Amplitude demodulation of interferometric signals with a 2D Hilbert transform

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The Hilbert transform and the analytic signal are widely known tools of 1D signal processing. They are useful for many applications, such as AM-FM demodulation or edge detection. Developing the multidimensional generalization of this method is particularly important for the purposes of image processing. Unfortunately, it is not obvious how to generalize the transform, keeping its essential properties. In this paper I survey some ideas: basic approaches with spectral masks imitating 1D signum function, the spiral phase operator method and the method involving quaternionic Fourier transform. I present and compare how these algorithms are useful for the amplitude demodulation of typical interferometric images.

Keywords and phrases: Hilbert transform, image analysis, Fourier transform, demodulation.

2000 Mathematics Subject Classification: Primary 94A14; Secondary 42B10, 30G35, 65T50.

Introduction

The Hilbert transform in one dimension is given by

$$H\{s(x)\} = s_H(x) = s(x) * \frac{1}{\pi x} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s(u)}{x-u}\, du \quad (1)$$

Convolution is denoted by $*$ and integral exists in a sense of Cauchy principal value. In the frequency domain this corresponds to

$$\begin{aligned} S_H(\omega) &= \mathbb{F}\{s_H(x)\} = -i\,\mathrm{sign}(\omega)\mathbb{F}\{s(x)\} = \\ &= -i\,\mathrm{sign}(\omega) S(\omega) \end{aligned} \quad (2)$$

The analytic signal (AS), introduced by Gabor in 1946 [1], is defined as

$$\begin{aligned} s_A(x) &= s(x) + i\, s_H(x) = \left[\delta(x) + \frac{1}{\pi x}\right] * s(x) = \\ &= A(x)\exp[i\Psi(x)] \end{aligned} \quad (3)$$

with the real functions $A(x)$ and $\Psi(x)$. Therefore

$$S_A(\omega) = \mathbb{F}\{s_A(x)\} = [1 + \mathrm{sign}(\omega)]\, S(\omega) \quad (4)$$

AS possesses several interesting properties:

- $SA(\omega) = 0$ for $\omega < 0$ and $SA(\omega) = 2S(\omega)$ for $\omega > 0$;
- the original real signal $s(t)$ is a real part of the corresponding analytic signal $sA(t)$;
- the envelope (*instantaneous amplitude*) of $s(t)$ is an absolute value of $sA(t)$. The third property is particularly important for the purposes of amplitude demodulation.

Typical problem of interferometry is to analyse signals in a form of

$$I(x, y) = a(x, y) + b(x, y)\cos[\psi(x, y)] \quad (5)$$

to extract the phase $\psi(x, y)$ or the local modulation $b(x, y)$, which encode measured physical quantity. Interesting example is given by the time-average interferometry [2], where the modulation contains information about a vibrating object amplitude. In this paper I obtain $b(x, y)$ from synthetic interferometric images by the means of different generalizations of the Hilbert transform, to compare their errors. For comparison, I also present results given by the 1D Hilbert transform (denoted as H1D), scanning image line by line [3].

Simple masks in the frequency domain

Basic approach to a 2D Hilbert transform is to simply extend the domain of the convolution

$$H\{s(x,y)\} = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\frac{s(u,v)}{(x-u)(y-v)}\,dudv = \\ = \frac{1}{\pi^2 xy} * s(x,y) \tag{6}$$

which leads to a following definition of the 2D analytic signal (in the frequency domain)

$$S_A(\zeta,\eta) = [1 - i\ \mathrm{sign}(\zeta)\mathrm{sign}(\eta)]\,S(\zeta,\eta) \tag{7}$$

Obviously so defined analytic signal does not possess first property — the negative frequencies are not suppressed. It was however effectively used in image processing for the purposes like corners detection [4].

A slightly different possibility is to define a 2D analytic signal with a given mask in the frequency domain, which would somehow resemble 1D mask 1 + sign(ω). The easiest example is to use only one of the directions

$$S_A(\zeta,\eta) = [1 + \mathrm{sign}(\zeta)]\,S(\zeta,\eta) \tag{8}$$

Another approach is the single orthant analytic signal, developed by Hahn [5]

$$S_A(\zeta,\eta) = [1 + \mathrm{sign}(\zeta)]\,[1 + \mathrm{sign}(\eta)]\,S(\zeta,\eta) \tag{9}$$

One can also define the analytic signal with averaged x and y directional Hilbert transform

$$S_A(\zeta,\eta) = \left[1 + \frac{1}{2}\left(\mathrm{sign}(\zeta) + \mathrm{sign}(\eta)\right)\right]S(\zeta,\eta) \tag{10}$$

The drawback of all above definitions of the 2D analytic signal is that they prefer certain directions over different ones and there is no reason for that if data is anisotropic or possesses no visible symmetry. Closer discussion of these methods is given in [6].

Spiral phase operator

This approach was developed by Larkin et al. in [7]. Instead of anisotropic 2D signum function analog they define spectral spiral phase function

$$P(\zeta,\eta) = \frac{\zeta + i\eta}{\sqrt{\zeta^2+\eta^2}} \tag{11}$$

It was shown by the means of a stationary phase method [8] that for signal which takes a form of $s(x, y) = b(x, y) \cos[\psi(x, y)]$ (unbiased interferometric image) following approximation holds

$$s_H(x,y) = b(x,y)\sin[\psi(x,y)] \approx \\ \approx -i\ \exp[-i\beta(x,y)]\,\mathbb{F}^{-1}\{P(\zeta,\eta)S(\zeta,\eta)\} \tag{12}$$

which serves as a definition of $sH(x, y)$. $\beta(x, y)$ denotes the angle of a local fringe orientation. There are many methods of extracting β from the fringe images. In this work I use a robust algorithm recently developed by Yang et al. [9]. Phase demodulation of interferograms with the spiral phase method is discussed in [10].

In [7] Larkin points the relationship between the spiral phase method and the Riesz transform, which is a tool of complex analysis, introduced to the signal processing by Felsberg and Sommer [11], who defined so-called *monogenic signal*.

Quaternionic Fourier Transform

Another approach is based on the definition of the Quaternionic Fourier Transform (QFT), which is in this case

$$S^q(\zeta,\eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i2\pi x\zeta}s(x,y)e^{-j2\pi y\eta}\,dxdy \tag{13}$$

where i and j denote the first and the second imaginary unit of a hypercomplex number — quaternion. QTF splits real function $s(x, y)$ into four different channels (as there are four different units in quaternion algebra), which was previously found to be useful in colour image processing. In [12] Bülow and Sommer introduced the quaternionic analytic signal

$$S_A^q(\zeta,\eta) = [1 + \mathrm{sign}(\zeta)][1 + \mathrm{sign}(\eta)]S^q(\zeta,\eta) \tag{14}$$

Note the similarity between the equations (9) and (14).

Different frequency masks are also possible. I test the mask from the equation (10)

$$S_A^q(\zeta,\eta) = \left[1 + \frac{1}{2}(\mathrm{sign}(\zeta) + \mathrm{sign}(\eta))\right]S^q(\zeta,\eta) \tag{15}$$

One of the papers which deal with the efficient numerical implementation of QFT is [13], where methods based on the FFT algorithm are developed.

AM demodulation efficiency comparison

In this section I perform amplitude demodulation of a synthetic radial chirp signal with certain envelope $b(x; y)$, where $r = \sqrt{x^2+y^2}$ and $(x; y) \in [-6; 6] \times [-6; 6]$.

$$s(x,y) = b(x,y)\cos\left(r + r^2\right) \tag{16}$$

This is one of the closed carrier type fringe patterns occurring in time-average two-beam interferometry (testing a vibrating unflat membrane) [14]. Example modulations were

$$b_1(x,y) = \mathrm{J}_0\left(\frac{r}{5}\right) \tag{17}$$

Table 1. Errors of presented methods applied to chirp signal, number of corresponding equation in brackets

b(x, y)	H1D	H1 (7)	H2 (8)	H3 (9)	H4 (10)	HS (12)	HQ (14)	HQ2 (15)
b1(x, y)	134.9	23.7	18.0	67.6	18.9	2.1	37.6	17.6
b2(x, y)	51.2	43.0	36.3	104.0	36.6	10.3	64.3	34.6
b3(x, y)	605.8	806.0	81.1	2612	497.8	31.4	1326	423.2
b4(x, y)	73.3	72.9	38.4	238.9	68.5	14.4	158.4	55.3

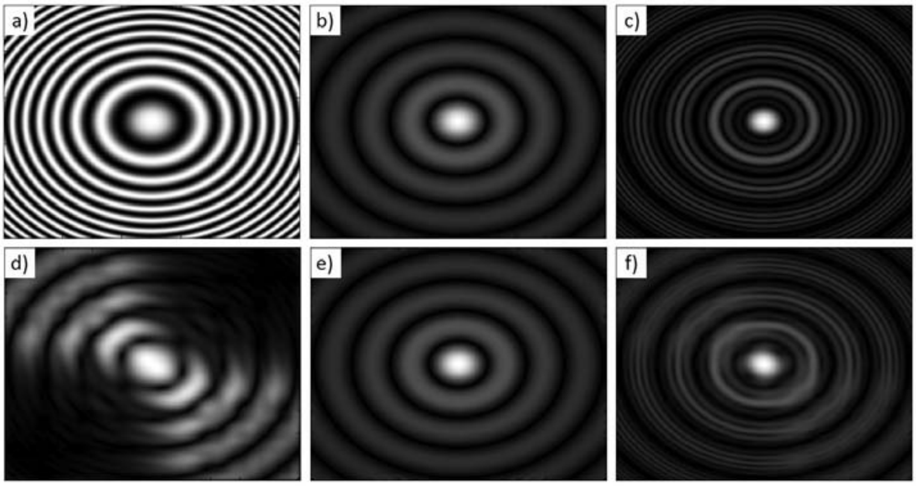

Fig. 1. Example results, a) chirp signal; b) modulation b1; c) modulated chirp signal; d) demodulation result given by H3 algorithm; e) result given by HS algorithm; f) result given by HQ2 algorithm

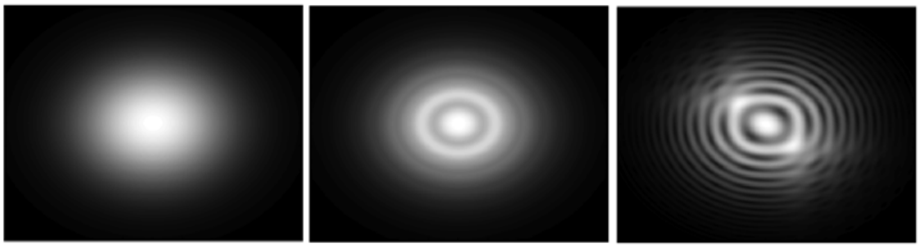

Fig. 2. Example results, from left to right: Modulation b2; result given by HS algorithm; result given by HQ2 algorithm

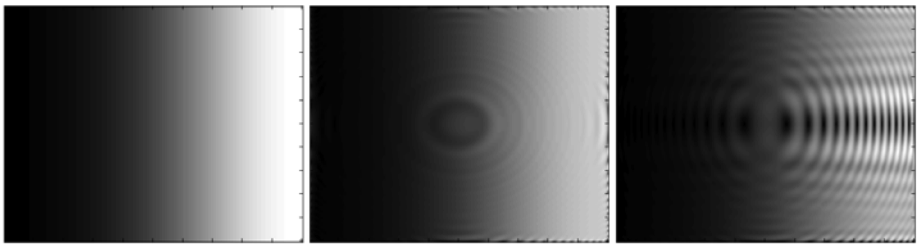

Fig. 3. Example results, from left to right: Modulation b3; result given by HS algorithm; result given by H4 algorithm

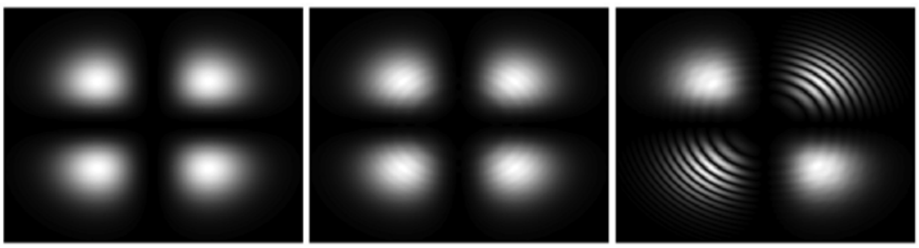

Fig. 4. Example results, from left to right: Modulation b4; result given by HS algorithm; result given by H2 algorithm

$$b_2(x,y) = \exp\left(\frac{-r^2}{10}\right) \tag{18}$$

$$b_3(x,y) = 5x^2y^2\exp\left(\frac{-r^2}{5}\right) \tag{19}$$

$$b_4(x,y) = \exp\left(\frac{-(x-6)^2}{35}\right) \tag{20}$$

$J_0(x)$ denotes a zeroth order Bessel function of the first kind. Error is measured by

$$Er(s_A) = \sqrt{\sum_{(x,y)\in\Omega}(|s_A(x,y)| - |b(x,y)|)^2} \tag{21}$$

where Ω stands for the image domain.

The relative errors (divided by the norm of the real modulation) for the spiral phase method, which gave the smallest absolute errors, equal respectively: 3.2%, 6.3%, 1.1%, 5.2%.

All implementations were based on the FFT algorithm rather than on the convolutions in the spatial domain.

Conclusion

Many approaches to the 2D Hilbert transform and the analytic signal were developed for the purposes of signal processing. It is clear that among all presented algorithms one which gives the best results in the amplitude demodulation of given synthetic data is the spiral phase method. In fact it is the only method which returned acceptably small relative error. For the real applications it is crucial to initially prepare data by removing bias and performing efficient denoising before passing to the demodulation [15].

Acknowledgments

This work has been supported in part by the Ministry of Science and Higher Education Grant No. N N505 464 238.

The author thanks Professor K. Patorski for reading the manuscript.

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