, assume that an input-output behavior a nonlinear system considered can be described by a nonlinear operator; that is by such an operator $H$ that does not obey (1). Volterra shown that under some conditions this operator can be expanded in a series of the so-called Volterra operators as

$$y\left(t\right) = H\left(x\left(t\right)\right) = \sum_{n=0}^{\infty} H^{(n)}\left(x\left(t\right)\right) = \sum_{n=0}^{\infty} y^{(n)}\left(x\left(t\right)\right), \tag{2}$$

where $x\left(t\right)$ and $y\left(t\right)$ are the input and output signal, respectively. Moreover, by $y^{(n)}\left(x\left(t\right)\right) = H^{(n)}\left(x\left(t\right)\right)$, we define the partial $n$-th order system's response, where $H^{(n)}\left(x\left(t\right)\right)$ means the $n$-th order Volterra operator. Further, note that for a fixed value of time $t$ this operator is simply a functional, called respectively the $n$-th order Volterra functional.

The successive Volterra operators are given by the following iterated integrals

$$y^{(0)}(t) = h^{(0)}, \tag{3a}$$

$$y^{(1)}(t) = \int_{-\infty}^{\infty} h^{(1)}(\tau)\, x(t-\tau)\, d\tau, \tag{3b}$$

$$y^{(2)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(2)}(\tau_1, \tau_2)\, x(t-\tau_1) x(t-\tau_2)\, d\tau_1 d\tau_2, \tag{3c}$$

. . . . . . . . . .,

$$\begin{aligned} y^{(n)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h^{(n)}(\tau_1, \tau_2, \tau_3, \dots, \tau_n)\, \times \\ \times x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) \dots x(t-\tau_n)\, d\tau_1 d\tau_2 d\tau_3 \dots d\tau_n \end{aligned}, \tag{3d}$$

. . . . . . . . . .,

where $h^{(0)}$ is the system impulse response of the zero-th order (in terms of currents or voltages, it is the dc component in the expansion). Further, the function $h^{(n)}(\tau_1, \tau_2, \tau_3, \dots, \tau_n),\ n = 1, 2, 3, \dots,$ means the $n$-th order nonlinear impulse response of a nonlinear system considered. Note that for $n$=1 this is a standard linear impulse response.

Looking at (3b), and then at (2) with the next components in this expansion given by (3c), and generally by (3d), we see that the Volterra series (2) is an extension of the well-known convolution integral for linear time-invariant (LTI) systems.

Obviously, for description of nonlinear systems without memory, instead of a Volterra series, we use a Taylor series.

Furthermore, it can be shown (Schetzen M. 1980) that for the stability reasons of the Volterra series description a sufficient condition is the following:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left| h^{(n)}(\tau_1, \tau_2, \tau_3, \dots, \tau_n) \right| d\tau_1 d\tau_2 d\tau_3 \dots d\tau_n < \infty \tag{4}$$

for $n$ = 1, 2, 3, ... . It is not a necessary one for $n \geq 2$. In his papers (Sandberg I. W. 1985, Sandberg I. W. 1990), Sandberg showed that in the above case for nonlinear impulse responses that are physically realizable, it has the form

$$\sup_{J \in \Psi} \left| \int_{J_n} .. \int_{J_1} h^{(n)}(\tau_1, .., \tau_n) d\tau_1 .. d\tau_n \right| < \infty \quad , \tag{5}$$

where $\Psi$ means a set of all general $n$-vectors $[J_1 \;.\;.\; J_n]$ having elements being finite sums of bounded subintervals of the set $\langle 0, \infty)$.

Moreover, for causal nonlinear systems, we have (Schetzen M. 1981)

$$h^{(n)}(\tau_1, \tau_2, .., \tau_n) \equiv 0 \text{ for any } \tau_i,\ i = 1, 2, ..., n, \text{ and } n = 1, 2, .... \tag{6}$$

Finally, it can be shown (Borys A. 2007) that the Volterra series converges if the following:

$$\|x\| < \frac{1}{\lim\limits_{n \to \infty} \sqrt[n]{\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} .. \int\limits_{-\infty}^{\infty} \left| h^{(n)}(\tau_1, \tau_2, .., \tau_n) \right| d\tau_1 d\tau_2 .. d\tau_n}} \tag{7}$$

holds, where $\|x\|$ means the norm of an input signal. In derivation of (7) in (Borys A. 2007), it was assumed $\|x\| \overset{df}{=} \sup\limits_{-\infty \leq t \leq \infty} |x(t)|$.

### 3.2 *Volterra series for time-varying systems with memory*

In practice, there occur also situations where we have to with nonlinear physical systems of which parameters change with time. Obviously, they cannot be treated as stationary in this case. Then, when describing them by a Volterra series, we must assume that their nonlinear impulse responses depend upon time. And this is a correct approach.

Concluding, we can say that the structure of equations (2) and (3) remains unchanged in this case, but we shall have $H\left(x(t), t'\right)$, $H^{(n)}\left(x(t), t'\right)$, and $h^{(n)}(\tau_1, \tau_2, \tau_3, ..., \tau_n, t')$, $n = 0, 1, 2, 3, ...$, dependent upon an additional time variable $t'$.

One very prominent example of such the systems as sketched above are wireless communication channels, whose characteristics vary with time and position, and which are additionally, in most cases, nonlinear ones, as for example satellite channels.

For more details regarding modelling of nonlinear time-varying systems by Volterra or related series, see papers of Sandberg (Sandberg I. W. 1982, Sandberg I. W. 1983) and cited therein.

### 3.3 *Volterra series for discrete-time nonlinear systems with memory*

A variant of the Volterra series for discrete-time (digital) nonlinear systems is named the discrete Volterra series (Borys A. 2000). For nonlinear time-invariant systems, it has the following form:

$$y(k) = H\left(x(k)\right) = \sum_{n=0}^{\infty} H^{(n)}\left(x(k)\right) = \sum_{n=0}^{\infty} y^{(n)}\left(x(k)\right) \tag{8}$$

with

$$y^{(0)}(k) = h^{(0)} \quad , \tag{9a}$$

$$y^{(1)}(k) = \sum_{i=-\infty}^{\infty} h^{(1)}(i)\, x(k-i) \quad , \tag{9b}$$

$$y^{(2)}(k) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} h^{(2)}(i_1, i_2) x(k-i_1) x(k-i_2) \quad , \tag{9c}$$

$$y^{(3)}(k) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \sum_{i_3=-\infty}^{\infty} h^{(3)}(i_1, i_2, i_3) x(k-i_1) x(k-i_2) x(k-i_3) \,, \tag{9d}$$

. . . . . . . . . . ,

$$y^{(n)}(k) = \sum_{i_1=-\infty}^{\infty} ... \sum_{i_n=-\infty}^{\infty} h^{(n)}(i_1, .., i_n) x(k-i_1) ... x(k-i_n) \,, \tag{9e}$$

. . . . . . . . . ,

where $k$ means a discrete time and $h^{(0)}$ is the zero-th order impulse response (constant component). Moreover, $h^{(1)}(i)$ is the $i$-th sample of the system first order impulse response (linear one). And further, $h^{(n)}(i_1, i_2, ..., i_n)$, $n = 2, ...$, mean the corresponding samples of the multidimensional impulse responses of orders greater than 1, related with the Volterra operators of higher order terms $(n \geq 2)$ in equation (8).

Conditions for stability of the Volterra operators in the discrete Volterra series given by (8), for their causality, and finally for the convergence of the whole series (8) are analogous to those given respectively by (4) or (5), by (6), and by (7), for the case of a continuous time. They and details of their derivation can be found, for example, in (Borys A. 2000) and reference cited therein.

Similarly, extension of the discrete Volterra series (8) for stationary systems of the discrete time to that for non-stationary ones can be easily done in a similar way as shortly described in subsection 2.2 for the continuous time case.

## 4 WIENER SERIES

### 4.1 *Reasons for searching for an orthogonal series*

We can view the Volterra series as a mathematical tool of general type for approximation of behavior of nonlinear systems in steady state. That means that in this case, we do not adjust the above description to a certain type (class) of input signals from a set of admissible ones. The only limitation here is the amplitude of these signals of which increase causes convergence problems. Moreover, in the case of description of a nonlinear system with memory by a Volterra series, in almost all cases, the structure and elements of this system are known. From this, it possible to deduce the form of functions describing system's nonlinear impulses, or equivalently in the multidimensional frequency domain, of its nonlinear transfer functions of the corresponding orders (Bussgang J. J. & Ehrman L. & Graham J. W. 1974, Bedrosian E. & Rice S. O. 1971).

Another approximation philosophy stands behind an expansion we call here the Wiener series (Schetzen M. 1980). In opposite to the previous approach, sketched in section 2, we adjust in this case the form of the series components to a specific class of input signals used in a given application - to achieve better convergence properties and adjustment to measured data. In other words, having records of data measured at input and output of a given system and knowing nothing about its internal structure, we approximate behavior of this system so good as only it is possible for a class of input signals chosen.

Basic ideas of the above two schemes of approximation can be illustrated by comparison of approximation of a given function of, say one variable $t$, on an interval $t_1 \le t \le t_2$ (obviously having no memory) by polynomials. We have two choices: 1. we can expand this function in a Taylor series and truncate it at the $n$-th component ($n$ depending upon a required accuracy) or 2. we can expand the considered function in a series of the first $m$ orthogonal polynomials, as for example, Legendre, Hermite or Chebyshev polynomials ($m$ depending, as before, upon a required accuracy). And now note that the first approach (1.) corresponds with the approximation of a nonlinear operators (systems) with memory by a Volterra series, but the second (2.) with a Wiener series. As we shall see in the next subsection, the Wiener series uses Hermite polynomials for orthogonalization of Volterra series components.

### 4.2 *Notion of Wiener G-functionals*

To define the so-called Wiener G-functionals (G-operators), we need first to explain the notion of nonhomogeneous Volterra operators. And to this end, note that a Volterra operator of the n-th order is homogeneous if the following: $H^{(n)}\left(c \cdot x(t)\right) = c^n \cdot H^{(n)}\left(x(t)\right)$ holds with c meaning a constant. If this does not hold, a given Volterra operator is a nonhomogeneous one.

Using similar nomenclature as in (Schetzen M. 1981), we define a nonhomogeneous Volterra operator of the first degree (order), $g^{(1)}[\cdot]$, as

$$g^{(1)}\left[h^{(1)}, h^{(0)1}; x(t)\right] = H^{(1)}\left(x(t)\right) + h^{(0)1} = \int_{-\infty}^{\infty} h^{(1)}(\tau)\, x(t-\tau) d\tau + h^{(0)1} , \tag{10}$$

where the double superscript (0)1 at $h^{(0)1}$ means that the zero-th order (the words "degree" and "order" are used interchangeably in this paper) homogeneous Volterra operator is a component of the first order nonhomogeneous Volterra operator $g^{(1)}[\cdot]$. We see that the operator $g^{(1)}[\cdot]$ is a sum of two components, of a homogeneous Volterra operator of the first order and of $h^{(0)1}$ (a constant component).

Similarly, the nonhomogeneous Volterra operator of the second order will have the form

$$g^{(2)}\left[h^{(2)}, h^{(1)2}, h^{(0)2}; x(t)\right] = H^{(2)}\left(x(t)\right) + H^{(1)2}\left(x(t)\right) + h^{(0)2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(2)}(\tau_1,\tau_2)\, x(t-\tau_1) \cdot x(t-\tau_2) d\tau_1 d\tau_2 + \int_{-\infty}^{\infty} h^{(1)2}(\tau)\, x(t-\tau) d\tau + h^{(0)2} , \tag{11}$$

where now $g^{(2)}[\cdot]$ is a sum of three homogeneous Volterra operators: $H^{(2)}\left(x(t)\right)$ and $H^{(1)2}\left(x(t)\right)$ being, respectively, second and first order convolutions, and $H^{(0)2}\left(x(t)\right) = h^{(0)2}$ being a constant.

So, in generally, we can write

$$g^{(n)}\left[h^{(n)}, h^{(n-1)n}, .., h^{(0)n}; x(t)\right] = H^{(n)}\left(x(t)\right) + \sum_{i=n-1}^{1} H^{(i)n}\left(x(t)\right) + h^{(0)n} . \tag{12}$$

For orthogonalization of the Volterra series, Wiener chose (Schetzen M. 1981) the Hermite polynomials; they have, after normalization, the following form:

$$H_0(x) = 1,\ H_1(x) = \frac{1}{\sigma}x,\ H_2(x) = \frac{1}{\sigma^2\sqrt{2}}\left(x^2 - \sigma^2\right),\ H_3(x) = \frac{1}{\sigma^3\sqrt{6}}\left(x^3 - 3\sigma^2 x\right),\ ..... , \tag{13}$$

where a subscript by $H$ denotes degree (order) of a given polynomial. A recursion formula describing these polynomials is given by

$$H_{n+1}(x) = \frac{1}{\sigma\sqrt{n+1}}\left(xH_n(x) - \sigma^2 \frac{d}{dx} H_n(x)\right). \tag{14}$$

As the orthonormal polynomials, they satisfy the following equality:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w^2(x) dx = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \tag{15}$$

with $w(x)$ meaning a weighting function. In the case of (15), that is of the Hermite polynomials, the weighting function $w(x)$ is such that

$$w^2(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad , \tag{16}$$

where $\sigma$ is a constant. In the means of approximation by the Wiener *G*-functionals, *x* in equations (13 – 16) stands for a white Gaussian time function applied as the input signal at the system's input. The parameter $\sigma$ in (16) plays a role of a time variance of the input signal, that is

$$\sigma^2 = Av\left(\left[x(t)\right]^2\right) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(x(t)\right)^2 dt \quad , \tag{17a}$$

where the operation of calculation of the time average is denoted by the symbol *Av*. Moreover, it was assumed in (17a) that the signal average, $Av\left(x(t)\right)$, is equal to zero. That is

$$Av\left(x(t)\right) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt = 0 \quad . \tag{17b}$$

The property of Gaussianity of the system input signal means that its amplitude distribution in time is described by the bell-shaped Gaussian function (16). More, the property of being "white" means that its autocorrelation function, let denote it $R_{xx}(\tau)$, is equal to the Dirac impulse $\delta(\tau)$ multiplied by a constant, say $N_0$, that is

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t+\tau) dt = N_0 \delta(\tau) \cdot \tag{18a}$$

Then, the Fourier transform of $R_{xx}(\tau)$ meaning the so-called power density spectrum, let denote it $G_{xx}(j\omega)$, is constant. That is

$$G_{xx}(j\omega) = N_0 \quad , \tag{18b}$$

where variable $\omega$ means the angular frequency.

For this class of input signals, being white and Gaussian, Wiener coined his *G*-functionals. They are defined as a set of nonhomogeneous Volterra functionals $g^{(n)}\left[k^{(n)}, k^{(n-1)n}, .., k^{(0)n}; x(t)\right]$ for which the following orthogonality principle

$$Av\left(H^{(m)}\left(x(t)\right) \cdot g^{(n)}\left[k^{(n)}, .., k^{(0)n}; x(t)\right]\right) = 0 \quad \text{for } m < n \tag{19}$$

holds (Schetzen M. 1981, Rugh W. J. 1981), where, as mentioned before, $x(t)$ is assumed to be Gaussian and having the autocorrelation function given by (18a). Moreover, $H^{(m)}\left(x(t)\right)$ in (19) means any *m*-th order homogeneous Volterra operator. In what follows, we will denote the nonhomogeneous Volterra functionals $g^{(n)}[\cdot]$ satisfying (19) by a capital letter *G*. For this reason, they will be called Wiener *G*-functionals (for a given *t*) or Wiener *G*-operators (for all values of *t*, that is considered as a function of *t*).

Assuming that the first Wiener *G*-operator, $G^{(0)}$, equals a constant and applying condition (19) for successive $n = 1, 2, 3, ....$, we get a set of *G*-operators. The procedure is described in more detail in (Schetzen M. 1980, Rugh W. J. 1981). Here, we present for illustration the first four Wiener *G*-operators. They have the following form:

$$G^{(0)}\left[k^{(0)}; x(t)\right] = k^{(0)} \quad , \tag{20a}$$

$$G^{(1)}\left[k^{(1)}; x(t)\right] = \int_{-\infty}^{\infty} k^{(1)}(\tau)\, x(t-\tau) d\tau \quad , \tag{20b}$$

$$\begin{aligned} G^{(2)}\left[k^{(2)}; x(t)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{(2)}(\tau_1, \tau_2)\, x(t-\tau_1) \cdot \\ &\cdot x(t-\tau_2) d\tau_1 d\tau_2 - N_0 \int_{-\infty}^{\infty} k^{(2)}(\tau, \tau)\, d\tau \end{aligned} \quad , \tag{20c}$$

$$\begin{aligned} G^{(3)}\left[k^{(3)}; x(t)\right] &= K^{(3)}\left(x(t)\right) + K^{(1)3}\left(x(t)\right) = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{(3)}(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1) x(t-\tau_2) \cdot \\ &\cdot x(t-\tau_3) d\tau_1 d\tau_2 d\tau_3 + \int_{-\infty}^{\infty} k^{(1)3}(\tau_1) x(t-\tau_1) d\tau_1 \end{aligned} \quad , \tag{20d}$$

where $k^{(1)3}(\tau_1)$ in (20d) is given by

$$k^{(1)3}(\tau_1) = -3N_0 \int_{-\infty}^{\infty} k^{(3)}(\tau_1, \tau_2, \tau_2) d\tau_2 \quad . \tag{20e}$$

Note that in equations (20) in $G^{(n)}\left[k^{(n)}; x(t)\right]$, $n = 0, 1, 2, 3$, only the leading component $k^{(n)}$ is shown in the square brackets, for shortening the notation. The functions $k^{(n)}$ are called the Wiener kernels (Schetzen M. 1981). Furthermore, observe that the numerical coefficients accompanying the components on the right hand sides of equations (20) are the same as those by the consecutive terms of Hermite polynomials (13). This is not fortuitous; for more details see, for example, (Schetzen M. 1980).

Using relation (19), it can be shown that the following orthogonality relationship between the Wiener $G$-operators

$$Av\left(G^{(m)}\left[k^{(m)};x\left(t\right)\right]\cdot G^{(n)}\left[k^{(n)};x\left(t\right)\right]\right)=0 \qquad (21)$$

holds for all $m\neq n$.

### 4.3 *Wiener description of a nonlinear system*

Using the properties of his G-operators, which were presented in the previous subsection, Wiener showed that the response $y\left(t\right)$ of a nonlinear system to a white Gaussian signal $x\left(t\right)$ can be described by an orthogonal series of the form

$$y\left(t\right)=\sum_{n=0}^{\infty}G^{(n)}\left[k^{(n)};x\left(t\right)\right] \quad . \qquad (22)$$

The expansion given by (22) was named, after his founder, the Wiener series.

Means of modelling of nonlinear systems driven by input signal being realizations of stochastic white Gaussian processes is illustrated in Fig. 1.

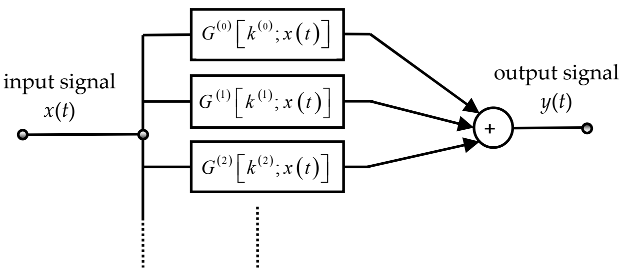


Figure 1. Nonlinear system modelling with the use of the Wiener operators and input signals being realizations of the white Gaussian processes.

Let a true output signal at the nonlinear system output be $z(t)$, and its approximate by the truncated Wiener series (given by (22)) with the first $p$ components (including $G^{(0)}$ ) $y_p\left(t\right)$ . Then, the mean-square value of the error $e_p\left(t\right)$ between the system's output signal $z(t)$ and $y_p\left(t\right)$ can be expressed in the following way (Schetzen M. 1981):

$$\begin{aligned}Av\left(\left[e_p\left(t\right)\right]^2\right)&=Av\left(\left[z\left(t\right)\right]^2\right)-Av\left(\left[y_p\left(t\right)\right]^2\right)=\\&=\sigma_z^2-\sum_{n=1}^{p}n!N_0^n\int_0^{\infty}\ldots\int_0^{\infty}\left(k^{(n)}\left(\tau_1,\ldots,\tau_n\right)\right)^2d\tau_1\cdots d\tau_n\end{aligned} \quad . \qquad (23)$$

In (23), $\sigma_z^2=Av\left(\left[z\left(t\right)\right]^2\right)-\left(Av\left[z\left(t\right)\right]\right)^2$ is the variance of the true system's response.

Obviously, in accordance with the rules of orthogonal approximation, the approximation error $Av\left(\left[e_p\left(t\right)\right]^2\right)$ decreases with the increase of the number of elements used, that is with the increase of the upper index $p$ in the sum symbol in (23). Its smallest value is given by

$$Av\left(\left[e_\infty\left(t\right)\right]^2\right)=\lim_{p\to\infty}Av\left(\left[e_p\left(t\right)\right]^2\right)\cdot \qquad (24)$$

### 4.4 *Orthogonal expansion of the Wiener kernels in the Wiener series*

Observe from (23) that the Wiener kernels satisfy the following equality

$$\int_0^{\infty}..\int_0^{\infty}\left(k^{(n)}\left(\tau_1,..,\tau_n\right)\right)^2d\tau_1\cdot\cdot d\tau_n<\infty \text{ , } n=1,2,3,.. \quad . \qquad (25)$$

The condition given by (25) is sufficient for expanding the Wiener kernels in a set of orthogonal functions. However, the orthonormal Laguerre functions are usually chosen in the literature because they can be easily physically realized, as all-pass filters. For example, see (Schetzen M. 1980).

The expansion of the Wiener kernel $k^{(n)}$ with the use of Laguerre functions $l_m(t),\ m=0,1,2,...,$ has the following form (Schetzen M. 1981):

$$k^{(n)}\left(\tau_1,..,\tau_n\right)=\sum_{m_1=0}^{\infty}..\sum_{m_2=0}^{\infty}a_{m_1..m_n}l_{m_1}\left(\tau_1\right)\cdot\cdot l_{m_n}\left(\tau_n\right) \quad , \qquad (26)$$

where the coefficients $a_{m_1..m_n}$ are given by

$$a_{m_1..m_n}=\int_0^{\infty}..\int_0^{\infty}k^{(n)}\left(\tau_1,..,\tau_n\right)l_{m_1}\left(\tau_1\right)\cdot\cdot l_{m_n}\left(\tau_n\right)d\tau_1\cdot\cdot d\tau_n\cdot \qquad (27)$$

In particular, the Laguerre expansion of the first order Wiener kernel $k^{(1)}$, restricted to the first $p+1$ components, assumes the form

$$k^{(1)}\left(\tau\right)=\sum_{m=0}^{p}a_m\cdot l_m\left(\tau\right) \qquad (28a)$$

with

$$a_m=\int_0^{\infty}k^{(1)}\left(\tau\right)\cdot l_m\left(\tau\right)d\tau \quad \cdot \qquad (28b)$$

### 4.5 *The Wiener model*

Using the results of derivations presented in the previous subsections 3.1 – 3.4, it can be further shown that a very general model for description of nonlinear systems follows from these outcomes. This model or its variants were used in a vast number of research papers dealing with the nonlinear systems. It is called the Wiener model (Schetzen M. 1981) and its structure is presented in Fig. 2.

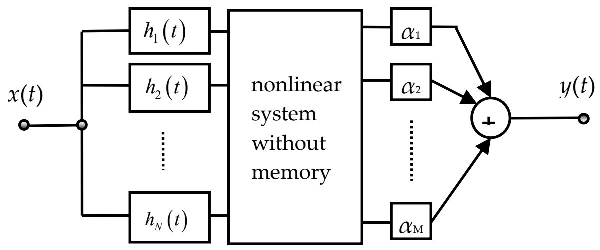


Figure 2. The Wiener model of a nonlinear system.

In Fig. 2, the first part of the model consisting of *N* blocks of linear subsystems having (linear) impulse responses denoted $h_1(t),...,h_N(t)$ is a single-input multi-output system. Elements of the above set of impulse responses are orthonormal. The next part of the model is a multi-input (*N* inputs) multi-output (*M* outputs) nonlinear system without memory, using multidimensional Hermite functions. And finally, the last part of the model in Fig. 2, consists of a set of *M* multipliers, $\alpha_1,...,\alpha_M$ , and a summing unit. Furthermore, note that all the memory of a given nonlinear system that is modelled according to the structure of Fig. 2 is concentrated solely in its first (linear) part.

### 4.6 *Remarks on stochastic functional Fourier series, Cameron-Martin type expansion and some other related ones*

Obviously, the scheme of modelling of nonlinear systems excited by signals being realizations of white Gaussian stochastic processes can be extended for other ones, for example Poisson processes (Marmarelis V. Z. & Berger T.W. 2005).

It is interesting that formulation of a stochastic version of the Fourier series is possible on the basis orthogonal functionals in a random environment (for random processes). This was done by Yasui in (Yasui S. 1979). In this paper, the relationships existing between the Wieneer kernels, Volterra kernels (nonlinear impulse responses), and coefficients in the so-called Cameron-Martin functional expansion (Cameron R. H. & Martin W. T. 1947) are found and discussed very thoroughly.

It is also worth noting here an algebraic approach to nonlinear functional expansions (Fliess M. & Lamnabhi M. & Lamnabhi-Lagarrigue F. 1983) leading to the expansions of the Volterra series type. This method relies upon the use of a formal power series in several non-commutative variables and of iterated integrals. For more details, see (Fliess M. & Lamnabhi M. & Lamnabhi-Lagarrigue F. 1983).

## 5 BEST APPROXIMATION OF LARGE-SCALE NONLINEAR SYSTEMS USING VOLTERRA OPERATORS IN WEIGHTED FOCK SPACES

One of the important problems with the Volterra series (as well as with the Wiener series) applications is that the number of calculations to be performed grows exponentially with the order (degree) of system's nonlinearities, which have to be taken into account to achieve good enough accuracy of the approximation. The number of the needed calculations grows also in a similar way with the increase of system input and/or output dimensionalities. The above facts cause that the Volterra series applications are limited to rather low-dimensional systems and/or such ones with mild nonlinearities.

As shown in (De Figueiredo R. J. P. & Dwyer III T. A. W. 1980), the above problem can be largely circumvented by reformulating the Volterra series with the use of a special mathematical tool called a reproducing kernel Hilbert space (RKHS). In the above paper, this tool was used in an appropriately chosen weighted Fock space. For more details, see (De Figueiredo R. J. P. & Dwyer III T. A. W. 1980) and references cited therein.

## 6 SOME INTERESTING APPLICATIONS OF THE VOLTERRA AND WIENER THEORIES

In this final section, because of lack of space, we present only examples of some interesting applications of the Volterra and Wiener theories in telecommunications, biological sciences, oceanology, and physics.

*Telecommunications*. Volterra series and an orthogonal series derived from it have been used for description of nonlinear distortions occurring in satellite communication channels. On this basis, the corresponding schemes for equalization of these nonlinear channels and compensation of distortions have been worked out in (Benedetto S. & Biglieri E. & Daffara R. 1979, Gutierrez A. & Ryan W. 2000). Another examples of applications for solving nonlinear problems in radio communication are presented in (Bedrosian E. & Rice S. O. 1971, Bussgang J. J. & Ehrman L. & Graham J. W. 1974).

*Biological sciences*. Examples of applications of the Wiener theory in this area can be found in articles (Dijk P. & Wit H. P. & Segenhout J. M. 1994, Marmarelis P. Z. & Naka K.-I. 1972, Marmarelis V. Z. & Zhao, Sclabassi R. J. & Risch H. A. & Hinman C. L. & Kroin J. S. & Enns N. F. & Namerow N. S. 1977).

*Oceanology*. Nonlinear ocean wave modelling with the use of the Volterra and other mathematical tools is described in (Maltz F. 2009).

*Physics*. The use of the Volterra and Wiener series in magnetic resonance spectroscopy has been exploited in (Blümich B 1985). Very interesting and promising is the application of the Volterra series to description of objects and phenomena in quantum physics (Zhang J. & Liu Y. &, Wu R.-B. & Jacobs K. & Ozdemir S. K. & Lan Y. & Tarn T.-J. & Nori F. 2014).

*Hydrology*. Interesting applications of the Volterra series are presented, for example, in (Napiórkowski J. J. & Strupczewski W.G.) and papers cited therein.

*Navigation*. Application of the Volterra filters in solving nonlinear problems of navigation can be found, for example, in (Park S. H. 2007).

*Transportation*. Nonlinear problems of transport-tation are tackled with the use of Wiener measure in (Feyel D. & A. S. Üstünel 2004).

## 7 CONCLUSIONS

First, a concise introduction to the Volterra and Wiener series has been made in this paper. Second, a general model of nonlinear systems, called the Wiener model after his founder, has been presented. Also, a model for description of very large nonlinear systems, based on the Volterra series and the so-called reproducing kernel Hilbert space, has been described. Finally, numerous applications of the above mathematical tools in such areas as tele-communications, biological sciences, oceanology, physics, hydrology, navigation, and transportation have been enumerated. However, because a lack of space, they are not presented here in more detail, with some needed illustrations. This will be done during an oral presentation at the conference. Nevertheless, we hope, all the examples given witness strongly great usefulness of the Volterra and Wiener theories in engineering.

## REFERENCES


Bedrosian E. & Rice S. O. 1971. The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs. *Proceedings of the IEEE*, 59(12): 1688-1707.

Benedetto S. & Biglieri E. & Daffara R. 1979. Modeling and performance evaluation of nonlinear satellite links. - A Volterra series approach. *IEEE Trans. Aerospace Electron. Syst.*, 15: 494-507.

Blümich B 1985. Stochastic NMR spectroscopy. *Bulletin of Magnetic Resonance*, 7(1): 5-26.

Borys A. 2000. *Nonlinear Aspects of Telecommunications: Discrete Volterra Series and Nonlinear Echo Cancellation*. Boca Raton, Florida: CRC Press.

Borys A. 2007, *Podstawowe Analogowe i Cyfrowe Układy Elektroniczne z Małymi Nieliniowościami Stosowane w Telekomunikacji*. Bydgoszcz: Wydawnictwa Uczelniane UTP w Bydgoszczy (in Polish).

Boudec, Le J.-Y. & Thiran P. 2004. *Network Calculus. A Theory of Deterministic Queuing Systems for the Internet*. Berlin: Springer Verlag.

Bussgang J. J. & Ehrman L. & Graham J. W. 1974. Analysis of nonlinear systems with multiple inputs, *Proceedings of the IEEE*, 62: 1088-1119.

Cameron R. H. & Martin W. T. 1947. The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. *Ann. Math.*, 48: 385-392.

De Figueiredo R. J. P. & Dwyer III T. A. W. 1980. A best approximation framework and implementation for simulation of large-scale nonlinear systems. *IEEE Trans. on Circuits and Systems*, 27(11): 1005-1013.

Dijk P. & Wit H. P. & Segenhout J. M. 1994. Wiener kernel analysis of inner ear function in the American bullfrog. *J. Acoust. Soc. Am.*, 95(2): 904-919.

Feyel D. & A. S. Üstünel 2004. Solution of the Monge-Ampère equation on Wiener space for log-concave measures, arXiv:math/0403497v1, 1-24.

Fliess M. & Lamnabhi M. & Lamnabhi-Lagarrigue F. 1983. An algebraic approach to nonlinear functional expansions. *IEEE Trans. on Circuits and Systems*, 30(8): 554-570.

Gutierrez A. & Ryan W. 2000. Performance of Volterra and MLSD receivers for nonlinear band-limited satellite systems. *IEEE Trans. on Communications*, 48(7): 1171-1177.

Maltz F. 2009. Nonlinear Ocean Wave Modeling: Beyond The Volterra Series Expansion and Fokker-Planck Equation. *Proceedings of the Conference OCEANS 2009, MTS/IEEE Biloxi - Marine Technology for Our Future: Global and Local Challenges*, 1-7.

Marmarelis P. Z. & Naka K.-I. 1972. White-noise analysis of a neuron chain: an application of the Wiener theory. *Science*, 175: 1276-1278.

Marmarelis V. Z. & Zhao. X. 1997. Volterra Models and Three-Layer Perceptrons. *IEEE Trans. on Communications* on Neural Networks, 8(6): 1421-1433.

Marmarelis V. Z. & Berger T.W. 2005. General methodology for nonlinear modeling of neural systems with Poisson point-process inputs, *Mathematical Biosciences*, 196: 1–13.

Napiórkowski J. J. & Strupczewski W.G. 1979. The analytical determination of the kernels of the Volterra series describing the cascade of nonlinear reservoirs, Journal of Hydrologic Sciences, 6(3-4): 121-142.

Oppenheim, A. V. 1965. *Superposition in a class of nonlinear systems*. Technical Report 432, Cambridge, Massachusetts: MIT.

Park S. H. 2007. Nonlinear trajectory navigation. Ph. D. A. E. thesis, The University of Michigan, 1-191.

Rugh W. J. 1981. *Nonlinear System Theory: The Volterra/Wiener Approach*. Baltimore (MD): Johns Hopkins Univ. Press.

Sandberg I. W. 1982. Volterra expansions for time-varying nonlinear systems. *Bell System Technical Journal*, 61(2): 201-225.

Sandberg I. W. 1983. On Volterra expansions for time-varying nonlinear systems. *IEEE Transactions on Circuits and Systems*, 30(2): 61-67.

Sandberg I. W. 1985. Multilinear maps and uniform boundedness. *IEEE Transactions on Circuits and Systems*, 32(4): 332-336.

Sandberg I. W. 1990. Criteria for the stability of $p$-linear maps and $p$-powers. *Nonlinear Analysis: Theory, Methods, and Applications*, 15(1): 69-84.

Schetzen M. 1980. *The Volterra and Wiener Theories of Nonlinear Systems*, New York: John Wiley & Sons.

Schetzen M. 1981. Nonlinear system modeling based on the Wiener theory. *Proceedings of the IEEE*, 69: 1557-1573.

Sclabassi R. J. & Risch H. A. & Hinman C. L. & Kroin J. S. & Enns N. F. & Namerow N. S. 1977. Complex pattern evoked somatosensory responses in the study of multiple sclerosis. *Proceedings of the IEEE*, 65(5): 626- 633.

Volterra V. 1959. *Theory of Functionals and of Integral and Integro-Differential Equations*. New York: Dover Publ.

Wiener N. 1942. *Response of a non-linear device to noise*. Technical Report 129, Cambridge, Massachusetts: MIT.

Wiener N. 1958. *Nonlinear Problems in Random Theory*. Cambridge, Massachusetts: MIT Press.

Yasui S. 1979. Stochastic functional Fourier series, Volterra series, and nonlinear systems analysis. *IEEE Trans. Automatic Control*, 24(2): 230-241.

Zhang J. & Liu Y. &, Wu R.-B. & Jacobs K. & Ozdemir S. K. & Lan Y. & Tarn T.-J. & Nori F. 2014. Nonlinear quantum input-output analysis using Volterra series, submitted to *IEEE Trans. Automatic Control*.