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Problems of steady vibrations in the coupled linear theory of double-porosity viscoelastic materials

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In the present paper the coupled linear theory of double-porosity viscoelastic materials is considered and the basic boundary value problems (BVPs) of steady vibrations are investigated. Indeed, in the beginning, the systems of equations of motion and steady vibrations are presented. Then, Green's identities are established and the uniqueness theorems for classical solutions of the BVPs of steady vibrations are proved. The fundamental solution of the system of steady vibration equations is constructed and the basic properties of the potentials (surface and volume) are given. Finally, the existence theorems for classical solutions of the above mentioned BVPs are proved by using the potential method (the boundary integral equations method) and the theory of singular integral equations.

Key words: viscoelasticity, double-porosity materials, uniqueness and existence theorems, potential method.

Mathematics subject classification: 74F10, 74G25, 74G30.

1. Introduction

Viscoelastic effects are described for a wide variety of materials including biopolymers, amorphous and semicrystalline polymers, metals, high damping alloys, rock, piezoelectric materials, cellular solids, dense composite materials, and biological materials (see, e.g., Brinson and Brinson [1], Lakes [2]). Viscoelastic materials can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies. Various theories of viscoelastic materials have been proposed and studied in the series of works (for details, see Truesdell and Noll [3], Amendola *et al.* [4], Fabrizio and Morro [5] and the references therein).

It is noteworthy that the prediction of the physical properties of porous media has been one of hot topics of continuum mechanics (see, de Boer [6], Bear [7], Cheng [8], Coussy [9], Ichikawa and Selvadurai [10], Wang [11]). In the last two decades, there has been interest in formulation of the theories of viscoelasticity of differential type for porous materials. In this connection, a theory of thermoviscoelasticity for a composite as a mixture of porous elastic solid and Kelvin-Voigt material is presented by IEŞAN [12]. The theory of porous thermoviscoelastic mixtures is developed by IEŞAN and QUINTANILLA [13]. The steady vibrations problems in the theory of thermoviscoelastic porous mixtures studied by SVANADZE [14].

Nowadays, mathematical models for porous media are mostly based on one of the following two phenomena: (i) *the concept of Darcy's classical law* and (ii) *the concept of volume fraction*.

Indeed, in the pioneering work [15], Biot firstly introduced the hydraulic and mechanical coupling phenomenon and the 3D consolidation theory of isotropic porous solids based on Darcy's law was presented. Then, this classical theory of poroelasticity is developed by using several coupling processes. The coupled thermo-hydro-mechanical models based on Darcy's law are introduced by many investigators (see, e.g., BOOKER and SAVVIDOU [16], DERSKI and KOWALSKI [17], NGUYEN and SELVADURAI [18], SCHIFFMAN [19], etc.). Much of the theoretical progress in the above mentioned theories for porous materials based on Darcy's law are discussed in the books by SELVADURAI and SUVOROV [20], STRAUGHAN [21, 22], SVANADZE [23]. Plane waves and problems of quasi-statics and steady vibrations in the theory of viscoelasticity for Kelvin-Voigt materials with double porosity are studied by SVANADZE [24, 25].

On the other hand, considerable progress has been made in the study of the mathematical theories of porous materials based on the volume fraction concept. In particular, in the papers [26] and [27], on the basis of this concept Nunziato and Cowin introduced an alternative linear and nonlinear theory of elasticity for deformable porous materials (also called materials with voids). Then, this mathematical model is extended and the linear theory of thermoelastic materials with voids is presented by IEŞAN [28]. More general models of elasticity and thermoelasticity for materials with voids based on the volume fraction concept are introduced and studied by several research groups. Basic results in this subject of investigation may be found in the books by CIARLETTA and IEŞAN [29], IEŞAN [30], STRAUGHAN [21, 22, 31].

In addition, on the basis of the volume fraction concept, the theory of thermoviscoelasticity for Kelvin-Voigt materials with voids are established by IEŞAN [32]. Then, this theory is extended and the basic results for single and double porosity materials are obtained in the series of papers (see, e.g., IEŞAN [33], CHIRIŢĂ [34], CHIRIŢĂ and DANESCU [35], D'APICE and CHIRIŢĂ [36], JAIANI [37], KUMAR and KUMAR [38], TOMAR [39], BUCUR [40], SVANADZE [41, 42]).

Moreover, we encounter various new coupled processes in mechanics of viscoelastic porous media and therefore, it is required to consider several coupled mechanical concepts simultaneously. In [43], a mathematical model of viscoelastic single-porosity materials is presented, in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered and then, steady vibrations problems of this model are investigated. In the present paper, on the basis of this coupled phenomenon of above mentioned two concepts a mathematical model of viscoelastic double-porosity materials is developed and the basic BVPs of steady vibration of this model are investigated.

Note that the linear models of elasticity and thermoelasticity for single-porosity materials are proposed by Svanadze [44, 45] in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is introduced. The basic BVPs of steady vibrations in the quasi-static case of these models are studied by Bitsadze [46, 47] and Mikelashvili [48, 49]. Recently, this coupled phenomenon has been extended for double-porosity elastic materials in [50].

This work is articulated as follows. In Section 2, the basic equations of motion and steady vibrations in the coupled linear theory of viscoelasticity for double-porosity materials are presented. The system of governing equations is expressed in terms of the displacement vector field, the changes of volume fractions of pores and fissures, and the changes of the fluid pressures in pore and fissure networks. In Section 3, the basic BVPs of steady vibrations are formulated and Green's first identity is established. In Section 4, the uniqueness theorems for the regular (classical) solutions of the above mentioned BVPs are proved. Then, in Section 5, the fundamental solution of the system of steady vibration equations is constructed explicitly by means of elementary functions and its basic properties are established. In Section 6, Green's second and third identities of the considered theory are obtained. In Section 7, the surface (single-layer and double-layer) and volume potentials are constructed and their basic properties are given. In addition, in this section, the symbolic determinants and indexes of some useful singular integral operators are calculated. Finally, in Section 8, the existence theorems for classical solutions of the BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

2. Basic equations

In what follows we consider a material with double porosity in which the skeleton is an isotropic and homogeneous viscoelastic Kelvin-Voigt solid with pores on the macro scale and pores on a much smaller meso or micro (also called fissures) scale. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$, let t denote the time variable, $t \geq 0$ and the dot denotes differentiation with respect to t. Repeated Latin and Greek indices are summed over the ranges $(1, 2, 3)$ and $(1, 2)$, respectively.

Let $\hat{\mathbf{u}}(\mathbf{x}, t)$ be the three-component displacement vector field for the skeleton of a porous material, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$; $\hat{p}_1(\mathbf{x}, t)$ and $\hat{p}_2(\mathbf{x}, t)$ are the changes of the pore and fissure fluid pressures, respectively; $\hat{\varphi}_1(\mathbf{x}, t)$ and $\hat{\varphi}_2(\mathbf{x}, t)$ are the changes of the volume fractions from the reference configuration for the pore and fissure networks, respectively.

Following the concepts of the deformation of a solid and the volume fraction, the basic equations of motion in the linear coupled theory of double-porosity viscoelastic materials we can write as

$$
\begin{aligned}
&\hat{t}_{lj,j} = \rho(\ddot{\hat{u}}_l - \hat{F}'_l), \qquad \hat{\sigma}^{(1)}_{j,j} + \hat{\xi}^{(1)} = \rho_1 \ddot{\hat{\varphi}}_1 - \rho \hat{s}_1, \\
&\hat{\sigma}^{(2)}_{j,j} + \hat{\xi}^{(2)} = \rho_2 \ddot{\hat{\varphi}}_2 - \rho \hat{s}_2, \qquad l = 1,2,3,
\end{aligned} \tag{2.1}
$$

where $\hat{t}_{lj}$ is the component of total stress tensor, $\hat{\mathbf{F}}' = (\hat{F}'_1, \hat{F}'_2, \hat{F}'_3)$ is the body force per unit mass; $\hat{\sigma}^{(1)}_j$, $\hat{\xi}^{(1)}$, $\hat{s}_1$, ρ_1 and $\hat{\sigma}^{(2)}_j$, $\hat{\xi}^{(2)}$, $\hat{s}_2$, ρ_2 are the component of the equilibrated stress, the intrinsic equilibrated body force, the extrinsic equilibrated body force, the coefficients of the equilibrated inertia associated to the pore and fissure networks, respectively; ρ is the reference mass density, $\rho > 0$, $\rho_1 > 0$, $\rho_2 > 0$,

$$
\begin{aligned}
\hat{\xi}^{(1)} &= -b_1 \hat{e}_{rr} - \zeta_1 \hat{\varphi}_1 - \zeta_3 \hat{\varphi}_2 + m_1 \hat{p}_1 + m_3 \hat{p}_2 - \gamma_1^* \dot{\hat{e}}_{rr} - \zeta_1^* \dot{\hat{\varphi}}_1 - \zeta_3^* \dot{\hat{\varphi}}_2, \\
\hat{\xi}^{(2)} &= -b_2 \hat{e}_{rr} - \zeta_3 \hat{\varphi}_1 - \zeta_2 \hat{\varphi}_2 + m_3 \hat{p}_1 + m_2 \hat{p}_2 - \gamma_2^* \dot{\hat{e}}_{rr} - \zeta_1^* \dot{\hat{\varphi}}_1 - \zeta_3^* \dot{\hat{\varphi}}_2,
\end{aligned} \tag{2.2}
$$

$\hat{e}_{lj}$ is the component of strain tensor and given by

$$
\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \qquad l,j = 1,2,3. \tag{2.3}
$$

Here the constitutive equations have the following form

$$
\begin{aligned}
\hat{t}_{lj} &= 2\mu \hat{e}_{lj} + \lambda \hat{e}_{rr} \delta_{lj} + (b_\alpha \hat{\varphi}_\alpha - \beta_\alpha \hat{p}_\alpha) \delta_{lj} \\
&\qquad + 2\mu^* \dot{\hat{e}}_{lj} + \lambda^* \dot{\hat{e}}_{rr} \delta_{lj} + (b_1^* \dot{\hat{\varphi}}_1 + b_2^* \dot{\hat{\varphi}}_2) \delta_{lj}, \\
\hat{\sigma}^{(1)}_l &= \alpha_1 \hat{\varphi}_{1,l} + \alpha_3 \hat{\varphi}_{2,l} + \alpha_1^* \dot{\hat{\varphi}}_{1,l} + \alpha_3^* \dot{\hat{\varphi}}_{2,l}, \\
\hat{\sigma}^{(2)}_l &= \alpha_3 \hat{\varphi}_{1,l} + \alpha_2 \hat{\varphi}_{2,l} + \alpha_3^* \dot{\hat{\varphi}}_{1,l} + \alpha_2^* \dot{\hat{\varphi}}_{2,l}, \quad l,j = 1,2,3,
\end{aligned} \tag{2.4}
$$

where δ_{lj} is Kronecker's delta.

Meanwhile, the equations of fluid mass conservation and Darcy's extended law can be expressed as follows

$$
\begin{aligned}
&\hat{v}^{(1)}_{j,j} + a_1 \dot{\hat{p}}_1 + a_3 \dot{\hat{p}}_2 + \beta_1 \dot{\hat{e}}_{rr} + m_1 \dot{\hat{\varphi}}_1 + m_3 \dot{\hat{\varphi}}_2 + \gamma_0 (\hat{p}_1 - \hat{p}_2) = 0, \\
&\hat{v}^{(2)}_{j,j} + a_3 \dot{\hat{p}}_1 + a_2 \dot{\hat{p}}_2 + \beta_2 \dot{\hat{e}}_{rr} + m_3 \dot{\hat{\varphi}}_1 + m_2 \dot{\hat{\varphi}}_2 - \gamma_0 (\hat{p}_1 - \hat{p}_2) = 0
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
\hat{\mathbf{v}}^{(1)} &= -\frac{\kappa_1}{\mu''} \nabla \hat{p}_1 - \frac{\kappa_3}{\mu''} \nabla \hat{p}_2 - \rho_3 \hat{\mathbf{s}}_3, \\
\hat{\mathbf{v}}^{(2)} &= -\frac{\kappa_3}{\mu''} \nabla \hat{p}_1 - \frac{\kappa_2}{\mu''} \nabla \hat{p}_2 - \rho_4 \hat{\mathbf{s}}_4,
\end{aligned} \tag{2.6}
$$

respectively, where $\hat{\mathbf{v}}^{(1)} = (\hat{v}_1^{(1)}, \hat{v}_2^{(1)}, \hat{v}_3^{(1)})$ and $\hat{\mathbf{v}}^{(2)} = (\hat{v}_1^{(2)}, \hat{v}_2^{(2)}, \hat{v}_3^{(2)})$ are the fluid flux vectors associated to the pore and fissure networks, respectively; γ_0 is the internal transport coefficient, $\gamma_0 \geq 0$, μ'' is the fluid viscosity, $\rho_3, \hat{\mathbf{s}}_3$ and $\rho_4, \hat{\mathbf{s}}_4$ are the density of fluid, the external force (such as gravity) for the pore and fissure networks, respectively; ∇ is the gradient operator, the values $\lambda, \mu, b_l, \beta_l, a_j, \alpha_j, \zeta_j, m_j, \kappa_j$ are the constitutive coefficients associated to the elasticity and porosity of materials, but the values $\lambda^*, \mu^*, b_l^*, \gamma_l^*, \alpha_j^*, \zeta_j^*$ $(l = 1,2,\ j = 1,2,3)$ are the viscosity constitutive coefficients.

Substituting Eqs. (2.2)–(2.4) and (2.6) into (2.1) and (2.5) we obtain the following system of equations of motion in the linear coupled theory of viscoelastic double-porosity materials expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fractions $\hat{\varphi}_1$, $\hat{\varphi}_2$ and the pressures $\hat{p}_1$, $\hat{p}_1$:

$$
\begin{aligned}
&\tilde{\mu}\Delta\hat{\mathbf{u}} + (\tilde{\lambda}+\tilde{\mu})\nabla\mathrm{div}\,\hat{\mathbf{u}} + \tilde{b}_\alpha\nabla\hat{\varphi}_\alpha - \beta_\alpha\nabla\hat{p}_\alpha = \rho(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}'),\\
&\tilde{\alpha}_1\Delta\hat{\varphi}_1 + \tilde{\alpha}_3\Delta\hat{\varphi}_2 - \tilde{\zeta}_1\hat{\varphi}_1 - \tilde{\zeta}_3\hat{\varphi}_2 - \tilde{\gamma}_1\,\mathrm{div}\,\hat{\mathbf{u}} + m_1\hat{p}_1 + m_3\hat{p}_2 = \rho_1\ddot{\hat{\varphi}}_1 - \rho\hat{s}_1,\\
&\tilde{\alpha}_3\Delta\hat{\varphi}_1 + \tilde{\alpha}_2\Delta\hat{\varphi}_2 - \zeta_3\hat{\varphi}_1 - \zeta_2\hat{\varphi}_2 - \tilde{\gamma}_2\,\mathrm{div}\,\hat{\mathbf{u}} + m_3\hat{p}_1 + m_2\hat{p}_2 = \rho_2\ddot{\hat{\varphi}}_2 - \rho\hat{s}_2,\\
&k_1\Delta\hat{p}_1 + k_3\Delta\hat{p}_2 - a_1\dot{\hat{p}}_1 - a_3\dot{\hat{p}}_2 - \beta_1\,\mathrm{div}\,\dot{\hat{\mathbf{u}}} - m_1\dot{\hat{\varphi}}_1 - m_3\dot{\hat{\varphi}}_2\\
&\qquad\qquad\qquad\qquad - \gamma_0(\hat{p}_1 - \hat{p}_2) = -\rho_3\,\mathrm{div}\,\hat{\mathbf{s}}_3,\\
&k_3\Delta\hat{p}_1 + k_2\Delta\hat{p}_2 - a_3\dot{\hat{p}}_1 - a_2\dot{\hat{p}}_2 - \beta_2\,\mathrm{div}\,\dot{\hat{\mathbf{u}}} - m_3\dot{\hat{\varphi}}_1 - m_2\dot{\hat{\varphi}}_2\\
&\qquad\qquad\qquad\qquad + \gamma_0(\hat{p}_1 - \hat{p}_2) = -\rho_4\,\mathrm{div}\,\hat{\mathbf{s}}_4,
\end{aligned}
\tag{2.7}
$$

where Δ is the Laplacian operator, $k_j = \kappa_j/\mu''$ and

$$
\begin{aligned}
&\tilde{\lambda} = \lambda + \lambda^*\frac{\partial}{\partial t}, \qquad \tilde{\mu} = \mu + \mu^*\frac{\partial}{\partial t}, \qquad \tilde{b}_l = b_l + b_l^*\frac{\partial}{\partial t}, \qquad \tilde{\alpha}_j = \alpha_j + \alpha_j^*\frac{\partial}{\partial t},\\
&\tilde{\zeta}_j = \zeta_j + \zeta^*\frac{\partial}{\partial t}, \qquad \tilde{\gamma}_l = b_l + \gamma_l^*\frac{\partial}{\partial t}, \qquad l = 1,2,\ j = 1,2,3.
\end{aligned}
$$

If $\hat{\mathbf{u}}, \hat{\mathbf{F}}', \hat{\varphi}_l, \hat{p}_l, \hat{s}_l$ and $\hat{\mathbf{s}}_{l+2}$ $(l = 1,2)$ are postulated to have a harmonic time variation, that is,

$$
\{\hat{\mathbf{u}}, \hat{\mathbf{F}}', \hat{\varphi}_l, \hat{p}_l, \hat{s}_l, \hat{\mathbf{s}}_{l+2}\}(\mathbf{x}, t) = \mathrm{Re}[\{\mathbf{u}, \mathbf{F}', \varphi_l, p_l, s_l, \mathbf{s}_{l+2}\}(\mathbf{x})e^{-i\omega t}],
$$

then from (2.7) we obtain the following system of equations of steady vibrations in the linear coupled theory of viscoelasticity for materials with double porosity

$$
\begin{aligned}
&(\mu'\Delta + \rho\omega^2)\mathbf{u} + (\lambda' + \mu')\nabla\mathrm{div}\,\mathbf{u} + b'_\alpha\nabla\varphi_\alpha - \beta_\alpha\nabla p_\alpha = -\rho\mathbf{F}',\\
&(\alpha_1\Delta + c_1)\varphi_1 + (\alpha_3\Delta - \zeta'_3)\varphi_2 - \gamma'_1\,\mathrm{div}\,\mathbf{u} + m_1p_1 + m_3p_2 = -\rho s_1,\\
&(\alpha_3\Delta - \zeta_3)\varphi_1 + (\alpha_2\Delta + c_2)\varphi_2 - \gamma'_2\,\mathrm{div}\,\mathbf{u} + m_3p_1 + m_2p_2 = -\rho s_2,\\
&(k_1\Delta + a'_1)p_1 + (k_3\Delta + a'_3)p_2 + \beta'_1\,\mathrm{div}\,\mathbf{u} + m'_1\varphi_1 + m'_3\varphi_2 = -\rho_3\,\mathrm{div}\,\mathbf{s}_3,\\
&(k_3\Delta + a'_3)p_1 + (k_2\Delta + a'_2)p_2 + \beta'_2\,\mathrm{div}\,\mathbf{u} + m'_3\varphi_1 + m'_2\varphi_2 = -\rho_4\,\mathrm{div}\,\mathbf{s}_4,
\end{aligned}
\tag{2.8}
$$

where ω is the oscillation frequency, $\omega > 0$,

$$
\begin{aligned}
&\lambda' = \lambda - \omega\lambda^*, \quad \mu' = \mu - i\omega\mu^*, \quad b'_l = b_l - i\omega b^*_l, \qquad c_l = \rho_l\omega^2 - \zeta'_l,\\
&\gamma'_l = b_l - i\omega\gamma^*_l, \quad a'_l = i\omega a_l - \gamma_0, \quad a'_3 = i\omega a_l + \gamma_0, \quad \zeta'_j = \zeta_j - i\omega\zeta^*_j,\\
&\alpha'_j = \alpha_j - i\omega\alpha^*_j, \quad \beta'_l = i\omega\beta_l, \qquad m'_j = i\omega m_j, \quad l = 1,2, \; j = 1,2,3.
\end{aligned}
\tag{2.9}
$$

For later convenience we introduce the following second order matrix differential operator

$$
\begin{aligned}
&\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{7\times 7}, \quad A_{lj} = (\mu'\Delta + \rho\omega^2)\delta_{lj} + (\lambda' + \mu')\frac{\partial^2}{\partial x_l \partial x_j},\\
&A_{l;r+3} = b'_r\frac{\partial}{\partial x_l}, \quad A_{l;r+5} = -\beta_r\frac{\partial}{\partial x_l}, \quad A_{r+3;l} = -\gamma'_r\frac{\partial}{\partial x_l},\\
&A_{44} = \alpha'_1\Delta + c_1, \quad A_{45} = A_{54} = \alpha'_3\Delta + \zeta'_3, \quad A_{55} = \alpha'_2\Delta + c_2,\\
&A_{r+3;6} = m_1, \quad A_{r+3;7} = A_{56} = m_3, \quad A_{r+5;l} = \beta'_r\frac{\partial}{\partial x_l}, \quad A_{64} = m'_1,\\
&A_{65} = A_{74} = m'_3, \quad A_{75} = m'_2, \quad A_{66} = k_1\Delta + a'_1,\\
&A_{67} = A_{76} = k_3\Delta + a'_3, \quad A_{77} = k_2\Delta + a'_2,\\
&\mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad l, j = 1,2,3, \; r = 1,2.
\end{aligned}
$$

Now we can rewrite the system (2.8) into the following

$$
\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.10}
$$

where $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ and $\mathbf{F} = (-\rho\mathbf{F}', -\rho s_1, -\rho s_2, -\rho_3 \operatorname{div}\mathbf{s}_3, -\rho_4 \operatorname{div}\mathbf{s}_4)$ are seven-component vector functions, $\mathbf{x} \in \mathbb{R}^3$.

The purpose of this paper is to prove the existence and uniqueness theorems for the classical solutions of the basic BVPs of steady vibrations in the linear coupled theory of viscoelastic double-porosity materials using the potential method.

In particular, for prove the uniqueness theorems of classical solutions we need Green's first identity. Moreover, the proof of the existence theorems by the potential method requires: (i) the fundamental solution of the system (2.8), (ii) Green's third identity, (iii) the basic properties of the surface and volume potentials. After obtaining these results, we are able to reduce the basic BVPs to the equivalent singular integral equations for which Fredholm's theorems will be valid.

3. Boundary value problems and Green's first identity

Let S be the smooth closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$. We denote by $\mathbf{n}(\mathbf{z})$ the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$. The scalar product of two vectors $\mathbf{U} = (U_1, U_2, \dots, U_7)$ and $\mathbf{V} = (V_1, V_2, \dots, V_7)$ is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum_{j=1}^{7} U_j \bar{V}_j$, where $\bar{V}_j$ is the complex conjugate of V_j.

DEFINITION 1. A vector function $\mathbf{U} = (U_1, U_2, \dots, U_7)$ is called *regular* in Ω^- (or Ω^+) if

(i) $\quad U_j \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \quad (\text{or } U_j \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$

(ii)

$$U_j(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad U_{j,l}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \quad \text{for } |\mathbf{x}| \gg 1, \tag{3.1}$$

where $j = 1, \dots, 7$, $l = 1, 2, 3$.

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (P_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{7\times 7},$$

where

$$\begin{gathered}
P_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \mu' \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \mu' n_j \frac{\partial}{\partial x_l} + \lambda' n_l \frac{\partial}{\partial x_j}, \quad P_{lr}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = b'_{r-3} n_l,\\
P_{l;r+2}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\beta_{r-3} n_l, \quad P_{44}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha'_1 \frac{\partial}{\partial \mathbf{n}},\\
P_{45}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{54}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha'_3 \frac{\partial}{\partial \mathbf{n}},\\
P_{55}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = \alpha'_2 \frac{\partial}{\partial \mathbf{n}}, \qquad P_{66}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = k_1 \frac{\partial}{\partial \mathbf{n}},\\
P_{67}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{76}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = k_3 \frac{\partial}{\partial \mathbf{n}}, \quad P_{77}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = k_2 \frac{\partial}{\partial \mathbf{n}},\\
P_{sj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{r;m}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = P_{mr}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = 0,\\
l, j = 1, 2, 3, \; r = 4, 5, \; m = 6, 7, \; s = 4, 5, 6, 7.
\end{gathered}$$

$\partial/\partial\mathbf{n}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the coupled linear theory of double-porosity viscoelastic materials are formulated as follows.

Find a regular (classical) solution $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ to system (2.10) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are known seven-component smooth vector functions.

Find a regular (classical) solution $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ to system (2.10) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}(\mathbf{D}_\mathbf{x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are known seven-component smooth vector functions, $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

In what follows, we assume that the constitutive coefficients satisfy the conditions

$$\begin{aligned}
&\mu^* > 0, \quad 3\lambda^* + 2\mu^* > 0, \quad \frac{1}{3}(3\lambda^* + 2\mu^*)\zeta_1^* > \frac{1}{4}(b_1^* + \gamma_1^*)^2,\\
&\frac{1}{3}(3\lambda^* + 2\mu^*)(\zeta_1^*\zeta_2^* - \zeta_3^{*^2})\\
&\qquad > \frac{1}{4}\big[\zeta_2^*(b_1^* + \gamma_1^*)^2 - 2\zeta_3^*(b_1^* + \gamma_1^*)(b_2^* + \gamma_2^*) + \zeta_1^*(b_2^* + \gamma_2^*)^2\big],\\
&\alpha_1^* > 0, \quad \alpha_1^*\alpha_2^* - \alpha^{*^2} > 0, \quad k_1 > 0, \quad k_1k_2 - k_3^2 > 0,\\
&a_1 > 0, \quad a_1a_2 - a_3^2 > 0.
\end{aligned} \tag{3.2}$$

REMARK 1. Obviously, we may derive from (3.2) the following useful inequalities

$$\begin{aligned}
&\lambda^* + 2\mu^* > 0, \quad \zeta_1^* > 0, \quad \zeta_2^* > 0, \quad \zeta_1^*\zeta_2^* - \zeta_3^{*^2} > 0,\\
&\frac{1}{3}(3\lambda^* + 2\mu^*)\zeta_2^* > \frac{1}{4}(b_2^* + \gamma_2^*)^2, \quad \alpha_2^* > 0, \quad k_2 > 0, \quad a_2 > 0.
\end{aligned}$$

Let $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2')$ be a seven-component smooth vector function, $\mathbf{u}' = (u_1', u_2', u_3')$. We introduce the notation

$$\begin{aligned}
W^{(0)}(\mathbf{u}, \mathbf{u}') = {}& \frac{1}{3}(3\lambda' + 2\mu')\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{u}'}\\
&+ \frac{\mu'}{2}\sum_{l,j=1;\, l\neq j}^{3}\left(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial \overline{u_j'}}{\partial x_l} + \frac{\partial \overline{u_l'}}{\partial x_j}\right)\\
&+ \frac{\mu'}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \overline{u_l'}}{\partial x_l} - \frac{\partial \overline{u_j'}}{\partial x_j}\right),
\end{aligned} \tag{3.3}$$

$$
(3.3)_{[\text{cont.}]}\qquad
\begin{aligned}
W^{(1)}(\mathbf{U},\mathbf{u}') &= W^{(0)}(\mathbf{u},\mathbf{u}') - \rho\omega^2\mathbf{u}\cdot\mathbf{u}' + (b'_\alpha\varphi_\alpha - \beta_\alpha p_\alpha)\operatorname{div}\overline{\mathbf{u}'},\\
W^{(2)}(\mathbf{U},\varphi'_1) &= (\alpha'_1\nabla\varphi_1 + \alpha'_3\nabla\varphi_2)\cdot\nabla\varphi'_1\\
&\quad + [\gamma'_1\operatorname{div}\mathbf{u} - c_1\varphi_1 + \zeta'_3\varphi_2 - m_1p_1 - m_3p_2]\overline{\varphi'_1},\\
W^{(3)}(\mathbf{U},\varphi'_2) &= (\alpha'_3\nabla\varphi_1 + \alpha'_2\nabla\varphi_2)\cdot\nabla\varphi'_2\\
&\quad + [\gamma'_2\operatorname{div}\mathbf{u} + \zeta'_3\varphi_1 - c_2\varphi_2 - m_3p_1 - m_2p_2]\overline{\varphi'_2},\\
W^{(4)}(\mathbf{U},p'_1) &= (k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p'_1\\
&\quad - (\beta'_1\operatorname{div}\mathbf{u} + m'_1\varphi_1 + m'_3\varphi_2 + a'_1p_1 + a'_3p_2)\overline{p'_1},\\
W^{(5)}(\mathbf{U},p'_2) &= (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p'_2\\
&\quad - (\beta'_2\operatorname{div}\mathbf{u} + m'_3\varphi_1 + m'_2\varphi_2 + a'_3p_1 + a'_2p_2)\overline{p'_2},\\
W(\mathbf{U},\mathbf{U}') &= W^{(1)}(\mathbf{U},\mathbf{u}') + W^{(2)}(\mathbf{U},\varphi'_1) + W^{(3)}(\mathbf{U},\varphi'_2)\\
&\quad + W^{(4)}(\mathbf{U},p'_1) + W^{(5)}(\mathbf{U},p'_2).
\end{aligned}
$$

In our further analysis we need the following matrix differential operators:

$$
\begin{aligned}
\mathbf{A}^{(1)}(\mathbf{D_x}) &= (A^{(1)}_{lr}(\mathbf{D_x}))_{3\times 7}, & A^{(1)}_{lr}(\mathbf{D_x}) &= A_{lr}(\mathbf{D_x}),\\
\mathbf{A}^{(m)}(\mathbf{D_x}) &= (A^{(m)}_{1r}(\mathbf{D_x}))_{1\times 7}, & A^{(m)}_{1r}(\mathbf{D_x}) &= A_{m+2;r}(\mathbf{D_x}),\\
\mathbf{P}^{(1)}(\mathbf{D_x},\mathbf{n}) &= (P^{(1)}_{lr}(\mathbf{D_x},\mathbf{n}))_{3\times 7}, & P^{(1)}_{lr}(\mathbf{D_x},\mathbf{n}) &= P_{lr}(\mathbf{D_x},\mathbf{n}),
\end{aligned}
$$

where $l = 1,2,3$, $m = 2,3,4,5$ and $r = 1,\ldots,7$.

It is not very difficult to prove the following result.

LEMMA 1. *If* $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$ *is a regular vector in* Ω^+, $u'_j,\varphi'_l,p'_l \in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, $j = 1,2,3$, $l = 1,2$, *then*

$$
(3.4)\qquad
\begin{aligned}
&\int_{\Omega^+}[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(1)}(\mathbf{U},\mathbf{u}')]\,d\mathbf{x}\\
&\qquad\qquad = \int_S \mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,\\
&\int_{\Omega^+}[\mathbf{A}^{(2)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\overline{\varphi'_1(\mathbf{x})} + W^{(2)}(\mathbf{U},\varphi'_1)]\,d\mathbf{x}\\
&\qquad\qquad = \int_S \frac{\partial(\alpha'_1\varphi_1 + \alpha'_3\varphi_2)}{\partial\mathbf{n}}\overline{\varphi'_1(\mathbf{z})}\,d_{\mathbf{z}}S,
\end{aligned}
$$

$$\int\limits_{\Omega^+} [\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\overline{\varphi_2'(\mathbf{x})} + W^{(3)}(\mathbf{U},\varphi_2')]\,d\mathbf{x} = \int\limits_S \frac{\partial(\alpha_3'\varphi_1 + \alpha_2'\varphi_2)}{\partial \mathbf{n}}\overline{\varphi_2'(\mathbf{z})}\,d_{\mathbf{z}}S,$$

(3.4)$_{\text{[cont.]}}$

$$\int\limits_{\Omega^+} [\mathbf{A}^{(4)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\overline{p_1'(\mathbf{x})} + W^{(4)}(\mathbf{U},p_1')]\,d\mathbf{x} = \int\limits_S \frac{\partial(k_1p_1 + k_3p_2)}{\partial \mathbf{n}}\overline{p_1'(\mathbf{z})}\,d_{\mathbf{z}}S,$$

$$\int\limits_{\Omega^+} [\mathbf{A}^{(5)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\overline{p_2'(\mathbf{x})} + W^{(5)}(\mathbf{U},p_2')]\,d\mathbf{x} = \int\limits_S \frac{\partial(k_3p_1 + k_2p_2)}{\partial \mathbf{n}}\overline{p_2'(\mathbf{z})}\,d_{\mathbf{z}}S.$$

Combining the relations of (3.4), we can obtain the following.

THEOREM 1. *If* $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$ *is a regular vector in* Ω^+, u_j', $\varphi_l', p_l' \in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, $j=1,2,3$, $l=1,2$, *then*

$$\int\limits_{\Omega^+} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = \int\limits_S \mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{3.5}$$

where $W(\mathbf{U},\mathbf{U}')$ *is defined by* (3.3).

Theorem 1 and the condition (3.1) lead to the following consequence.

THEOREM 2. *If* $\mathbf{U}$ *and* $\mathbf{U}'$ *are seven-component regular vectors in* Ω^-, *then*

$$\int\limits_{\Omega^-} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = -\int\limits_S \mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S. \tag{3.6}$$

The formulas (3.5) and (3.6) are Green's first identities in the coupled linear theory of double-porosity viscoelastic materials for domains Ω^+ and Ω^-, respectively.

4. Uniqueness theorems

In this section the uniqueness theorems for classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{+}_{\mathbf{F},\mathbf{f}}$ are proved. We have the following results.

THEOREM 3. *The internal BVP* $(K)^+_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Proof. We suppose there are two regular solutions of problem $(K)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(K)^+_{\mathbf{0},\mathbf{0}}$. Therefore, $\mathbf{U}$ is a regular solution of the homogeneous system of equations

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^+ \tag{4.1}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}, \qquad \mathbf{z} \in S \tag{4.2}$$

for $K = I$, and

$$\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}, \qquad \mathbf{z} \in S \tag{4.3}$$

for $K = II$.

On the basis of the Eqs. (4.1)–(4.3), from (3.4) we may deduce that

$$\begin{gathered}\int\limits_{\Omega^+} W^{(1)}(\mathbf{U}, \mathbf{u})d\mathbf{x} = 0, \qquad \int\limits_{\Omega^+} W^{(l+1)}(\mathbf{U}, \varphi_l)d\mathbf{x} = 0, \\ \int\limits_{\Omega^+} W^{(l+3)}(\mathbf{U}, p_l)d\mathbf{x} = 0, \quad l = 1, 2.\end{gathered} \tag{4.4}$$

With the help of the relations (3.3) we obtain

$$\begin{aligned}
W^{(0)}(\mathbf{u}, \mathbf{u}) &= \frac{1}{3}(3\lambda' + 2\mu')\,|\mathrm{div}\,\mathbf{u}|^2 \\
&\quad + \frac{\mu'}{2} \sum_{l,j=1;\, l\neq j}^{3} \left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 + \frac{\mu'}{3} \sum_{l,j=1}^{3} \left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2, \\
W^{(1)}(\mathbf{U}, \mathbf{u}) &= W^{(0)}(\mathbf{u}, \mathbf{u}) - \rho\omega^2|\mathbf{u}|^2 + (b'_\alpha\varphi_\alpha - \beta_\alpha p_\alpha)\,\mathrm{div}\,\overline{\mathbf{u}}, \\
W^{(2)}(\mathbf{U}, \varphi_1) &= (\alpha'_1\nabla\varphi_1 + \alpha'_3\nabla\varphi_2)\cdot\nabla\varphi_1 \\
&\quad + [\gamma'_1\,\mathrm{div}\,\mathbf{u} - c_1\varphi_1 + \zeta'_3\varphi_2 - m_1p_1 - m_3p_2]\overline{\varphi_1}, \\
W^{(3)}(\mathbf{U}, \varphi_2) &= (\alpha'_3\nabla\varphi_1 + \alpha'_2\nabla\varphi_2)\cdot\nabla\varphi_2 \\
&\quad + [\gamma'_2\,\mathrm{div}\,\mathbf{u} + \zeta'_3\varphi_1 - c_2\varphi_2 - m_3p_1 - m_2p_2]\overline{\varphi_2}, \\
W^{(4)}(\mathbf{U}, p_1) &= (k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p_1 \\
&\quad - (\beta'_1\,\mathrm{div}\,\mathbf{u} + m'_1\varphi_1 + m'_3\varphi_2 + a'_1p_1 + a'_3p_2)\overline{p_1}, \\
W^{(5)}(\mathbf{U}, p_2) &= (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p_2 \\
&\quad - (\beta'_2\,\mathrm{div}\,\mathbf{u} + m'_3\varphi_1 + m'_2\varphi_2 + a'_3p_1 + a'_2p_2)\overline{p_2}.
\end{aligned} \tag{4.5}$$

In view of the relations of (2.9) we can easily verify that

$$
\begin{aligned}
&-\frac{1}{\omega}\mathrm{Im}\,W^{(0)}(\mathbf{u},\mathbf{u})=\frac{1}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2\\
&\qquad+\frac{\mu^*}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2+\frac{\mu^*}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2,\\
&-\frac{1}{\omega}\,\mathrm{Im}\,W^{(1)}(\mathbf{U},\mathbf{u})\\
&\quad=-\frac{1}{\omega}\,\mathrm{Im}\,W^{(0)}(\mathbf{u},\mathbf{u})+\frac{1}{\omega}b_\alpha\,\mathrm{Im}(\mathrm{div}\,\mathbf{u}\overline{\varphi_\alpha})+b_\alpha^*\,\mathrm{Re}(\mathrm{div}\,\mathbf{u}\overline{\varphi_\alpha})\\
&\qquad-\frac{1}{\omega}\beta_\alpha\,\mathrm{Im}(\mathrm{div}\,\mathbf{u}\overline{p_\alpha}),-\frac{1}{\omega}\,\mathrm{Im}[W^{(2)}(\mathbf{U},\varphi_1)+W^{(3)}(\mathbf{U},\varphi_2)]\\
&\quad=\alpha_1^*|\nabla\varphi_1|^2+2\alpha_3^*\,\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2)+\alpha_2^*|\nabla\varphi_2|^2\\
&\qquad+\gamma_\alpha^*\,\mathrm{Re}(\mathrm{div}\,\mathbf{u}\overline{\varphi_\alpha})-\frac{1}{\omega}b_\alpha\,\mathrm{Im}(\mathrm{div}\,\mathbf{u}\overline{\varphi_\alpha})+\zeta_1^*|\varphi_1|^2\\
&\qquad+2\zeta_3^*\,\mathrm{Re}(\varphi_1\overline{\varphi_2})+\zeta_2^*|\varphi_2|^2\\
&\qquad-\frac{1}{\omega}[m_1\,\mathrm{Im}(\varphi_1\overline{p_1})+m_3\,\mathrm{Im}(\varphi_1\overline{p_2}+\varphi_2\overline{p_1})+m_2\,\mathrm{Im}(\varphi_2\overline{p_2})],\\
&\mathrm{Re}[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)]\\
&\quad=k_1|\nabla p_1|^2+2k_3\,\mathrm{Re}(\nabla p_1\cdot\nabla p_2)+k_2|\nabla p_2|^2+\omega\beta_\alpha\,\mathrm{Im}(\mathrm{div}\,\mathbf{u}\overline{p_\alpha})\\
&\qquad+\omega[m_1\,\mathrm{Im}(\varphi_1\overline{p_1})+m_3\,\mathrm{Im}(\varphi_1\overline{p_2}+\varphi_2\overline{p_1})+m_2\,\mathrm{Im}(\varphi_2\overline{p_2})]+\gamma_0|p_1-p_2|^2.
\end{aligned}
$$

Combining these relations, by virtue of inequalities (3.2) we can obtain

$$
\begin{aligned}
&-\frac{1}{\omega}\big[\mathrm{Im}\,W^{(1)}(\mathbf{U},\mathbf{u})+W^{(2)}(\mathbf{U},\varphi_1)+W^{(3)}(\mathbf{U},\varphi_2)\big]\\
&\qquad\qquad\qquad+\frac{1}{\omega^2}\,\mathrm{Re}\big[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)\big]\\
&\quad=\frac{1}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2+(b_\alpha^*+\gamma_\alpha^*)\,\mathrm{Re}(\mathrm{div}\,\mathbf{u}\overline{\varphi_\alpha})\\
&\qquad+\zeta_1^*|\varphi_1|^2+2\zeta_3^*\,\mathrm{Re}(\varphi_1\overline{\varphi_2})+\zeta_2^*|\varphi_2|^2\\
&\qquad+\frac{\gamma_0}{\omega^2}|p_1-p_2|^2+\alpha_1^*|\nabla\varphi_1|^2+2\alpha_3^*\,\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2)+\alpha_2^*|\nabla\varphi_2|^2\\
&\qquad+\frac{1}{\omega^2}\big[k_1|\nabla p_1|^2+2k_3\,\mathrm{Re}(\nabla p_1\cdot\nabla p_2)+k_2|\nabla p_2|^2\big]\\
&\qquad+\frac{\mu^*}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2+\frac{\mu^*}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2\geq 0
\end{aligned}
$$

and from (4.4) it follows that

$$
\begin{aligned}
&\frac{1}{3}(3\lambda^* + 2\mu^*)\,|\operatorname{div}\mathbf{u}|^2 + (b_\alpha^* + \gamma_\alpha^*)\operatorname{Re}(\operatorname{div}\mathbf{u}\overline{\varphi_\alpha}) \\
&\qquad\qquad + \zeta_1^*|\varphi_1|^2 + 2\zeta_3^*\operatorname{Re}(\varphi_1\overline{\varphi_2}) + \zeta_2^*|\varphi_2|^2 = 0, \\
&\alpha_1^*|\nabla\varphi_1|^2 + 2\alpha_3^*\operatorname{Re}(\nabla\varphi_1\cdot\nabla\varphi_2) + \alpha_2^*|\nabla\varphi_2|^2 = 0, \\
&k_1|\nabla p_1|^2 + 2k_3\operatorname{Re}(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2 = 0, \\
&\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 = 0, \qquad \sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2 = 0.
\end{aligned}
\tag{4.6}
$$

Then, using again the inequalities (3.2) from (4.6) we get the following

$$
\begin{aligned}
&\operatorname{div}\mathbf{u}(\mathbf{x}) = 0, \qquad \frac{\partial u_j(\mathbf{x})}{\partial x_l} + \frac{\partial u_l(\mathbf{x})}{\partial x_j} = 0, \qquad \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j(\mathbf{x})}{\partial x_j} = 0, \\
&\nabla p_1(\mathbf{x}) = \nabla p_2(\mathbf{x}) = \mathbf{0}, \qquad l,j = 1,2,3
\end{aligned}
\tag{4.7}
$$

and

$$
\varphi_1(\mathbf{x}) = \varphi_2(\mathbf{x}) = 0 \tag{4.8}
$$

for $\mathbf{x}\in\Omega^+$.

On the other hand, based on the Eqs. (4.5) and (4.7) we have the following relation

$$
W^{(1)}(\mathbf{U},\mathbf{u}) = \rho\omega^2|\mathbf{u}(\mathbf{x})|^2 = 0
$$

and we obtain

$$
\mathbf{u}(\mathbf{x}) = \mathbf{0} \qquad \text{for } \mathbf{x}\in\Omega^+. \tag{4.9}
$$

Quite similarly, from (4.5) and (4.7) it follows that

$$
W^{(4)}(\mathbf{U},p_1) + W^{(5)}(\mathbf{U},p_2) = i\omega\big[a_1|p_1|^2 + 2a_3\operatorname{Re}(p_1\overline{p_2}) + a_2|p_2|^2\big] = 0
$$

and we get

$$
p_1(\mathbf{x}) = p_2(\mathbf{x}) = 0 \qquad \text{for } \mathbf{x}\in\Omega^+. \tag{4.10}
$$

Finally, upon combining the relations (4.8)–(4.10) we obtain the desired result $\mathbf{U}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x}\in\Omega^+$. ■

Using the condition (3.1) the next theorem can be proved similarly.

THEOREM 4. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Hence, each of the basic BVPs $(I)^\pm_{\mathbf{F},\mathbf{f}}$ and $(II)^\pm_{\mathbf{F},\mathbf{f}}$ in the class of regular vectors has the unique classical solution.

5. Fundamental solution

In this section the fundamental solution of the system (2.8) is constructed and its basic properties are established.

Let us now suppose that $\mathbf{B}(\Delta) = (B_{lj}(\Delta)_{5\times 5}$ is the following matrix

$$\mathbf{B}(\Delta) = \begin{pmatrix} \mu_0'\Delta + \rho\omega^2 & -\gamma_1'\Delta & -\gamma_2'\Delta & \beta_1'\Delta & \beta_2'\Delta \\ b_1' & \alpha_1'\Delta + c_1 & \alpha_3'\Delta - \zeta_3' & m_1' & m_3' \\ b_2' & \alpha_3'\Delta - \zeta_3' & \alpha_2'\Delta + c_2 & m_3' & m_2' \\ -\beta_1 & m_1 & m_3 & k_1\Delta + a_1' & k_3\Delta + a_3' \\ -\beta_2 & m_3 & m_2 & k_3\Delta + a_3' & k_2\Delta + a_2' \end{pmatrix}_{5\times 5},$$

where $\mu_0' = \lambda' + 2\mu'$. Let $\lambda_1^2, \lambda_2^2, \dots, \lambda_5^2$ be the roots of algebraic equation $\Lambda_1(-\xi) = 0$ (with respect to ξ), where

$$\Lambda_1(\Delta) = \frac{1}{\mu_0'\alpha_0' k_0} \det \mathbf{B}(\Delta) = \prod_{j=1}^{5} (\Delta + \lambda_j^2),$$

$\alpha_0' = \alpha_1'\alpha_2' - (\alpha_3')^2$, $k_0 = k_1 k_2 - k_3^2$. We assume that the values $\lambda_1^2, \lambda_2^2, \dots, \lambda_6^2$ are distinct and $\operatorname{Im} \lambda_j > 0$ $(j = 1, \dots, 6)$, where $\lambda_6^2 = \rho\omega^2/\mu'$.

Let us introduce the following notation:

(i)

$$\begin{aligned} &\mathbf{M}(\mathbf{D_x}) = (M_{lj}(\mathbf{D_x}))_{7\times 7}, \qquad M_{lj}(\mathbf{D_x}) = \frac{1}{\mu'}\Lambda_1(\Delta)\,\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j}, \\ &M_{l;r+2}(\mathbf{D_x}) = n_{1r}(\Delta)\frac{\partial}{\partial x_l}, \qquad M_{r+2;l}(\mathbf{D_x}) = n_{r1}(\Delta)\frac{\partial}{\partial x_l}, \\ &M_{r+2;s+2}(\mathbf{D_x}) = n_{rs}(\Delta), \qquad l, j = 1, 2, 3, \; r, s = 2, 3, 4, 5, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} n_{l1}(\Delta) &= -\frac{1}{\mu'\mu_0'\alpha_0' k_0}\left[(\lambda' + \mu')B_{l1}^*(\Delta) - \gamma_\alpha' B_{l;\alpha+1}^*(\Delta) + \beta_\alpha' B_{l;\alpha+3}^*(\Delta)\right], \\ n_{lj}(\Delta) &= \frac{1}{\mu_0'\alpha_0' k_0} B_{lj}^*(\Delta), \qquad l = 1, \dots, 5, \qquad j = 2, 3, 4, 5 \end{aligned}$$

and B_{lj}^* is the cofactor of the element B_{lj} of matrix $\mathbf{B}$.

(ii)

$$\begin{aligned} &\mathbf{\Psi}(\mathbf{x}) = (\Psi_{lr}(\mathbf{x}))_{7\times 7}, \qquad \Psi_{11}(\mathbf{x}) = \Psi_{22}(\mathbf{x}) = \Psi_{33}(\mathbf{x}) = \sum_{j=1}^{6} \eta_{2j}\gamma^{(j)}(\mathbf{x}), \\ &\Psi_{44}(\mathbf{x}) = \Psi_{55}(\mathbf{x}) = \Psi_{66}(\mathbf{x}) = \Psi_{77}(\mathbf{x}) = \sum_{j=1}^{5} \eta_{1j}\gamma^{(j)}(\mathbf{x}), \\ &\Psi_{lr}(\mathbf{x}) = 0, \qquad l \neq r, \qquad l, r = 1, \dots, 7, \end{aligned} \tag{5.2}$$

where

$$\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{5.3}$$

and

$$\eta_{1r} = \prod_{l=1,\, l\neq r}^{5} (\lambda_l^2 - \lambda_r^2)^{-1}, \qquad \eta_{2j} = \prod_{l=1,\, l\neq j}^{6} (\lambda_l^2 - \lambda_j^2)^{-1},$$
$$r = 1, \dots, 5, \; j = 1, \dots, 6.$$

It is now easy to prove the following equality

$$\mathbf{A}(\mathbf{D_x})\mathbf{M}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta), \tag{5.4}$$

where

$$\begin{aligned}
&\mathbf{\Lambda}(\Delta) = (\Lambda_{lj}(\Delta))_{7\times 7},\\
&\Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_2(\Delta),\\
&\Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \Lambda_{66}(\Delta) = \Lambda_{77}(\Delta) = \Lambda_1(\Delta),\\
&\Lambda_{lj}(\Delta) = 0, \quad\; l \neq j, \; l, j = 1, \dots, 7.
\end{aligned}$$

Furthermore, by direct calculation we may see that $\mathbf{\Psi}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$, i.e.,

$$\mathbf{\Lambda}(\Delta)\mathbf{\Psi}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{5.5}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{7\times 7}$ is the unit matrix and $\mathbf{x} \in \mathbb{R}^3$.

Let us introduce the matrix $\mathbf{\Gamma}(\mathbf{x})$ by

$$\mathbf{\Gamma}(\mathbf{x}) = \mathbf{M}(\mathbf{D_x})\mathbf{\Psi}(\mathbf{x}), \tag{5.6}$$

where $\mathbf{M}(\mathbf{D_x})$ and $\mathbf{\Psi}(\mathbf{x})$ are defined by (5.1) and (5.2) respectively. In view of the relations (5.4) and (5.5), we can write

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Hence, the following theorem is valid.

Theorem 5. *The matrix* $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{7\times 7}$ *defined by* (5.6) *is the fundamental solution of system* (2.8).

Remark 2. Thus, the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed explicitly by means of six elementary functions $\gamma^{(j)}$ $(j = 1, \dots, 6)$ (see (5.3)).

We introduce the notation:

(i)

$$\hat{\mathbf{A}}(\mathbf{D_x}) = \Big(\hat{A}_{lj}(\mathbf{D_x})\Big)_{7\times 7}, \qquad \hat{A}_{lj}(\mathbf{D_x}) = \mu'\Delta\delta_{lj} + (\lambda' + \mu')\frac{\partial^2}{\partial x_l \partial x_j},$$

$$\hat{A}_{44}(\mathbf{D_x}) = \alpha_1'\Delta, \qquad \hat{A}_{45}(\mathbf{D_x}) = \hat{A}_{45}(\mathbf{D_x}) = \alpha_3'\Delta, \qquad \hat{A}_{55}(\mathbf{D_x}) = \alpha_2'\Delta,$$

$$\hat{A}_{66}(\mathbf{D_x}) = k_1\Delta, \qquad \hat{A}_{67}(\mathbf{D_x}) = \hat{A}_{76}(\mathbf{D_x}) = k_3\Delta, \qquad \hat{A}_{77}(\mathbf{D_x}) = k_2\Delta,$$

$$\hat{A}_{ls}(\mathbf{D_x}) = \hat{A}_{sl}(\mathbf{D_x}) = \hat{A}_{m;r+2}(\mathbf{D_x}) = \hat{A}_{m+2;r}(\mathbf{D_x}) = 0.$$

(ii)

$$\mathbf{\Gamma}^{(0)}(\mathbf{x}) = \big(\Gamma_{lj}^{(0)}(\mathbf{x})\big)_{7\times 7}, \qquad \Gamma_{lj}^{(0)}(\mathbf{x}) = -\frac{\lambda' + 3\mu'}{8\pi\mu_0'\mu'}\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{\lambda' + \mu'}{8\pi\mu_0'\mu'}\frac{x_l x_j}{|\mathbf{x}|^3},$$

$$\Gamma_{44}^{(0)}(\mathbf{x}) = \frac{\alpha_2'}{\alpha_0'}\gamma^{(0)}(\mathbf{x}), \qquad \Gamma_{45}^{(0)}(\mathbf{x}) = \Gamma_{54}^{(0)}(\mathbf{x}) = -\frac{\alpha_3'}{\alpha_0'}\gamma^{(0)}(\mathbf{x}),$$

$$\Gamma_{55}^{(0)}(\mathbf{x}) = \frac{\alpha_1'}{\alpha_0'}\gamma^{(0)}(\mathbf{x}), \qquad \Gamma_{66}^{(0)}(\mathbf{x}) = \frac{k_2}{k_0}\gamma^{(0)}(\mathbf{x}),$$

$$\Gamma_{67}^{(0)}(\mathbf{x}) = \Gamma_{76}^{(0)}(\mathbf{x}) = -\frac{k_3}{k_0}\gamma^{(0)}(\mathbf{x}), \qquad \Gamma_{77}^{(0)}(\mathbf{x}) = \frac{k_1}{k_0}\gamma^{(0)}(\mathbf{x}),$$

$$\Gamma_{ls}^{(0)}(\mathbf{x}) = \Gamma_{sl}^{(0)}(\mathbf{x}) = \Gamma_{m;r+2}^{(0)}(\mathbf{x}) = \Gamma_{m+2;r}^{(0)}(\mathbf{x}) = 0, \qquad \gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|},$$

where $l, j = 1, 2, 3$, $m, r = 4, 5$ and $s = 4, 5, 6, 7$.

Theorem 5 leads to the following basic properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$.

THEOREM 6.

(i) *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of homogeneous equation* $\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{0}$ *at every point* $\mathbf{x} \in \mathbb{R}^3$ *except the origin of* $\mathbb{R}^3$.

(ii) *The relations*

$$\Gamma_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{mr}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{m+2;r+2}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),$$

$$\Gamma_{ls}(\mathbf{x}) = O(1), \qquad \Gamma_{sl}(\mathbf{x}) = O(1), \qquad \Gamma_{m;r+2}(\mathbf{x}) = O(1),$$

$$\Gamma_{m+2;r}(\mathbf{x}) = O(1), \qquad l, j = 1, 2, 3, \ m, r = 4, 5, \ s = 4, 5, 6, 7$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

(iii) *The matrix* $\mathbf{\Gamma}^{(0)}(\mathbf{x})$ *is the fundamental solution of differential operator* $\hat{\mathbf{A}}(\mathbf{D_x})$ *and the following relations*

$$\Gamma_{lj}^{(0)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \Gamma_{mr}^{(0)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}),$$

$$\Gamma_{m+2;r+2}^{(0)}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad l, j = 1, 2, 3, \ m, r = 4, 5$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

(iv) *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Gamma^{(0)}_{lj}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \qquad l,j = 1,\ldots,7 \tag{5.7}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

Thus, on the basis of Theorem 6, the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin of $\mathbb{R}^3$.

6. Green's second and third identities

Now we introduce the following matrix differential operators:

(i) $\tilde{\mathbf{A}}(\mathbf{D}_\mathbf{x}) = \mathbf{A}^\top(-\mathbf{D}_\mathbf{x})$, where the superscript $\top$ denotes transposition and

(ii)

$$\begin{aligned}
&\tilde{\mathbf{P}}(\mathbf{D}_\mathbf{x},\mathbf{n}) = (\tilde{P}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}))_{7\times 7}, \qquad \tilde{P}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}) = P_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}),\\
&\tilde{P}_{l;m+3}(\mathbf{D}_\mathbf{x},\mathbf{n}) = \gamma'_m n_l, \qquad \tilde{P}_{l;m+5}(\mathbf{D}_\mathbf{x},\mathbf{n}) = -\beta'_m n_l,\\
&\tilde{P}_{sr}(\mathbf{D}_\mathbf{x},\mathbf{n}) = P_{sr}(\mathbf{D}_\mathbf{x},\mathbf{n}),\\
&\qquad\qquad l,j = 1,2,3,\;\; r = 1,\ldots,7,\;\; m = 1,2,\;\; s = 4,5,6,7.
\end{aligned} \tag{6.1}$$

Let $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$, the vector $\tilde{\mathbf{U}}_j$ is the j-th column of the matrix $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{7\times 7}$. By direct calculation we get the following results.

THEOREM 7. *If* $\mathbf{U}$ *and* $\tilde{\mathbf{U}}_j$ $(j = 1,\ldots,7)$ *are regular vectors in* $\Omega^\pm$, *then*

$$\begin{aligned}
&\int\limits_{\Omega^\pm} \left\{[\tilde{\mathbf{A}}(\mathbf{D}_\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D}_\mathbf{y})\mathbf{U}(\mathbf{y})\right\} d\mathbf{y}\\
&\qquad = \pm \int\limits_S \left\{[\tilde{\mathbf{P}}(\mathbf{D}_\mathbf{z},\mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{P}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_\mathbf{z}S.
\end{aligned} \tag{6.2}$$

The formula (6.2) is Green's second identity in the considered theory for domain $\Omega^\pm$.

Let $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ is the fundamental matrix of operator $\tilde{\mathbf{A}}(\mathbf{D}_\mathbf{x})$. Clearly, the matrix $\tilde{\mathbf{\Gamma}}(\mathbf{x})$ satisfies the following condition

$$\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{\Gamma}^\top(-\mathbf{x}), \tag{6.3}$$

where $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental solution of the system (2.9). Pick $\varepsilon > 0$ such that $\overline{\mathcal{B}(\mathbf{x},\varepsilon)} \subset \Omega^+$, where $\mathcal{B}(\mathbf{x},\varepsilon)$ is the (open) ball of radius ε and center $\mathbf{x}$ and $\mathbf{x} \in \Omega^+$. Applying the formula (6.2) in the domain $\Omega^+ \backslash \overline{\mathcal{B}(\mathbf{x},\varepsilon)}$ with $\tilde{\mathbf{U}}(\mathbf{y}) = \tilde{\mathbf{\Gamma}}(\mathbf{y}-\mathbf{x})$ and letting $\varepsilon \to 0$. Then, using Eq. (6.3) we get the following results.

THEOREM 8. *If* $\mathbf{U}$ *is a regular vector in* Ω^+, *then*

$$\mathbf{U}(\mathbf{x}) = \int\limits_S \big\{[\tilde{\mathbf{P}}(\mathbf{D_z},\mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{z})]^\top\mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\big\}\, d_{\mathbf{z}}S + \int\limits_{\Omega^+} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\, d\mathbf{y} \quad \textit{for } \mathbf{x} \in \Omega^+. \tag{6.4}$$

THEOREM 9. *If* $\mathbf{U}$ *is a regular vector in* Ω^-, *then*

$$\mathbf{U}(\mathbf{x}) = -\int\limits_S \big\{[\tilde{\mathbf{P}}(\mathbf{D_z},\mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{z})]^\top\mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\big\}\, d_{\mathbf{z}}S + \int\limits_{\Omega^-} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})\, d\mathbf{y} \quad \textit{for } \mathbf{x} \in \Omega^-. \tag{6.5}$$

The formulas (6.4) and (6.5) are Green's third identities (integral representations of a regular vector) in the coupled linear theory of double-porosity viscoelastic materials for domains Ω^+ and Ω^-, respectively.

7. Potentials and singular integral operators

We introduce the following notation:

(i) $\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}) = \int_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ is the single-layer potential,

(ii) $\mathbf{Z}^{(2)}(\mathbf{x},\mathbf{g}) = \int_S [\widetilde{\mathbf{P}}(\mathbf{D_y},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top\mathbf{g}(\mathbf{y})\, d_{\mathbf{y}}S$ is the double-layer potential,

(iii) $\mathbf{Z}^{(3)}(\mathbf{x},\mathbf{h},\Omega^\pm) = \int_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})\, d\mathbf{y}$ is the volume potential, where $\mathbf{g}$ and $\mathbf{h}$ are seven-component vector functions, $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ and defined by (5.6) and $\widetilde{\mathbf{P}}(\mathbf{D_y},\mathbf{n}(\mathbf{y}))$ is given by (6.1).

In the next four theorems, by virtue of the results of Theorems 5 and 6 the basic properties of these potentials are established.

THEOREM 10. *If* $S \in C^{2,p}$, $\mathbf{g} \in C^{1,p'}(S)$, $0 < p' < p \le 1$, *then:*

(a) $\mathbf{Z}^{(1)}(\cdot,\mathbf{g}) \in C^{0,p'}(\mathbb{R}^3) \cap C^{2,p'}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm)$;

(b) $\mathbf{A}(\mathbf{D_x})\,\mathbf{Z}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0}$;

(c) $\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})$ *is a singular integral;*

(d)

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Z}^{(1)}(\mathbf{z},\mathbf{g}), \tag{7.1}$$

where $\mathbf{x} \in \Omega^\pm$ *and* $\mathbf{z} \in S$.

THEOREM 11. *If* $S \in C^{2,p}$, $\mathbf{g} \in C^{1,p'}(S)$, $0 < p' < p \leq 1$, *then:*
(a) $\mathbf{Z}^{(2)}(\cdot, \mathbf{g}) \in C^{1,p'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm})$,
(b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}$,
(c) $\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})$ *is a singular integral,*
(d)

$$\{\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \pm\tfrac{1}{2}\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}), \tag{7.2}$$

(e) $\{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{+} = \{\mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})\}^{-}$, *where* $\mathbf{x} \in \Omega^{\pm}$ *and* $\mathbf{z} \in S$.

THEOREM 12. *If* $S \in C^{1,p}$, $\mathbf{h} \in C^{0,p'}(\Omega^{+})$, $0 < p' < p \leq 1$, *then:*
(a) $\mathbf{Z}^{(3)}(\cdot, \mathbf{h}, \Omega^{+}) \in C^{1,p'}(\mathbb{R}^3) \cap C^{2}(\Omega^{+}) \cap C^{2,p'}(\overline{\Omega_0^{+}})$,
(b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{h}, \Omega^{+}) = \mathbf{h}(\mathbf{x})$, *where* $\mathbf{x} \in \Omega^{+}$, Ω_0^{+} *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{+}} \subset \Omega^{+}$.

THEOREM 13. *If* $S \in C^{1,p}$, $\operatorname{supp} \mathbf{h} = \Omega \subset \Omega^{-}$, $\mathbf{h} \in C^{0,p'}(\Omega^{-})$, $0 < p' < p \leq 1$, *then:*
(a) $\mathbf{Z}^{(3)}(\cdot, \mathbf{h}, \Omega^{-}) \in C^{1,p'}(\mathbb{R}^3) \cap C^{2}(\Omega^{-}) \cap C^{2,p'}(\overline{\Omega_0^{-}})$,
(b) $\mathbf{A}(\mathbf{D_x})\, \mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{h}, \Omega^{-}) = \mathbf{h}(\mathbf{x})$, *where* $\mathbf{x} \in \Omega^{-}$, Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{-}} \subset \Omega^{-}$.

Let us introduce the integral operators by

$$\begin{aligned}
\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}^{(4)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{Z}^{(1)}(\mathbf{z}, \mathbf{g}),\\
\mathcal{K}_{\varsigma}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \varsigma\,\mathbf{Z}^{(2)}(\mathbf{z}, \mathbf{g})
\end{aligned} \tag{7.3}$$

for $\mathbf{z} \in S$, where ς is a complex parameter. Taking into account Theorems 10 and 11, $\mathcal{K}^{(j)}$ $(j = 1, 2, 3, 4)$ and $\mathcal{K}_{\varsigma}$ are singular integral operators.

Let $\boldsymbol{\sigma}^{(j)} = (\sigma_{lm}^{(j)})_{7\times 7}$ be the symbol (symbolic matrix) of the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1, 2, 3, 4)$. On the basis of Theorem 6, the relations (3.2), (5.7) and (7.3) for $\det \boldsymbol{\sigma}^{(j)}$ we obtain

$$\det \boldsymbol{\sigma}^{(1)} = -\det \boldsymbol{\sigma}^{(2)} = -\det \boldsymbol{\sigma}^{(3)} = \det \boldsymbol{\sigma}^{(4)} = -\frac{(\lambda' + \mu')(\lambda' + 3\mu')}{128(\lambda' + 2\mu')^2}. \tag{7.4}$$

In view of the inequalities of (3.2) from (7.4) we get $\det \boldsymbol{\sigma}^{(j)} \neq 0$ which proves that the singular integral operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1, 2, 3, 4$.

Furthermore, let $\boldsymbol{\sigma}_\varsigma$ and $\operatorname{ind}\mathcal{K}_\varsigma$ be the symbol and the index of the operator $\mathcal{K}_\varsigma$, respectively. It may be easily shown that

$$\det \boldsymbol{\sigma}_\varsigma = -\frac{(\lambda' + \varsigma\mu')(\lambda' + 3\varsigma\mu')}{128(\lambda' + 2\mu')^2}$$

and $\det \boldsymbol{\sigma}_\varsigma$ vanishes only at two points

$$\varsigma_1 = -\frac{\lambda'}{\mu'}, \qquad \varsigma_2 = -\frac{\lambda'}{3\mu'}$$

of the complex plane. By virtue of (7.4) and $\det \boldsymbol{\sigma}_1 = \det \boldsymbol{\sigma}^{(1)}$ we get $\varsigma_l \neq 1$ $(l = 1, 2)$ and we obtain

$$\operatorname{ind}\mathcal{K}^{(1)} = \operatorname{ind}\mathcal{K}_1 = 0.$$

The relation $\operatorname{ind}\mathcal{K}^{(2)} = 0$ is proved in a quite similar manner. Clearly, the operators $\mathcal{K}^{(3)}$ and $\mathcal{K}^{(4)}$ are the adjoint operators for $\mathcal{K}^{(2)}$ and $\mathcal{K}^{(1)}$, respectively. Hence,

$$\operatorname{ind}\mathcal{K}^{(3)} = -\operatorname{ind}\mathcal{K}^{(2)} = 0, \qquad \operatorname{ind}\mathcal{K}^{(4)} = -\operatorname{ind}\mathcal{K}^{(1)} = 0.$$

Thus, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1, 2, 3, 4)$ is of the normal type with an index equal to zero and consequently, Fredholm's theorems are valid for $\mathcal{K}^{(j)}$.

REMARK 3. The definitions of a normal type singular integral operator, the symbol and the index of this operator are given in the book by KUPRADZE *et al.* [51].

8. Existence theorems

We are now in a position to prove the existence theorems for regular (classical) solutions of the BVPs of steady vibrations in the coupled linear theory of double-porosity viscoelastic materials using the potential method.

It is noteworthy that the volume potential $\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^\pm)$ is a partial regular solution of nonhomogeneous Eq. (2.10) (see Theorems 12 and 13), where $\mathbf{F} \in C^{0,p'}(\Omega^\pm)$, $0 < p' \leq 1$ and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. Therefore, further we consider the problem $(K)^\pm_{\mathbf{0},\mathbf{f}}$ for $K = I, II$.

PROBLEM $(I)^+_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to problem $(I)^+_{\mathbf{0},\mathbf{f}}$ in the form of double-layer potential

$$\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \text{for } \mathbf{x} \in \Omega^+, \tag{8.1}$$

where $\mathbf{g}$ is the required seven-component vector function. By Theorem 11 the vector function $\mathbf{V}$ is a solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{V}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \tag{8.2}$$

for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition and the identity (7.2), from (8.1) we have, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \mathbf{z} \in S \tag{8.3}$$

for which the Fredholm's theorems are valid. We prove that (8.3) is always solvable for an arbitrary vector $\mathbf{f}$. Let us consider the adjoint homogeneous equation

$$\mathcal{K}^{(4)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S, \tag{8.4}$$

where $\mathbf{h}_0$ is the required seven-component vector function. It is sufficient to show that the integral equation (8.4) has only the trivial solution.

Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (8.4). On the basis of Theorem 10 and Eq. (8.4) the vector function $\mathbf{R}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{0}}$. In view of Theorem 4, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{R}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^-. \tag{8.5}$$

Furthermore, by using Theorem 10 and (8.5) we can write

$$\{\mathbf{R}(\mathbf{z})\}^+ = \{\mathbf{R}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for } \mathbf{z} \in S,$$

i.e., the vector $\mathbf{R}(\mathbf{x})$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Now in view of Theorem 3, the problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is

$$\mathbf{R}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+. \tag{8.6}$$

By virtue of (8.5), (8.6) and the identity (7.1) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{R}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{R}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (8.4) has only the trivial solution and therefore (8.3) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following theorem.

THEOREM 14. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of problem* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by double-layer potential* (8.2), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (8.3) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

PROBLEM $(II)^-_{\mathbf{0},\mathbf{f}}$. Now we are looking for a regular solution to problem $(II)^-_{\mathbf{0},\mathbf{f}}$ in the form of single-layer potential

$$\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for } \mathbf{x} \in \Omega^-, \tag{8.7}$$

where $\mathbf{h}$ is the required seven-component vector function. Clearly, by Theorem 10 the vector function $\mathbf{V}$ is a solution of (8.2) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition and using (7.1), from (8.7) we obtain, for determining the unknown vector $\mathbf{h}$, a singular integral equation

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \tag{8.8}$$

It has been proved above that the corresponding homogeneous equation (8.4) has only the trivial solution. Hence, on the basis of Fredholm's theorems it follows that (8.8) is always solvable. We have thereby proved the following consequence.

THEOREM 15. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of problem* $(II)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by single-layer potential* (8.7), *where* $\mathbf{h}$ *is a solution of the singular integral equation* (8.8) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Quite similarly, we can prove the following results.

THEOREM 16. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of problem* $(II)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by single-layer potential*

$$\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) \quad \text{for } \mathbf{x} \in \Omega^+,$$

where $\mathbf{g}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

THEOREM 17. *If* $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \le 1$, *then a regular solution of problem* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by double-layer potential*

$$\mathbf{V}(\mathbf{x}) = \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{h}) \quad \text{for } \mathbf{x} \in \Omega^-,$$

where $\mathbf{h}$ *is a solution of the singular integral equation*

$$\mathcal{K}^{(3)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

9. Concluding remarks

1. In this paper the coupled linear theory of double-porosity viscoelastic materials is considered and the basic BVPs of steady vibrations of this theory are investigated by using the potential method. Indeed, the following results are obtained:

(i) The basic systems of equations of motion and steady vibrations are established.

(ii) Green's identities are obtained for bounded and unbounded domains.

(iii) Then, the uniqueness theorems for classical solutions of the basic internal and external BVPs of steady vibrations are proved.

(iv) The fundamental solution of the system of steady vibration equations is constructed explicitly by elementary functions and its basic properties are established.

(v) The surface (single-layer and double-layer) and volume potentials are introduced and their properties are given.

(vi) Some useful singular integral operators are studied for which Fredholm's theorems are valid.

(vii) Finally, the existence theorems for classical solutions of the above mentioned basic BVPs are proved by using the potential method and the theory of singular integral equations.

2. On the basis of results of this paper are possible:

(i) To present the basic equations of the coupled linear theory of thermoviscoelasticity for materials with double porosity.

(ii) To investigate the nonclassical BVPs of steady vibrations of this theory by using the potential method.

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