Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

SOME COUNTING FORMULAS FOR FINITE DISTRIBUTIVE LATTICES

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Abstract. In the paper we show that the weighted double skeleton of a finite distributive lattice is a sufficient structure to characterize the lattice numerically. We prove some combinatorial formulas for the number of all elements of a finite distributive lattice with the given weighted double skeleton, all its elements with exactly k lower covers and all its covering pairs. Introducing some simple examples, we show how the formulas work.

1. Introduction

In the case of big finite lattices it is often impossible to represent them by diagrams. To simplify their description it is useful to introduce the method given by Herrmann in [6], called gluing of lattices, which in fact is a way of building a lattice by means of smaller structures. It is particularly useful in the case of a finite distributive lattice, which turns out to be glued from its maximal Boolean intervals according to some factor structure (being also a lattice) called its skeleton.

However, knowing only the skeleton and Boolean lattices – bricks from which an original distributive lattice $\mathcal{D}$ is built – does not mean that we know how the lattice $\mathcal{D}$ looks like. To make the description complete we introduced in [5] the notion of weighted double skeleton.

Here we are going to show how to compute some combinatorial values of a finite distributive lattice, whose weighted double skeleton is known.

Let us start with introducing some basic notions. It was proved in [2] that maximal Boolean intervals which constitute a finite distributive lattice are in fact blocks of the smallest glued tolerance relation of the lattice.

A tolerance relation on a lattice $\mathcal{L}$ is a reflexive and symmetric binary relation on $\mathcal{L}$ compatible with lattice operations. A block of a tolerance relation Θ is a maximal subset B of L such that every pair of elements of B belongs to Θ. In the case of finite lattices the blocks of any tolerance relation Θ on a lattice $\mathcal{L}$ are intervals and by introducing an order of blocks compatible with the order of their largest elements we get a lattice called the factor lattice $\mathcal{L}/\Theta$.

It is clear that a congruence relation is a special case of a tolerance relation. However, while dealing with congruences we get a partition of the underlying set, here we are rather concerned with overlapping subsets determined by so called glued tolerances. A tolerance relation on $\mathcal{L}$ is called glued if its transitive closure is the total relation on $\mathcal{L}$. It can be proved (see [3]) that blocks of the smallest glued tolerance relation $\Sigma(L)$ are generated by the covering relation on $\mathcal{L}$. The factor lattice $\mathcal{L}/\ \Sigma(L)$ is called the skeleton of $\mathcal{L}$, and it will be denoted by $S(L)$.

Let $\mathcal{L}$ be a finite lattice and denote by $J_k(L)$ (resp. $M_k(L)$) the set of elements of $\mathcal{L}$ with exactly k lower (resp. upper) covers, i.e.

$$\begin{aligned} J_k(L) &= \{a \in L;\ \ |\{b \in L;\ \ b \prec a\}| = k\},\\ M_k(L) &= \{a \in L;\ \ |\{b \in L;\ \ a \prec b\}| = k\}. \end{aligned}$$

It is clear that the zero of $\mathcal{L}$ is the only element of $J_0(L)$ and $J_1(L)$ is the set of all join-irreducible elements of $\mathcal{L}$ (except the zero).

Let $Cov(L)$ denote the set of all covering pairs in $\mathcal{L}$, i.e.

$$Cov(L) = \{(x, y) :\ \ x \prec y,\ x, y \in L\}.$$

In [7], using the Möbius function, Reuter proved a formula counting the numbers of elements in $J_k(L)$ ($M_k(L)$) for any finite lattice with a given glued tolerance relation. Let us recall that the Möbius function μ_P of a poset P can be given by the recursive formula (see e.g. [1]):

$$\begin{cases} \mu_P(x,x) = 1 & \text{for} \quad x \in P,\\ \mu_P(x,y) = -\Sigma_{x \le z < y}\mu_P(x,z) & \text{for} \quad x < z;\ x, z \in P. \end{cases}$$

Theorem 1. *([7]) Let Θ be a glued tolerance relation on a finite lattice $\mathcal{L}$ with the factor lattice P and blocks $\{L_p\}_{p\in P}$. Then for any $k \ge 0$*

$$\begin{aligned} |J_k(L)| &= \sum_{r \le s} \mu_P(r,s)|J_k(L_r \cap L_s)|;\\ |M_k(L)| &= \sum_{r \le s} \mu_P(r,s)|M_k(L_r \cap L_s)|. \end{aligned}$$

Moreover,

$$|Cov(L)| = \sum_{r \leq s} \mu_P(r,s) |Cov(L_r \cap L_s)|.$$

As we see, to count elements of a lattice $\mathcal{L}$ we have to know not only the factor lattice P (the skeleton, for example) and blocks of the glued tolerance relation but intersections of all blocks, as well. All the information in the case of finite distributive lattices can be provided by the weighted double skeleton of the lattice, the notion of which we introduced in [5].

2. The main result

Let $\mathcal{D}$ be a finite distributive lattice with skeleton S. The blocks of the skeleton tolerance Θ are the maximal Boolean intervals of $\mathcal{D}$, we can denote them by $B_x = [0_x, 1_x]$ for $x \in S$. One can show that the subset $\{0_x\}_{x \in S}$ with the order inherited from $\mathcal{D}$ is a lattice isomorphic to the skeleton S (although the meet operations of these lattices may not agree). The same can be said about the subset $\{1_x\}_{x \in S}$ (now, the operations of join in $\mathcal{D}$ and the lattice of units can be different). Thus, these subsets need not form sublattices of $\mathcal{D}$. Let us consider the partially ordered subset $S^d = \{0_x\}_{x \in S} \cup \{1_x\}_{x \in S}$ of D. We shall call it the double skeleton of $\mathcal{D}$.

For simplicity we will write x instead of 0_x and x' instead of 1_x for $x \in S$. Thus, we can regard the double skeleton as a digraph, whose vertices are labeled by elements of some set S and its copy S' and whose arcs are determined just by the covering relation in the poset S^d. Let us observe that S and S' are not necessarily disjoint, hence some vertices can have two labels. It is also clear that if $a \prec b$ in the poset S^d, then $a < b$ in the lattice $\mathcal{D}$, and since all the maximal chains from a to b in a distributive lattice are of the same length, which will be denoted by $l[a,b]$, then in the digraph S^d we can introduce the weight function w assigning to every arc (a,b) the length of the interval $[a,b]$ in $\mathcal{D}$, i.e. $w(a,b) = l[a,b]$. The pair (S^d, w) is called the weighted double skeleton of $\mathcal{D}$.

Let $a \leq b$ in the poset S^d. Then there is a directed path from a to b in the weighted double skeleton and let $\bar{w}(a,b)$ denote the weight of the shortest path from a to b. In fact, in that case all the directed paths are of the same weight and $\bar{w}(a,b) = l[a,b]$.

Theorem 2. *If $\mathcal{D}$ is a finite distributive lattice with the weighted double skeleton (S^d, w), then for any $k \geq 0$*

$$|J_k(D)| = |M_k(D)| = \sum_{\substack{x \leq y \leq x' \\ x,y \in S}} \mu_S(x,y) \binom{\bar{w}(y,x')}{k}.$$

In particular,

$$|D| = \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) 2^{\bar{w}(y,x')}.$$

Moreover,

$$|Cov(D)| = \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) \bar{w}(y,x') 2^{\bar{w}(y,x')-1}.$$

Proof. Let $\mathcal{D}$ be a finite distributive lattice with the weighted double skeleton (S^d, w). Then maximal Boolean intervals of $\mathcal{D}$ can be written in the form $\mathcal{B}_x = [x, x']$ for $x \in S$. Let us observe that

$$\dim \mathcal{B}_x = l[x,x'] = w(x,x')$$

for any $x \in S$.

Moreover, if $x < y$ in S, then

$$B_x \cap B_y \neq \emptyset \text{ iff } y \le x'.$$

In that case $\mathcal{B}_x \cap \mathcal{B}_y$ is also a Boolean interval of the dimension $l[y,x'] = w(y,x')$.

On the other hand, for any Boolean algebra $\mathcal{B}$ and any $0 \le k \le \dim B$ we have

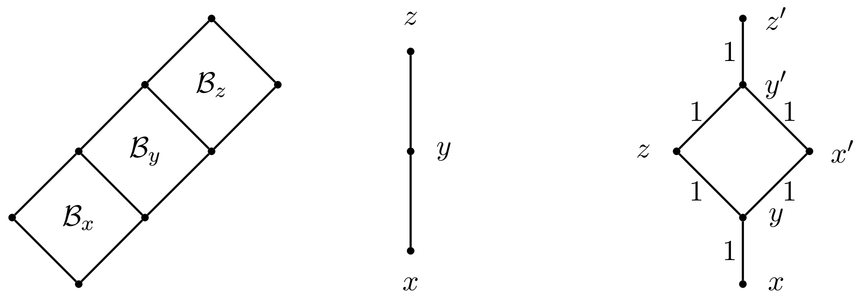
$$|J_k(B)| = |M_k(B)| = \binom{\dim B}{k}.$$

Thus, using Theorem 1, we get

$$|J_k(D)| = |M_k(D)| = \sum_{\substack{x \le y \\ x,y \in S}} \mu_S(x,y) |J_k(B_x \cap B_y)| = \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) \binom{\bar{w}(y,x')}{k}.$$

In particular,

$$\begin{aligned} |D| = \sum_{k \ge 0} |J_k(D)| &= \sum_{k \ge 0} \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) \binom{\bar{w}(y,x')}{k} \\ &= \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) \sum_{k \ge 0} \binom{\bar{w}(y,x')}{k} = \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y) 2^{\bar{w}(y,x')}. \end{aligned}$$



$\mathcal{D}$ $\mathcal{S}$ $\mathcal{S}^d$

Figure 1:

Now, let us notice that for any m-dimensional Boolean algebra $\mathcal{B}$ we have

$$|Cov(B)| = m2^{m-1}.$$

Therefore, by Theorem 1,

$$|Cov(D)| = \sum_{\substack{x \le y \\ x,y \in S}} \mu_S(x,y)|Cov(B_x \cap B_y)| = \sum_{\substack{x \le y \le x' \\ x,y \in S}} \mu_S(x,y)\bar{w}(y,x')2^{\bar{w}(y,x')-1}.$$

Example 1. Let us consider the distributive lattice $\mathcal{D}$ from Figure 1. Its skeleton S is the three-element chain.

For every poset P being a chain $x_1 \prec x_2 \prec \ldots \prec x_n$ we have

$$\mu_P(x_1, x_i) = \begin{cases} 1 & \text{if} \qquad\qquad i = 1, \\ -1 & \text{if} \qquad\qquad i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The weighted double skeleton S^d of $\mathcal{D}$ can be found in Figure 1. Thus, the number $|J_1(D)|$ of join-irreducible elements of $\mathcal{D}$ is counted by the formula:

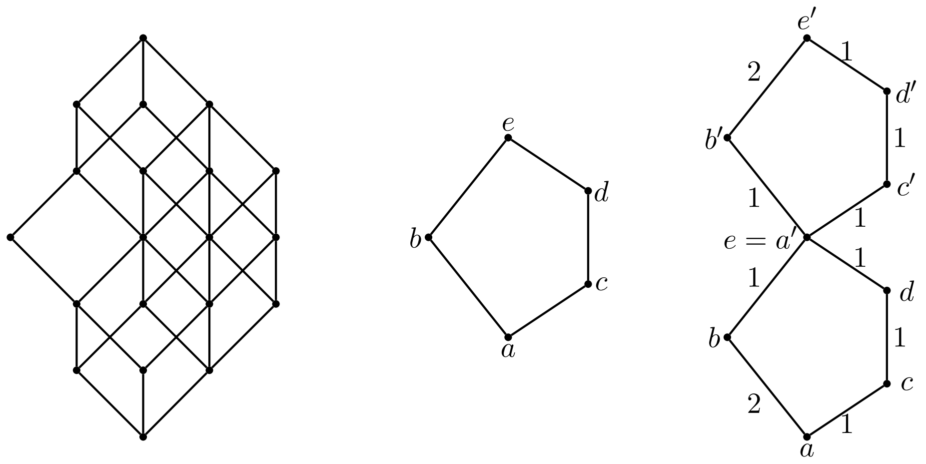
$$\begin{aligned} |J_1(D)| &= \bar{w}(x,x') + \bar{w}(y,y') + \bar{w}(z,z') - \bar{w}(y,x') - \bar{w}(z,y') \\ &= 2 + 2 + 2 - 1 - 1 = 4, \end{aligned}$$

and the total number of elements of $\mathcal{D}$ is given by

$$|D| = 2^2 + 2^2 + 2^2 - 2^1 - 2^1 = 8.$$

Moreover,

$$|Cov(D)| = 2 \cdot 2^1 + 2 \cdot 2^1 + 2 \cdot 2^1 - 1 \cdot 2^0 - 1 \cdot 2^0 = 10.$$



$\mathcal{D}$ $\mathcal{S}$ $\mathcal{S}^d$

Figure 2:

Example 2. Let us consider the distributive lattice $\mathcal{D}$ from Figure 2, whose skeleton S is a pentagon. Since the skeleton of the pentagon is the trivial lattice, then $\mathcal{D}$ is an H-irreducible lattice (see [4]) and its double skeleton S^d consists of two copies of the skeleton having one element in common – the top element of the lattice of zeroes is at the same time the bottom element of the lattice of units of the maximal Boolean intervals of $\mathcal{D}$. The weighted double skeleton of $\mathcal{D}$ can be seen in Figure 2.

The Möbius function for the pentagon is given by the table below:

	a	b	c	d	e
a	1	-1	-1	0	1
b	x	1	x	x	-1
c	x	x	1	-1	0
d	x	x	x	1	-1
e	x	x	x	x	1

where x means that the value of μ for the given pair of elements does not exist.

Thus, the number of elements of $\mathcal{D}$ can be counted by the following formula:

$$\begin{aligned}|D| =& 2^{\bar{w}(a,a')} - 2^{\bar{w}(b,a')} - 2^{\bar{w}(c,a')} + 2^{\bar{w}(e,a')} + 2^{\bar{w}(b,b')} \\ &- 2^{\bar{w}(e,b')} + 2^{\bar{w}(c,c')} - 2^{\bar{w}(d,c')} + 2^{\bar{w}(d,d')} - 2^{\bar{w}(e,d')} + 2^{\bar{w}(e,e')} \\ =& 2^3 - 2^1 - 2^2 + 2^0 + 2^2 - 2^1 + 2^3 - 2^2 + 2^3 - 2^2 + 2^3 = 21.\end{aligned}$$

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